**PSEUDO-SYMMETRIC CONTACT 3-MANIFOLDS III**

BY

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**Abstract.** A trans-Sasakian 3-manifold is pseudo-symmetric if and only if it is $\eta$-Einstein. In particular, a quasi-Sasakian 3-manifold is pseudo-symmetric if and only if it is a coKähler manifold or a homothetic Sasakian manifold. Some examples of non-Sasakian pseudo-symmetric contact 3-manifolds are exhibited.

**Introduction.** A Riemannian 3-manifold $(M, g)$ is said to be a proper pseudo-symmetric space if its Ricci eigenvalues $\{\varrho_1, \varrho_2, \varrho_3\}$ satisfy the relation $\varrho_1 = \varrho_2 \neq \varrho_3$ ($\varrho_3 \neq 0$) up to numbering [14]. In particular, a proper pseudo-symmetric 3-space $(M, g)$ is said to be of constant type if $\varrho_3$ is a nonzero constant.

Such spaces have been studied from different motivations. For instance, in hypersurface geometry of nonflat 4-dimensional Riemannian space forms, it is shown that isometrically deformable hypersurfaces of type number two are pseudo-symmetric spaces of constant type [20].

O. Kowalski explained some other motivations of the study of pseudo-symmetric 3-spaces with constant principal Ricci curvatures in [28].

In our previous paper [11], we have investigated pseudo-symmetry of contact Riemannian 3-manifolds. In particular, we have shown that every Sasakian 3-manifold is constant type pseudo-symmetric. Moreover, in [12], we proved that tangent sphere bundles over Riemannian 2-manifolds are pseudo-symmetric if and only if the base manifolds are of constant curvature.

As is well known, odd-dimensional spheres are typical examples of Sasakian manifolds. On the other hand, odd-dimensional hyperbolic spaces cannot admit a Sasakian structure, but have a so-called Kenmotsu structure. K. Kenmotsu manifolds are normal (noncontact) almost contact Riemannian manifolds. Kenmotsu [25] investigated fundamental properties and local structure of such manifolds. Kenmotsu manifolds are locally isometric to warped product spaces with 1-dimensional base and Kähler fiber.


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As a generalization of both Sasakian manifolds and Kenmotsu manifolds, J. A. Oubiña [36] introduced the notion of trans-Sasakian manifold. An almost contact Riemannian manifold \((M; \varphi, \xi, \eta, g)\) is said to be a trans-Sasakian manifold if it satisfies
\[
(\nabla_X \varphi)Y = \alpha \{g(X,Y)\xi - \eta(Y)X\} + \beta \{g(\varphi X,Y)\xi - \eta(Y)\varphi X\}
\]
for some functions \(\alpha\) and \(\beta\). Here \(\nabla\) denotes the Levi-Civita connection.

J. C. Marrero [30] has proven that there are no proper trans-Sasakian manifolds in higher dimensions. Moreover, Marrero has shown the existence of proper trans-Sasakian 3-manifolds.

N. Hashimoto and M. Sekizawa [21] investigated conformally flat (irreducible) pseudo-symmetric 3-spaces of constant type. Their (local) classification says such spaces are warped products with 1-dimensional base and constant curvature fiber. One can see that every 3-dimensional warped product with 1-dimensional base and 2-dimensional fiber admits a trans-Sasakian structure with \(\alpha = 0\).

In this paper, motivated by these observations, we study pseudo-symmetry of trans-Sasakian 3-manifolds.

As another generalization of Sasakian manifolds, generalized \((\kappa, \mu)\)-spaces have been extensively studied ([5], [6], [9], [16], [17], [24], [26], [27]).

A contact Riemannian manifold is said to be a generalized \((\kappa, \mu)\)-space if
\[
R(X,Y)\xi = (\kappa I + \mu h)\{\eta(Y)X - \eta(X)Y\}, \quad X,Y \in \mathfrak{X}(M),
\]
for some functions \(\kappa\) and \(\mu\). Here \(h\) is an endomorphism field defined by \(h = \mathcal{L}_\xi \varphi / 2\). If both \(\kappa\) and \(\mu\) are constants, \(M\) is called a \((\kappa, \mu)\)-space. One can see that Sasakian manifolds are \((\kappa, \mu)\)-spaces with \(\kappa = 1\) and \(h = 0\).

In the final section, we shall study pseudo-symmetry of 3-dimensional generalized \((\kappa, \mu)\)-spaces.

Throughout this paper we assume that all manifolds are connected.

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1. Preliminaries. Let \((M, g)\) be a Riemannian manifold with its Levi-Civita connection \(\nabla\). Denote by \(R\) the Riemannian curvature of \(M\):
\[
R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}, \quad X,Y \in \mathfrak{X}(M).
\]
Here \(\mathfrak{X}(M)\) is the Lie algebra of all vector fields on \(M\). A tensor field \(F\) of type \((1,3)\),
\[
F : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M),
\]
is said to be curvature-like provided that \(F\) has the symmetry properties of \(R\). For example,
\[
(X \wedge Y)Z = g(Y,Z)X - g(Z,X)Y, \quad X,Y \in \mathfrak{X}(M),
\]
defines a curvature-like tensor field on \( M \). Note that the curvature \( R \) of a Riemannian manifold \((M, g)\) of constant curvature \( c \) satisfies the formula \( R(X, Y) = c(X \wedge Y) \).

As is well known, every curvature-like tensor field \( F \) acts on the algebra \( T^1_s(M) \) of all tensor fields on \( M \) of type \((1, s)\) as a derivation [35, p. 44]:

\[
(F \cdot P)(X_1, \ldots, X_s; Y, X) = F(X, Y)\{P(X_1, \ldots, X_s)\}
\]

\[
- \sum_{j=1}^s P(X_1, \ldots, F(X, Y)X_j, \ldots, X_s),
\]

\( X_1, \ldots, X_s \in \mathfrak{X}(M), P \in T^1_s(M). \)

The derivative \( F \cdot P \) of \( P \) with respect to \( F \) is a tensor field of type \((1, s + 2)\).

For a tensor field \( P \) of type \((1, s)\), we denote by \( Q(g, P) \) the derivative of \( P \) with respect to the curvature-like tensor defined by (1.1):

\[
Q(g, P)(X_1, \ldots, X_s; Y, X) = (X \wedge Y)P(X_1, \ldots, X_s)
\]

\[
- \sum_{j=1}^s P(X_1, \ldots, (X \wedge Y)X_j, \ldots, X_s).
\]

A Riemannian manifold \((M, g)\) is said to be semi-symmetric if \( R \cdot R = 0 \). Obviously, locally symmetric spaces \((\nabla R = 0)\) are semi-symmetric.

More generally, a Riemannian manifold \((M, g)\) is said to be pseudo-symmetric if

\[
R \cdot R = L Q(g, R)
\]

for some function \( L \). In particular, if \( L \) is constant, then \( M \) is called a pseudo-symmetric space of constant type [29]. A pseudo-symmetric space is said to be proper if it is not semi-symmetric.

For Riemannian 3-manifolds, the following characterizations of pseudo-symmetry are known (cf. [29]).

**Proposition 1.1.** A Riemannian 3-manifold \((M, g)\) is pseudo-symmetric if and only if it is quasi-Einstein. This means that there exists a one-form \( \omega \) such that the Ricci tensor field \( \varrho \) has the form

\[
\varrho = ag + b\omega \otimes \omega.
\]

Here \( a \) and \( b \) are functions.

