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FUNCTIONS HAVING THE DARBOUX PROPERTY AND SATISFYING SOME FUNCTIONAL EQUATION

BY

ELIZA JABŁOŃSKA (Rzeszów)

Abstract. Let X be a real linear topological space. We characterize solutions $f : X \to \mathbb{R}$ and $M : \mathbb{R} \to \mathbb{R}$ of the equation $f(x + M(f(x))y) = f(x)f(y)$ under the assumption that f and M have the Darboux property.

Let X be a real linear topological space. We know that a function $f : X \to \mathbb{R}$ having the Darboux property and satisfying the exponential equation

$$f(x + y) = f(x)f(y) \tag{1}$$

need not be continuous (see [9], cf. also [4, Remark 2]), and also that every solution $f : X \to \mathbb{R}$ of the Gołąb–Schinzel equation

$$f(x + f(x)y) = f(x)f(y) \tag{2}$$

having the Darboux property is continuous (see [4]). These two classical functional equations, in spite of their different nature, can be "connected" in the following equation:

$$f(x + M(f(x))y) = f(x)f(y), \tag{3}$$

i.e. (1) and (2) are particular cases of (3) with $M = 1$ and $M = \mathrm{id}_{\mathbb{R}}$, respectively. So, the study of the solutions $f : X \to \mathbb{R}$, $M : \mathbb{R} \to \mathbb{R}$ of (3) having the Darboux property seems to be interesting.

Equation (1) is well known; for further information on it see e.g. [2]. The first paper concerning (2) is due to S. Gołąb and A. Schinzel [7]. This equation and its generalizations have been studied by many authors (the relevant bibliography can be found in [6]). In particular, J. Brzdęk has studied extensively the equation

$$f(x + f(x)^n y) = f(x)f(y) \quad \text{for a positive integer } n, \tag{4}$$

which is tightly connected with some classes of subgroups of the Lie groups L_s^1 (see [3]) and some other groups. Referring to the second part of Hilbert's

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fifth problem (cf. [1, p. 153]), he determined solutions of (4) having the Darboux property. As a corollary he proved that such solutions are continuous.

Here we solve the equation (3), which also generalizes equation (4), under the assumption that $f : X \to \mathbb{R}$ and $M : \mathbb{R} \to \mathbb{R}$ are unknown functions having the Darboux property. Our results correspond to a problem of J. Brzdęk (see [6, Problem, p.21]) and extend [5, Corollary 1].

Let us recall that a function $f : X \to \mathbb{R}$ has the *Darboux property* whenever for every nonempty connected set $D \subset X$ the set $f(D)$ is connected in $\mathbb{R}$.

We start with some lemmas.

Lemma 1 ([4, Corollary 1]). *Every linear functional $h : X \to \mathbb{R}$ having the Darboux property is continuous.*

In the next lemma we collect several properties of functions satisfying (3). For their proofs we refer the reader to [8, Lemma 2(ii), Lemma 3, Proposition 2 and the proof of Corollary 1].

Lemma 2. *Let $f : X \to \mathbb{R}$ and $M : \mathbb{R} \to \mathbb{R}$ satisfy equation* (3), *and $f \neq 0$, $f \neq 1$. Set $A = f^{-1}(\{1\})$ and $W = f(X) \setminus \{0\}$. Then*

(i) *A is a subgroup of $(X, +)$ and $A \setminus \{0\}$ is the set of periods of f;*
(ii) *W is a subgroup of $(\mathbb{R} \setminus \{0\}, \cdot)$;*
(iii) *$M(a)A = A$ for $a \in W$;*
(iv) $(M \circ f)^{-1}(\{0\}) = f^{-1}(\{0\})$;
(v) *if, moreover, $M(1) = 1$ and $M \circ f \neq 1$, then $0 \in f(X)$;*
(vi) *f and $\widetilde{M}$ satisfy* (3), *where*

$$\widetilde{M}(a) = \frac{M(a)}{M(1)} \quad \text{for each } a \in \mathbb{R}. \tag{5}$$

Now, we give further properties of solutions of (3), especially a necessary form of functions f satisfying (3).

Lemma 3 (cf. [8, Lemma 4]). *Let $f : X \to \mathbb{R}$ and $M : \mathbb{R} \to \mathbb{R}$ satisfy equation* (3), *$f \neq 0$ and $M(1) = 1$. Set $W = f(X) \setminus \{0\}$ and $A = f^{-1}(\{1\}$. Suppose that $M(W) \setminus \{1\} \neq \emptyset$ and A is a linear subspace of X. Then there exists $x_0 \in X \setminus A$ such that*

$$f(x) = \begin{cases} a & \text{if } x \in (M(a) - 1)x_0 + A \text{ for some } a \in W, \\ 0 & \text{otherwise,} \end{cases} \tag{6}$$

and $M|_{f(X)}$ is injective and multiplicative.

Proof. By [8, Lemma 4] we only have to show the multiplicativity of $M|_{f(X)}$. Using Lemma 2(iv), (v), it is easy to see that

$$M(f(x)f(y)) = 0 = M(f(x))M(f(y))$$

for $x, y \in X$ with $f(x)f(y) = 0$. Now, take $x, y \in X$ such that $f(x)f(y) \neq 0$. Then, by (6),

$$x = (M(f(x)) - 1)x_0 + z_1 \quad \text{and} \quad y = (M(f(y)) - 1)x_0 + z_2$$

for some $z_1, z_2 \in A$. According