**Proposition 1.2.** Let \((M, g)\) be a Riemannian 3-manifold. Then \((M, g)\) is a pseudo-symmetric space of constant type if and only if there exists a one-form \( \omega \) such that the Ricci tensor field \( \varrho \) is expressed as \( \varrho = ag + b\omega \otimes \omega \), where \( a \) is a function and \( a + b|\omega|^2 \) is a constant (provided that \( \omega \neq 0 \)).

**Remark 1.** The preceding proposition can be rephrased as follows (see [29, Proposition 0.1]):
A Riemannian 3-manifold is a pseudo-symmetric space of constant type with $R \cdot R = LQ(g, R)$ if and only if the principal Ricci curvatures (eigenvalues of the Ricci tensor) locally satisfy the following relations (up to numbering):

$$\varrho_1 = \varrho_2, \quad \varrho_3 = 2L.$$

2. Almost contact Riemannian manifolds

2.1. Let $M$ be an odd-dimensional manifold. An almost contact structure on $M$ is a quadruple of tensor fields $(\varphi, \xi, \eta, g)$, where $\varphi$ is an endomorphism field, $\xi$ is a vector field, $\eta$ is a one-form and $g$ is a Riemannian metric such that

$$\varphi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1,$$

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \quad X, Y \in \mathfrak{X}(M).$$

A $(2n+1)$-dimensional manifold together with an almost contact structure is called an almost contact Riemannian manifold (or almost contact manifold). The fundamental 2-form $\Phi$ of $M$ is defined by

$$\Phi(X, Y) := g(X, \varphi Y), \quad X, Y \in \mathfrak{X}(M).$$

If an almost contact Riemannian manifold $(M; \varphi, \xi, \eta, g)$ satisfies the condition

$$\varrho = ag + b\eta \otimes \eta$$

for some functions $a$ and $b$, then $M$ is said to be $\eta$-Einstein. Clearly, every $\eta$-Einstein almost contact 3-manifold is pseudo-symmetric.

2.2. Let $(M; \varphi, \xi, \eta, g)$ be an almost contact Riemannian manifold. A tangent plane at a point of $M$ is said to be a holomorphic plane if it is invariant under $\varphi$. The sectional curvature of a holomorphic plane is called its holomorphic sectional curvature. If the sectional curvature function of $M$ is constant on all holomorphic planes in $TM$, then $M$ is said to be of constant holomorphic sectional curvature.

On the other hand, if the sectional curvature function is constant on all planes in $TM$ which contain $\xi$, then $M$ is said to be of constant $\xi$-sectional curvature.

2.3. An almost contact Riemannian manifold $(M; \varphi, \xi, \eta, g)$ is called a contact Riemannian manifold if

$$\Phi = d\eta.$$

The formula (2.3) implies that the one-form $\eta$ is actually a contact form, namely $\eta$ satisfies $(d\eta)^n \wedge \eta \neq 0$. On a contact Riemannian manifold $M$, the structure vector field $\xi$ is traditionally called the characteristic vector field (or Reeb vector field).
2.4. An almost contact Riemannian manifold $M$ is said to be of rank $r = 2s$ ($> 0$) if $(d\eta)^s \neq 0$ and $\eta \wedge (d\eta)^s = 0$, and of rank $r = 2s + 1$ if $\eta \wedge (d\eta)^s \neq 0$ and $(d\eta)^{s+1} = 0$. Thus contact Riemannian manifolds are of rank $2n + 1$.

An almost contact Riemannian manifold $M$ is said to be normal if it satisfies $[\varphi, \varphi] + 2d\eta \otimes \xi = 0$, where $[\varphi, \varphi]$ is the Nijenhuis torsion of $\varphi$.

A normal almost contact Riemannian manifold is said to be a quasi-Sasakian manifold if its fundamental 2-form $\Phi$ is closed ($d\Phi = 0$) [1]. In particular, a contact Riemannian manifold is called a Sasakian manifold if it is normal. By definition, Sasakian manifolds are quasi-Sasakian manifolds of rank $2n + 1$.

2.5. According to Oubiña [36], an almost contact manifold $(M; \varphi, \xi, \eta, g)$ is said to be a trans-Sasakian manifold (of type $(\alpha, \beta)$) if

$$(\nabla_X \varphi)Y = \alpha\{g(X, Y)\xi - \eta(Y)X\} + \beta\{g(\varphi X, Y)\xi - \eta(Y)\varphi X\}$$

for some functions $\alpha$ and $\beta$.

In particular, a trans-Sasakian manifold is said to be a

- Sasakian manifold if $(\alpha, \beta) = (1, 0),$
- Kenmotsu manifold if $(\alpha, \beta) = (0, 1),$
- coKähler manifold if $(\alpha, \beta) = (0, 0).$

More generally a trans-Sasakian manifold of type $(\alpha, 0)$ with nonzero constant $\alpha$ is homothetic to a Sasakian manifold and called a homothetic Sasakian manifold or $\alpha$-Sasakian manifold. Analogously, a homothetic Kenmotsu manifold (or $\beta$-Kenmotsu manifold) is a trans-Sasakian manifold of type $(0, \beta)$ with nonzero constant $\beta$ [23].

**Remark 2.** Trans-Sasakian manifolds are normal [36].

There are two typical subclasses of the class of trans-Sasakian manifolds.

A trans-Sasakian manifold of type $(\alpha, \beta)$ is said to be of class $C_5$ if $\alpha = 0$. This class $C_5$ contains the class of $\beta$-Kenmotsu manifolds. On the other hand, a trans-Sasakian manifold is said to be of class $C_6$ if $\beta = 0$. $\alpha$-Sasakian manifolds and coKähler manifolds are of class $C_6$.

Let $(M; \varphi, \xi, \eta, g)$ be a trans-Sasakian manifold. Then from (2.1) and (2.4), we have

$$\nabla_X \xi = -\alpha \varphi X + \beta\{X - \eta(X)\xi\}, \quad X, Y \in \mathfrak{X}(M).$$

In particular, we have $\nabla_\xi \xi = 0$. Hence on trans-Sasakian manifolds, integral curves (trajectories) of $\xi$ are geodesics.

Moreover, trans-Sasakian manifolds satisfy the following formula ([7], [42, (4.9)]):

$$2\alpha \beta + \xi \alpha = 0.$$
The formula (2.6) implies the following characterization of $\alpha$-Sasakian manifolds.

**Lemma 2.1 ([7]).** Let $M$ be a trans-Sasakian manifold of type $(\alpha, \beta)$. If $\alpha$ is a nonzero constant, then $\beta = 0$ and hence $M$ is $\alpha$-Sasakian.

Marrero proved the following fundamental result (see also [42, Theorem 4.8]).

**Proposition 2.1 ([30]).** Trans-Sasakian manifolds of dimension $\geq 5$ are either of class $C_5$ or of class $C_6$ with constant $\alpha$.

From (2.5)–(2.6), one can deduce the following formulas:

$$\alpha = -(\nabla_X \Phi)(X, \xi), \quad \beta = -\frac{1}{2n} \delta \eta, \quad X \perp \xi, \quad |X| = 1.$$  

Here $\delta$ denotes the codifferential operator. The function $\delta \eta$ is defined by $\delta \eta = -\text{trace}(\nabla \eta)$.

### 3. Pseudo-symmetric trans-Sasakian 3-manifolds

3.1. Let $(M; \varphi, \xi, \eta, g)$ be an almost contact Riemannian 3-manifold. Then the covariant derivative $\nabla \varphi$ of $\varphi$ satisfies ([33])

$$\nabla_X \varphi Y = g(\varphi(\nabla_X \xi), Y)\xi - \eta(Y)\varphi \nabla_X \xi, \quad X, Y \in \mathfrak{X}(M).$$

In dimension 3, there exist *proper* trans-Sasakian manifolds, namely, trans-Sasakian manifolds which are neither of class $C_5$ or of class $C_6$ (see Proposition 3.7).

On the other hand, Olszak obtained the following characterization of trans-Sasakian 3-manifolds.

**Proposition 3.1.** Let $M$ be an almost contact Riemannian 3-manifold. Then the following three conditions are equivalent:

- $\nabla \xi \circ \varphi = \varphi \circ \nabla \xi$.
- $M$ is normal.
- $M$ is trans-Sasakian.

In that case, $M$ is a trans-Sasakian manifold of type $(\alpha, \beta)$ with

$$\alpha = \frac{1}{2} \text{trace}(\varphi \nabla \xi), \quad \beta = \frac{1}{2} \text{div} \xi.$$  

Moreover, Olszak gave the following characterization of quasi-Sasakian 3-manifolds.

**Proposition 3.2 ([33]).** Let $M$ be an almost contact Riemannian 3-manifold. Then $M$ is quasi-Sasakian if and only if $M$ is a trans-Sasakian manifold of type $(\alpha, 0)$ with $d\alpha(\xi) = 0$. 
In particular, every quasi-Sasakian 3-manifold is of class $C_6$.

The Ricci operator of a trans-Sasakian 3-manifold is given by the following formula due to Olszak [33].

**Proposition 3.3.** Let $M$ be a trans-Sasakian 3-manifold. Denote by $Q$ the Ricci operator of $M$ defined by

$$
\varrho(X,Y) = g(QX,Y), \quad X,Y \in \mathfrak{X}(M).
$$

Then $Q$ is given by

$$
QX = \frac{s}{2} + \xi \beta - (\alpha^2 - \beta^2)I + \{-(s/2 - \xi \beta + 3(\alpha^2 - \beta^2))\eta(X)\xi
$$

$$
- \eta(X)\{\text{grad } \beta - \varphi \text{ grad } \alpha\} - \{d\alpha(\varphi X) + d\beta(X)\}\xi,
$$

where $s = \text{tr } \varrho$ is the scalar curvature of $M$.

Now let $M$ be a pseudo-symmetric trans-Sasakian 3-manifold. Let us take a local orthonormal frame field $\{e_1, e_2, e_3\}$ such that $\eta(e_1) = 0$, $e_2 = \varphi e_1$, $e_3 = \xi$. Denote by $\varrho_{ij}$ the components of the Ricci tensor field $\varrho$ with respect to this frame;

$$
\varrho_{11} = \varrho_{22} = \frac{s}{2} - \alpha^2 + \beta^2 + d\beta(\xi), \quad \varrho_{33} = 2\alpha^2 - 2\beta^2 - 2d\beta(\xi),
$$

$$
\varrho_{12} = 0, \quad \varrho_{13} = d\alpha(\varphi e_1) + d\beta(e_1), \quad \varrho_{23} = d\alpha(\varphi e_2) + d\beta(e_2).
$$

Then the characteristic polynomial $\Psi(\lambda) = \det(\lambda \delta_{ij} - \varrho_{ij})$ for $\varrho$ is given by

$$
\Psi(\lambda) = (\lambda - \varrho_{11})F(\lambda),
$$

$$
F(\lambda) = \lambda^2 - (\varrho_{11} + \varrho_{33})\lambda + \varrho_{11}\varrho_{33} - 4 \sum_{i=1}^{2} \{d\alpha(\varphi e_i) + d\beta(e_i)\}^2.
$$

Hence $\varrho_0 := \varrho_{11} = \varrho_{22}$ is a Ricci eigenvalue. The solutions $\varrho_{\pm}$ to $F(\lambda) = 0$ are given by

$$
\varrho_{\pm} := \frac{1}{2} \left[ (\varrho_0 + \varrho_{33}) \pm \sqrt{(\varrho_0 - \varrho_{33})^2 + 4 \sum_{i=1}^{2} \{d\alpha(\varphi e_i) + d\beta(e_i)\}^2} \right].
$$

**Case 1:** $\varrho_0$ solves $F(\lambda) = 0$. In this case, $F(\varrho_0) = 0$ is equivalent to

$$
d\alpha(\varphi e_i) + d\beta(e_i) = 0, \quad i = 1, 2.
$$

In other words, $F(\varrho_0) = 0$ if and only if

$$
(3.2) \quad g(\text{grad } \beta - \varphi \text{ grad } \alpha, X) = 0
$$

for all $X \in \mathfrak{X}(M)$ orthogonal to $\xi$. In this case, the Ricci eigenvalues are $\varrho_0$, $\varrho_0$ and $\varrho_{33}$.

**Case 2:** $\varrho_+ = \varrho_-$. The trans-Sasakian manifold $M$ satisfies $\varrho_+ = \varrho_-$ if and only if $M$ satisfies (3.2) and $\varrho_{33} = \varrho_0$. In this case, all the Ricci eigenvalues are the same function. Hence $M$ is of constant curvature.

Hence we obtain the following result.
Lemma 3.1. Every pseudo-symmetric trans-Sasakian 3-manifold satisfies (3.2).

Here we give an interpretation of the condition (3.2).

Lemma 3.2. On a trans-Sasakian 3-manifold $M$, $\xi$ is an eigenvector field of the Ricci operator $Q$ if and only if $M$ satisfies (3.2).

Proof. Direct computations using Proposition 3.3 show that

$$Q\xi = 2(\alpha^2 - \beta^2 - d\beta(\xi))\xi - (\text{grad} \beta - \varphi \text{grad} \alpha).$$

Hence $\xi$ is an eigenvector field of $Q$ if and only if (3.2) holds. In that case, the following formulas hold:

$$\text{grad} \beta - \varphi \text{grad} \alpha = d\beta(\xi)\xi, \quad Q\xi = (2(\alpha^2 - \beta^2) - 3d\beta(\xi))\xi.$$

Lemma 3.3. Let $M$ be a trans-Sasakian 3-manifold. Then $M$ is pseudo-symmetric if and only if $M$ is $\eta$-Einstein.

Proof. ($\Leftarrow$) If $M$ is $\eta$-Einstein, then $M$ is pseudo-symmetric by Proposition 1.1.

($\Rightarrow$) Assume that $M$ is pseudo-symmetric. Then $M$ satisfies (3.2). Hence the Ricci tensor field is given by

$$\varrho = \{s/2 + \xi\beta - (\alpha^2 - \beta^2)\}g + \{-s/2 - 3\xi\beta + 3(\alpha^2 - \beta^2)\}\eta \otimes \eta.$$

This formula says $M$ is $\eta$-Einstein.

E. Vergara-Diaz and C. M. Wood gave the following characterization of (3.2).

Lemma 3.4 ([42]). A trans-Sasakian 3-manifold $M$ satisfies (3.2) if and only if $\xi$ is a harmonic section of the unit tangent sphere bundle $T_1M$ of $M$.

Hence we obtain the following result.

Theorem 3.1. Let $M$ be a trans-Sasakian 3-manifold. Then the following conditions are equivalent:

1. $M$ is pseudo-symmetric.
2. $M$ is $\eta$-Einstein.
3. $\xi$ is an eigenvector field of $Q$.
4. $\xi$ is a harmonic section of the unit tangent sphere bundle $T_1M$.
5. $M$ satisfies (3.2).

In this case, the Ricci tensor field of $M$ is given by

$$\varrho = \{s/2 + \xi\beta - (\alpha^2 - \beta^2)\}g + \{-s/2 - 3\xi\beta + 3(\alpha^2 - \beta^2)\}\eta \otimes \eta.$$

Example 3.1 (CoKähler 3-manifolds). Let $M$ be a coKähler 3-manifold. Then its Ricci operator is given by

$$Q = \frac{s}{2} I - \frac{s}{2} \eta \otimes \xi.$$
Thus the principal Ricci curvatures are
\[ \varrho_1 = \varrho_2 = \frac{s}{2}, \quad \varrho_3 = 0. \]

Hence \( M \) is semi-symmetric.

**Example 3.2 (Homothetic Kenmotsu manifolds).** Let \( M \) be a 3-dimensional almost contact Riemannian manifold of class \( C_5 \). Then its principal Ricci curvatures are
\[ \varrho_1 = \varrho_2 = \frac{s}{2} + \beta^2 + d\beta(\xi), \quad \varrho_3 = -2\beta^2 - 2d\beta(\xi). \]
Thus \( M \) is pseudo-symmetric if and only if \( d\beta(X) = 0 \) for all \( X \perp \xi \). In particular, every homothetic Kenmotsu 3-manifold is a pseudo-symmetric space of constant type.

**Example 3.3 (Homothetic Sasakian manifolds).** The principal Ricci curvatures of \( \alpha \)-Sasakian manifold \( M \) are
\[ \varrho_1 = \varrho_2 = \frac{s}{2} - \alpha^2, \quad \varrho_3 = 2\alpha^2 > 0. \]
Thus every \( \alpha \)-Sasakian 3-manifold is a pseudo-symmetric space of constant type.

**Remark 3.** Let \((M^3, g)\) be a locally symmetric Riemannian 3-manifold. Then \( M \) is (locally) isometric to one of the following spaces:
- Euclidean 3-space \( \mathbb{E}^3 \) (coKähler),
- the 3-sphere \( S^3(c^2) \) of curvature \( c^2 \) (homothetic Sasakian) or hyperbolic 3-space \( \mathbb{H}^3(-c^2) \) of curvature \( -c^2 \) (homothetic Kenmotsu),
- Riemannian products \( S^2(c^2) \times \mathbb{E}^1 \) or \( \mathbb{H}^2(-c^2) \times \mathbb{E}^1 \) (coKähler).

It is known that semi-symmetric Kenmotsu manifolds are locally symmetric and hence of constant curvature \(-1\) [25]. On the other hand, semi-symmetric Sasakian manifolds are locally symmetric and hence of constant curvature 1. Thus we obtain

**Corollary 3.1.**

1. \( \beta \)-Kenmotsu 3-manifolds other than hyperbolic space forms are proper pseudo-symmetric spaces of constant type.
2. \( \alpha \)-Sasakian 3-manifolds other than spherical space forms are proper pseudo-symmetric spaces of constant type.

Here we give a classification of pseudo-symmetric quasi-Sasakian 3-manifolds.

**Corollary 3.2.** A quasi-Sasakian 3-manifold is pseudo-symmetric if and only if it is a coKähler manifold or a homothetic Sasakian manifold.

**Proof.** For a quasi-Sasakian 3-manifold \( M \), (3.2) reduces to
\[ g(\varphi \text{grad } \alpha, e_1) = g(\varphi \text{grad } \alpha, e_2) = 0. \]
Since \( e_2 = \varphi e_1 \) and \( e_1 = -\varphi e_2 \), \((3.2)\) is equivalent to the equation
\[
e_1\alpha = e_2\alpha = 0.
\]
Thus \( M \) is pseudo-symmetric if and only if \( \alpha \) is constant, because \( \xi\alpha = 0 \) by Proposition 3.2.

Every Sasakian 3-manifold satisfies the condition \( Q\varphi = \varphi Q \). We consider
here the commutator \([Q, \varphi]\). Direct computation shows that
\[
(Q\varphi - \varphi Q)X = g(X, \text{grad} \alpha + \varphi \text{grad} \beta)\xi - \eta(X)(\text{grad} \alpha + \varphi \text{grad} \beta).
\]
From this formula, we get the following result.

**Proposition 3.4.** On a trans-Sasakian 3-manifold \( M \), the following three conditions are equivalent.

- \( \eta(Q\varphi - \varphi Q) = 0 \).
- \( Q\varphi = \varphi Q \).
- \( \text{grad} \alpha + \varphi \text{grad} \beta = 0 \).

In this case, \( \xi\alpha = -2\alpha\beta = 0 \) and \( M \) is \( \eta \)-Einstein with Ricci tensor field \((3.3)\).

**Proof.** It is clear that \( \eta([Q, \varphi]) = 0 \) if and only if \( Z := \text{grad} \alpha + \varphi \text{grad} \beta = 0 \). By \((2.6)\), we have \( \eta(Z) = \xi\alpha = -2\alpha\beta \).

**Example 3.4 (Warped products).** Let \((N, h, J)\) be a Riemannian 2-manifold together with the compatible orthogonal complex structure \( J \). Take a direct product \( M = E^1(t) \times N \) and denote by \( \pi \) and \( \sigma \) the natural projections onto the first and second factors, respectively.

Take the warped product \( M = E^1(t) \times f N \) and define \( \xi = \partial/\partial t \). Then the
Levi-Civita connection \( \nabla \) of \( M \) is given by (cf. [35])
\[
\nabla_{X^v}Y^v = (\nabla_{X^v}Y)^v - \frac{1}{f} g(X^v, Y^v) f' \xi,
\]
\[
\nabla_{\xi}X^v = \nabla_{X^v}\xi = \frac{f'}{f} X^v,
\]
\[
\nabla_{\xi}\xi = 0.
\]
Here the superscript \( v \) means the vertical lift operation of vector fields from \( N \) to \( M \). Define \( \varphi X = \{J(\sigma_\ast X)\}^v \). Then we get
\[
\nabla_X\xi = \beta(X - \eta(X)\xi),
\]
\[
(\nabla_X\varphi)Y = \beta \{g(\varphi X, Y) - \eta(Y)\varphi X\}, \quad \beta = f'/f.
\]
Hence \( M = E^1 \times f N \) is of class \( C_5 \).

Take a local orthonormal frame field \( \{\bar{e}_1, \bar{e}_2\} \) of \((N, h)\) such that \( \bar{e}_2 = J\bar{e}_1 \). Then we obtain a local orthonormal frame field \( \{e_1, e_2, e_3\} \) by
\[
e_1 = \frac{1}{f} \bar{e}_1^v, \quad e_2 = \frac{1}{f} \bar{e}_2^v = \varphi e_1, \quad e_3 = \xi.
\]
Then the holomorphic sectional curvature of $M$ is given by

$$H = K(e_1 \wedge e_2) = \frac{1}{f^2} \{K_N - (f')^2\}.$$ 

On the other hand, the sectional curvature of a plane containing $\xi$ is

$$K(e_1 \wedge e_3) = K(e_2 \wedge e_3) = -\frac{f''}{f}.$$ 

The Ricci tensor components $\varrho_{ij} = \varrho(e_i, e_j)$ are given by

$$\varrho_{11} = \varrho_{22} = \frac{K}{f^2} - \frac{f''}{f} - \left(\frac{f'}{f}\right)^2, \quad \varrho_{33} = -2\frac{f''}{f}.$$ 

Hence $M$ is a pseudo-symmetric space. In particular, $M$ is of constant type if and only if $f$ is a solution to $f'' = -Lf$ for some constant $L$.

The local structure of Kenmotsu manifolds is described as follows.

**Proposition 3.5 ([25]).**

- Kenmotsu manifolds of constant holomorphic sectional curvature are hyperbolic space forms of curvature $-1$.
- A Kenmotsu manifold $M$ is locally isomorphic to a warped product $I \times_f N$ whose base $I \subset \mathbb{E}^1(t)$ is an open interval and $N$ is a Kähler manifold with warping function $f(t) = e^{ct}$, $c \neq 0$. The structure vector field is $\xi = \partial/\partial t$.

As we saw before, warped products of the form $M = \mathbb{E}^1 \times_f N$ with 2-dimensional standard fiber are pseudo-symmetric trans-Sasakian 3-manifolds. In particular $M$ is of constant type if and only if the warping function $f$ satisfies the ODE $f'' = -Lf$ for some constant $L$. In particular, if we assume that, in addition, $N$ is of constant Gaussian curvature, the warped product is conformally flat. Conversely, 3-dimensional conformally flat irreducible pseudo-symmetric space of constant type are locally isometric to warped products as above. More precisely, Hashimoto and Sekizawa obtained the following result.

**Theorem 3.2 ([21]).** Let $(M, g)$ be a 3-dimensional conformally flat irreducible pseudo-symmetric space of constant type. Then $M$ is locally isometric to the warped product space $\mathbb{E}^1 \times_f N^2(k)$, whose base is the real line $\mathbb{E}^1$ and standard fiber $N^2(k)$ is a 2-dimensional space form of curvature $k$, respectively. The warping function $f$ is one of the following:

$$f(t) = \begin{cases} 
  t, & L = 0, \\
  \sinh(\lambda t) \text{ or } \cosh(\lambda t), & L = -\lambda^2 < 0, \\
  \sin(\lambda t), & L = \lambda^2 > 0.
\end{cases}$$
The principal Ricci curvatures are given by

$$\varrho_1 = \varrho_2 = \pm a^2 \frac{f(t)^2}{f(t)^2 + 2L}, \quad \varrho_3 = 2L,$$

where $a$ is a positive constant. The curvature constant $k$ is determined as follows:

- If $(M, g)$ is semi-symmetric, then $k = 1 \pm a^2$.
- If $L = -\lambda^2 < 0$, then $k = \lambda^2 \pm a^2$ when $f(t) = \sinh(\lambda t)$, and $k = -\lambda^2 \pm a^2$ when $f(t) = \cosh(\lambda t)$, respectively.
- If $L = \lambda^2 > 0$, then $k = \lambda^2 \pm a^2$.

**Remark 4.** M. S. Goto [15] studied global structures of compact conformally flat semi-symmetric spaces of dimension 3. Olszak [34] gave an example of a conformally flat quasi-Sasakian 3-manifold which is not pseudo-symmetric.

CoKähler manifolds are characterized as follows.

**Proposition 3.6 ([8, Lemma 2]).** Let $(M; \varphi, \xi, \eta, g)$ be an almost contact manifold such that $\xi$ is Killing and $d\eta = 0$. Then $M$ is locally isometric to a Riemannian product $N \times I$, where $I$ is an open interval and $N$ is an almost Hermitian manifold.

In particular, a coKähler manifold is locally isometric to a Riemannian product $N \times I$, where $I$ is an open interval and $N$ is a Kähler manifold.

Marrero [30] showed the nonexistence of proper trans-Sasakian manifolds of dimension greater than 3. On the other hand, he showed the following method of constructing proper trans-Sasakian 3-manifolds (see also [32]).

**Proposition 3.7 ([30], [32]).** Let $M$ be a Sasakian 3-manifold and $\sigma$ a nonconstant positive function on $M$. Then the pseudo-conformal deformation

$$g \mapsto g^\sigma := \sigma g + (1 - \sigma)\eta \otimes \eta$$

induces a trans-Sasakian manifold $(M; \varphi, \xi, \eta, g^\sigma)$ of type $(\alpha^\sigma, \beta^\sigma)$, where

$$\alpha^\sigma = \frac{1}{\sigma}, \quad \beta^\sigma = \frac{1}{2\sigma}d\sigma(\xi).$$

- If $d\sigma(\xi) \neq 0$, then $(M; \varphi, \xi, \eta, g^\sigma)$ is a proper trans-Sasakian manifold. Moreover, $(M; \varphi, \xi, \eta, g^\sigma)$ is neither of class $C_5$ nor of class $C_6$.
- If $d\sigma(\xi) = 0$, then $M$ is quasi-Sasakian. Conversely, every quasi-Sasakian 3-manifold can be obtained in this way ([32]).
Let $\mathbb{R}^3(-3)$ be the Heisenberg group

$$\left\{ \begin{pmatrix} 1 & y & z \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix} \right| (x, y, z) \in \mathbb{R}^3 \right\}$$

with the canonical Sasakian structure $(\varphi, \xi, \eta, g)$ of constant holomorphic sectional curvature $-3$:

$$g = \frac{1}{4} (dx^2 + dy^2) + \eta \otimes \eta, \quad \eta = \frac{1}{2} (dz - xdy), \quad \xi = 2 \frac{\partial}{\partial z},$$

$$\varphi = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ x & 0 & 0 \end{pmatrix}.$$  

We take a global orthonormal frame field:

$$e_1 = 2 \frac{\partial}{\partial x}, \quad e_2 = 2 \left( \frac{\partial}{\partial y} + x \frac{\partial}{\partial z} \right), \quad e_3 = 2 \frac{\partial}{\partial z} = \xi.$$  

Then the endomorphism field $\varphi$ satisfies $\varphi e_1 = e_2, \varphi e_2 = -e_1$ and $\varphi \xi = 0$.

Now let us take a positive function $\sigma$ on $\mathbb{R}^3(-3)$ such that $d\sigma(\xi) \neq 0$ and consider the pseudo-conformal deformation $g \mapsto \tilde{g} := g^\sigma$. The resulting proper trans-Sasakian 3-manifold is of type

$$\tilde{\alpha} = \frac{1}{\sigma}, \quad \tilde{\beta} = \frac{\sigma_z}{2\sigma}.$$  

We can take a global orthonormal frame field

$$\tilde{e}_1 = \frac{1}{\sqrt{\sigma}} e_1, \quad \tilde{e}_2 = \frac{1}{\sqrt{\sigma}} e_2, \quad \tilde{e}_3 = \xi.$$  

Let us consider the pseudo-symmetry condition:

$$\tilde{g}(\tilde{e}_i, \text{grad}_{\tilde{g}} \tilde{\beta} - \varphi \text{grad}_{\tilde{g}} \tilde{\alpha}) = 0, \quad i = 1, 2,$$

for the deformed manifold. Direct computation shows that the deformed manifold is pseudo-symmetric if and only if

$$\left( \frac{\sigma_z}{2\sigma} \right)_x + \left( \frac{1}{\sigma} \right)_y + x \left( \frac{1}{\sigma} \right)_z = 0, \quad (3.4)$$

$$- \left( \frac{1}{\sigma} \right)_x + \left( \frac{\sigma_z}{2\sigma} \right)_y + x \left( \frac{\sigma_z}{2\sigma} \right)_z = 0. \quad (3.5)$$

**Proposition 3.8.** Let $\sigma(x, y, z)$ be a positive solution to the system $(3.4)-(3.5)$ such that $\sigma_z \neq 0$. Then the pseudo-conformal deformation of $\mathbb{R}^3(-3)$ by $\sigma$ is a pseudo-symmetric proper trans-Sasakian 3-manifold.
For simplicity, we assume that $\sigma$ depends only on $z$. Then the pseudo-symmetry condition reduces to

$$
\left(\frac{1}{\sigma}\right)_z = \left(\frac{\sigma_z}{\sigma}\right)_z = 0.
$$

Hence $\sigma$ is a constant. Thus the example due to Marrero (pseudo-conformal deformation of $\mathbb{R}^3(-3)$ with $\sigma = e^z$) is not pseudo-symmetric.

4. Pseudo-symmetric homogeneous contact Riemannian 3-manifolds. A contact Riemannian manifold $(M; \phi, \xi, \eta, g)$ is said to be a homogeneous contact Riemannian manifold if there exists a connected Lie group $G$ acting transitively on $M$ as a group of isometries which leave the contact form $\eta$ invariant.

Assume that $M$ is simply connected. Then by a theorem due to Sekigawa [40], $M$ is a Riemannian symmetric space or a Lie group with a left invariant metric. By using the classification of 3-dimensional Lie groups with left invariant metric due to J. Milnor [31], D. Perrone classified all simply connected homogeneous contact Riemannian 3-manifolds.

**Proposition 4.1 ([37]).** Let $(M; \varphi, \xi, \eta, g)$ be a simply connected homogeneous contact Riemannian 3-manifold. Then $M$ is a Lie group $G$ together with a left invariant contact Riemannian structure $(\eta, g)$ and Webster scalar curvature $W = (s - \varrho(\xi, \xi) + 4)/8$ and torsion invariant $\tau = \mathcal{L}_\xi g$. Here $\mathcal{L}_\xi$ denotes the Lie differentiation with respect to $\xi$.

- **If $G$ is unimodular,** then $G$ is one of the following:
  1. the Heisenberg group $\mathbb{H}_3$ if $W = |\tau| = 0$;
  2. $SU(2)$ if $4\sqrt{2}W > |\tau|$;
  3. $\tilde{E}(2)$ if $4\sqrt{2}W = |\tau| > 0$;
  4. $\tilde{SL}(2, \mathbb{R})$ if $-|\tau| \neq 4\sqrt{2}W < |\tau|$;
  5. $E(1,1)$ if $4\sqrt{2}W = -|\tau| < 0$.

  The Lie algebra $\mathfrak{g}$ of $G$ is generated by an orthonormal basis $\{e_1, e_2 = \varphi e_1, e_3 = \xi\}$ with commutation relations

  $$
  [e_1, e_2] = 2e_3, \quad [e_2, e_3] = c_2 e_1, \quad [e_3, e_1] = c_3 e_2.
  $$

- **If $G$ is nonunimodular,** then the Lie algebra $\mathfrak{g}$ of $G$ satisfies the commutation relations

  $$
  [e_1, e_2] = \alpha e_2 + 2e_3, \quad [e_2, e_3] = 0, \quad [e_3, e_1] = \gamma e_2,
  $$

  where $e_3 = \xi, e_1, e_2 \in \text{Ker} \eta, e_2 = \varphi e_1, \alpha \neq 0$ and $4\sqrt{2}W < |\tau|$. If $\gamma = 0$ then the structure is Sasakian ($\tau = 0$) and $W = -\alpha^2/4$.

In our previous work [11], we obtained the following result.
Proposition 4.2. Every 3-dimensional unimodular Lie groups with special left invariant contact Riemannian structure is a pseudo-symmetric space of constant type.

On the other hand, unfortunately, our result on nonunimodular groups in [11] is not correct. We take this opportunity to give a correct classification of pseudo-symmetric nonunimodular Lie groups with left invariant contact Riemannian structure (cf. [22]).

Let $G$ be a 3-dimensional nonunimodular Lie group with a left invariant contact Riemannian structure. Then there exists an orthonormal basis $\{e_1, e_2 = \varphi e_1, e_3 = \xi\}$ of the Lie algebra $\mathfrak{g}$ such that

$$[e_1, e_2] = \alpha e_2 + 2e_3, \quad [e_2, e_3] = 0, \quad [e_3, e_1] = \gamma e_2,$$

where $\alpha \neq 0$. In particular, $\gamma = 0$ if and only if $G$ is a Sasakian manifold of constant holomorphic sectional curvature $-3 - \alpha^2$. In this case $G$ is isomorphic to $\tilde{\mathbb{SL}}(2, \mathbb{R})$ with left invariant Sasakian structure of some constant holomorphic curvature as a contact Riemannian manifold, but not isomorphic as a homogeneous contact manifold. The Ricci curvatures of $G$ are given in terms of $\{e_1, e_2, e_3\}$ as follows:

$$\rho_{11} = -\alpha^2 - 2 + 2\gamma - \gamma^2/2,$$
$$\rho_{22} = -\alpha^2 - 2 + \gamma^2/2,$$
$$\rho_{32} = \rho_{23} = \alpha\gamma, \quad \rho_{33} = 2 - \gamma^2/2.$$

The characteristic polynomial $\Psi(\lambda) = \det(\lambda\delta_{ij} - \rho_{ij})$ for the Ricci tensor field is given by

$$\Psi(\lambda) = (\lambda - \rho_{11})F(\lambda),$$
$$F(\lambda) = \lambda^2 + \alpha^2\lambda - \{\alpha^2(2 + \gamma^2/2) + (2 - \gamma^2/2)^2\}.$$

The discriminant $D$ of $F(\lambda) = 0$ is

$$D = \alpha^4 + 4\{\alpha^2(2 + \gamma^2/2) + (2 - \gamma^2/2)^2\} > 0.$$ 

Thus the equation $F(\lambda) = 0$ has no double roots. On the other hand, we have $F(\rho_{11}) = 2\gamma\{(\gamma + 2)^2 + \alpha^2\}$. Thus $F(\rho_{11}) = 0$ if and only if $\gamma = 0$. In this case,

$$\rho_{11} = \rho_{22} = -\alpha^2 - 2, \quad \rho_{33} = 2.$$

Thus we obtain the following result.

Theorem 4.1. A 3-dimensional nonunimodular Lie group with a left invariant contact Riemannian structure is pseudo-symmetric if and only if it is a Sasakian space form of constant holomorphic sectional curvature $-3 - \alpha^2 < -3$. 
5. **Pseudo-symmetric non-Sasakian contact Riemannian 3-manifolds.** As we saw in the preceding section, there exist many pseudo-symmetric homogeneous Riemannian 3-manifolds. Moreover, the unit tangent sphere bundle of a Riemannian 2-manifold of constant curvature is locally homogeneous and pseudo-symmetric. In fact, in our previous paper [12], we have shown that for every Riemannian 2-manifold of constant curvature $c$, its unit tangent sphere bundle $T_1M$ equipped with the standard contact Riemannian structure is a pseudo-symmetric space of constant type. In particular, if $c \neq 1$, the unit tangent sphere bundle is non-Sasakian. It was pointed out by D. E. Blair, Th. Koufogiorgos and B. J. Papantoniou [5] that the unit tangent sphere bundle of a surface with constant curvature $c$ with standard contact Riemannian structure is a so-called $(\kappa, \mu)$-space with $\kappa = c(2 - c)$ and $\mu = -2c$.

Note that non-Sasakian 3-dimensional $(\kappa, \mu)$-spaces are locally homogeneous and of constant holomorphic sectional curvature $H = -(\kappa + \mu)$.

On the other hand, O. Kowalski [28] gave examples of nonhomogeneous pseudo-symmetric 3-spaces. Nonhomogeneous Sasakian 3-manifolds provide examples of nonhomogeneous pseudo-symmetric spaces.

In view of the results of our previous papers, one may raise the following question:

*Are there examples of nonhomogeneous, non-Sasakian, pseudo-symmetric contact Riemannian 3-manifolds?*

In this section we exhibit some examples of non-Sasakian pseudo-symmetric contact Riemannian 3-manifolds.

**5.1.** Let $M$ be a contact Riemannian 3-manifold. Then the formula (3.1) reduces to ([41])

$$(\nabla_X \varphi)Y = g((I + h)X, Y)\xi - \eta(Y)(I + h)X, Y \quad X \in \mathfrak{X}(M),$$

where $I$ is the identity transformation and the endomorphism field $h$ is defined by $h = \mathcal{L}_\xi \varphi / 2$.

Now let us define an endomorphism field $\ell$ by

$$\ell(X) = R(\xi, X)\xi, \quad X \in \mathfrak{X}(M).$$

Then $\ell$ and $h$ satisfy the following relations:

$$h\xi = \ell(\xi) = 0, \quad \eta \circ h = 0, \quad \text{tr } h = \text{tr}(\varphi h) = 0, \quad h\varphi + \varphi h = 0,$$

$$\nabla_\xi h = \varphi(I - \ell - h^2), \quad \text{tr } \ell = 2 - \text{tr}(h^2).$$

**Lemma 5.1** (cf. [10]). *Let $M$ be a 3-dimensional contact Riemannian manifold. Then there exists a local orthonormal frame field $\mathcal{E} = \{e_1, e_2, e_3\}$ such that*

$$he_1 = \lambda e_1, \quad e_2 = \varphi e_1, \quad e_3 = \xi.$$
With respect to $\mathcal{E}$, the Levi-Civita connection $\nabla$ is given by
\begin{align*}
\nabla_{e_1}e_1 &= be_2, \quad \nabla_{e_1}e_2 = -be_1 + (1 + \lambda)\xi, \quad \nabla_{e_1}\xi = -(1 + \lambda)e_2, \\
\nabla_{e_2}e_1 &= -ce_2 + (\lambda - 1)e_3, \quad \nabla_{e_2}e_2 = ce_1, \quad \nabla_{e_2}\xi = (1 - \lambda)e_1, \\
\nabla_{\xi}e_1 &= \alpha e_2, \quad \nabla_{\xi}e_2 = -\alpha e_1, \quad \nabla_{\xi}\xi = 0.
\end{align*}

The Ricci operator $Q$ is given by
\begin{align*}
Qe_1 &= \varrho_{11}e_1 + \xi(\lambda)e_2 + (2b\lambda - e_2(\lambda))\xi, \\
Qe_2 &= \xi(\lambda)e_1 + \varrho_{22}e_2 + (2c\lambda - e_1(\lambda))\xi, \\
Q\xi &= (2b\lambda - e_2(\lambda))e_1 + (2c\lambda - e_1(\lambda))e_2 + 2(1 - \lambda^2)\xi,
\end{align*}
where
\begin{align*}
\varrho_{11} &= s/2 + \lambda^2 - 2\alpha\lambda - 1, \quad \varrho_{22} = s/2 + \lambda^2 + 2\alpha\lambda - 1.
\end{align*}

**Proposition 5.1 ([16]).** On a contact Riemannian 3-manifold with local orthonormal frame field $\mathcal{E}$ as in Lemma 5.1, $Q\varphi = \varphi Q$ if and only if $b = c = 0$.

**Proposition 5.2.** Let $M$ be a contact Riemannian 3-manifold with local orthonormal frame field $\mathcal{E}$ as in Lemma 5.1. Then $\varrho_{11} = \varrho_{22}$ if and only if $\alpha = 0$ or $M$ is Sasakian.

**Corollary 5.1** (cf. [18, Proposition 2]). If a contact Riemannian 3-manifold $M$ has constant $\xi$-sectional curvature, then $\alpha = 0$ or $M$ is Sasakian.

**Remark 5.** A contact Riemannian 3-manifold is said to be a $(3-\tau)$-manifold if $\nabla_{\xi}\tau = 0$ [3], [16]. Every contact Riemannian 3-manifold of constant $\xi$-sectional curvature is a $(3-\tau)$-manifold with constant $\text{tr} \ell$ [18, Proposition 2].

Now let $M$ be a contact Riemannian 3-manifold with constant $\xi$-sectional curvature. Then the Ricci operator has the form:
\begin{align*}
Qe_1 &= (s/2 + \lambda^2 - 1)e_1 + 2b\lambda\xi, \\
Qe_2 &= (s/2 + \lambda^2 - 1)e_2 + 2c\lambda\xi, \\
Q\xi &= 2b\lambda e_1 + 2c\lambda e_2 + 2(1 - \lambda^2)\xi.
\end{align*}

Hence the characteristic polynomial $\Psi(t) = \det(t\delta_{ij} - \varrho_{ij})$ for the Ricci tensor field $\varrho$ is
\[
\Psi(t) = (t - \varrho_{11})F(t),
\]
\[
F(t) = t^2 - (\varrho_{11} + 2 - 2\lambda^2)t + \{2(1 - \lambda^2)\varrho_{11} - 4\lambda^2(b^2 + c^2)\}.
\]

**Case 1:** $\varrho_{11}$ solves $F(t) = 0$. Direct computation shows that $F(\varrho_{11}) = 0$ if and only if $\lambda = 0$ (i.e., $M$ is Sasakian) or $b = c = 0$ (i.e., $Q\varphi = \varphi Q$).

**Case 2:** $F(t) = 0$ has real double solutions. The discriminant $\mathcal{D}$ of the equation $F(t) = 0$ is
\[
\mathcal{D} = (\varrho_{11} + 2\lambda^2 - 2)^2 + 16(b^2 + c^2).
\]
Hence $F(t) = 0$ has two equal real solutions if and only if $q_{11} + 2\lambda^2 - 2 = 0$ and $b = c = 0$.

5.2.

**Definition 5.1.** A contact Riemannian manifold is said to be a generalized $(\kappa, \mu)$-space if

$$R(X, Y)\xi = (\kappa I + \mu h)\{\eta(Y)X - \eta(X)Y\}, \quad X, Y \in \mathfrak{X}(M),$$

for some functions $\kappa$ and $\mu$. If both $\kappa$ and $\mu$ are constants, $M$ is called a $(\kappa, \mu)$-space. A generalized $(\kappa, \mu)$-space is said to be proper if $(d\kappa)^2 + (d\mu)^2 \neq 0$.

Sasakian manifolds are $(\kappa, \mu)$-spaces with $\kappa = 1$, $\mu = 0$ and $h = 0$. Generalized $(\kappa, \mu)$-spaces are of particular interest in dimension 3. In fact, the following results are known.

**Theorem 5.1** ([26]). Let $M$ be a non-Sasakian generalized $(\kappa, \mu)$-space of dimension greater than 3. Then $M$ is a $(\kappa, \mu)$-space.

**Proposition 5.3** ([27, Lemma 1]). Let $M$ be a 3-dimensional generalized $(\kappa, \mu)$-space. Then there exists a local orthonormal frame field $E = \{e_1, e_2, e_3\}$ such that

$$he_1 = \lambda e_1, \quad e_2 = \varphi e_1, \quad e_3 = \xi,$$

where $\lambda = \sqrt{1 - \kappa} > 0$. The Ricci operator $Q$ is given by

$$QX = aX + b\eta(X)\xi + \mu hX, \quad X \in \mathfrak{X}(M).$$

with

$$a = \frac{1}{2}(s - 2\kappa), \quad b = \frac{1}{2}(6\kappa - s).$$

Hence the principal Ricci curvatures of a 3-dimensional generalized $(\kappa, \mu)$-space are given by

$$\varrho_1 = \frac{1}{2}(s - 2\kappa) + \mu \sqrt{1 - \kappa},$$

$$\varrho_2 = \frac{1}{2}(s - 2\kappa) - \mu \sqrt{1 - \kappa},$$

$$\varrho_3 = 2\kappa.$$

From these we can see that

$$\varrho_1 = \varrho_2 \iff \mu = 0 \text{ or } \kappa = 1,$$

$$\varrho_1 = \varrho_3 \iff \mu = \frac{1}{\sqrt{1 - \kappa}}(3\kappa - s/2),$$

$$\varrho_2 = \varrho_3 \iff \mu = -\frac{1}{\sqrt{1 - \kappa}}(3\kappa - s/2).$$

**Proposition 5.4.** A 3-dimensional proper generalized $(\kappa, \mu)$-space is pseudo-symmetric if and only if $\mu = 0$ or $\mu = \pm \frac{1}{\sqrt{1 - \kappa}}(3\kappa - s/2)$. 
Perrone gave a characterization of “generalized \((\kappa, \mu)\)-property” as follows:

**Theorem 5.2** ([39]). On a contact Riemannian 3-manifold \(M\), its Reeb vector field \(\xi : M \to T_1M\) is a harmonic map with respect to the Sasaki-lift metric if and only if \(M\) satisfies the generalized \((\kappa, \mu)\)-condition on an everywhere dense open subset of \(M\).

For 3-dimensional \((\kappa, \mu)\)-spaces, the following characterization is known.

**Theorem 5.3** ([6]). Let \(M\) be a contact Riemannian 3-manifold. Then the following three conditions are equivalent:

1. \(M\) is \(\eta\)-Einstein.
2. \(Q\varphi = \varphi Q\).
3. \(M\) is a \((\kappa, 0)\)-space with \(\kappa \leq 1\).

In the third case, \(M\) is of constant holomorphic sectional curvature \(-\kappa\).

**Theorem 5.4** ([6]). Let \(M\) be a contact Riemannian 3-manifold. Then \(M\) satisfies \(Q\varphi = \varphi Q\) if and only if \(M\) is either

1. a Sasakian 3-manifold,
2. a flat contact Riemannian 3-manifold, or
3. a non-Sasakian contact Riemannian space form of constant holomorphic sectional curvature \(-\kappa\) and constant \(\xi\)-sectional curvature \(\kappa\).

In the third case, \(\kappa < 1\).

These results imply that every \((\kappa, 0)\)-space with \(\kappa \leq 1\) is a pseudo-symmetric space.

To close this paper we exhibit two examples.

**Example 5.1.** In [38], D. Perrone gave the following example of weakly \(\varphi\)-symmetric 3-space which is neither homogeneous nor strongly \(\varphi\)-symmetric. Let \(M\) be the open submanifold \(\{(x, y, z) \in \mathbb{R}^3 | x \neq 0\}\) of Cartesian 3-space \(\mathbb{R}^3\) together with a contact form \(\eta = xydx + dz\). The Reeb vector field of this contact 3-manifold is \(\xi = \partial/\partial z\). Take a global frame field \(e_1 = -2 \frac{\partial}{x \partial y}, e_2 = \frac{\partial}{x} - 4z \frac{\partial}{y} - xy \frac{\partial}{z}, e_3 = \xi\) and define a Riemannian metric \(g\) by the condition that \(\{e_1, e_2, e_3\}\) is orthonormal with respect to it. Moreover, define an endomorphism field \(\varphi\) by \(\varphi e_1 = e_2, \varphi e_2 = -e_1\) and \(\varphi \xi = 0\). Then \((\varphi, \xi, \eta, g)\) is the associated almost contact structure of \((M, \eta)\). The endomorphism field \(h\) satisfies \(he_1 = e_1, he_2 = -e_2\). Hence \(M\) is non-Sasakian. Perrone showed that this contact Riemannian 3-manifold is nonhomogeneous. The Ricci operator of \((M, g)\) is given by \(Q = -8\omega^1 \otimes e_1\), where \(\omega^1\) is the dual 1-form of \(e_1\). Hence
(M, g) is pseudo-symmetric. Thus Perrone’s example is a nonhomogeneous and non-Sasakian contact Riemannian 3-manifold which is pseudo-symmetric.

Next we recall an example of a generalized (κ, µ)-space constructed by Koufogiorgos and Ch. Tsichlias [26] (see also [24, Section 4.3]).

**Example 5.2.** Let \( M = \{(x, y, z) \in \mathbb{R}^3 \mid z \neq 0\} \). Define a frame field \( \mathcal{U} = \{u_1, u_2, u_3\} \) by

\[
\begin{align*}
    u_1 &= \frac{\partial}{\partial x}, \quad u_2 = -2yz \frac{\partial}{\partial x} + \frac{2x}{z^2} \frac{\partial}{\partial y} - \frac{1}{z^2} \frac{\partial}{\partial z}, \quad u_3 = \frac{1}{z} \frac{\partial}{\partial y},
\end{align*}
\]

Then we have

\[
[u_1, u_2] = \frac{2}{z^2} u_3, \quad [u_2, u_3] = 2u_1 + \frac{1}{z^3} u_3, \quad [u_3, u_1] = 0.
\]

Put \( \xi = u_1 \) and define a Riemannian metric \( g \) by \( g(u_i, u_j) = \delta_{ij} \). Then we have a contact Riemannian manifold \( M = (M; \varphi, \xi, \eta, g) \) with structure \( \eta = g(\xi, \cdot) \) and

\[
\begin{align*}
    \varphi u_1 &= 0, \quad \varphi u_2 = u_3, \quad \varphi u_3 = -u_2.
\end{align*}
\]

Then \( \mathcal{E} = \{e_1, e_2, e_3\} = \{u_2, u_3, u_1\} \) satisfies the condition

\[
he_1 = \lambda e_1, \quad he_2 = -\lambda e_2, \quad h\xi = 0,
\]

where \( \lambda = 1/z^2 \). Moreover this contact Riemannian 3-manifold is a generalized (κ, µ)-space with

\[
\begin{align*}
    \kappa &= \frac{z^4 - 1}{z^4}, \quad \mu = 2 \left(1 - \frac{1}{z^2}\right).
\end{align*}
\]

The Ricci operator \( Q \) is given by

\[
Qe_1 = \varrho_{11} e_1, \quad Qe_2 = \varrho_{22} e_2, \quad Q\xi = 2(1 - \lambda^2)\xi,
\]

where

\[
\begin{align*}
    \varrho_{11} &= s/2 + \lambda^2 - 2\alpha\lambda - 1, \quad \varrho_{22} = s/2 + \lambda^2 + 2\alpha\lambda - 1, \\
    \alpha &= -1 + 1/z^2, \quad b = 1/z^3, \quad c = 0.
\end{align*}
\]

The scalar curvature is

\[
s = \frac{6}{z^6} - \frac{2}{z^4} - \frac{2}{z^3} + \frac{4}{z^2} - 2.
\]

Hence this space is not pseudo-symmetric.

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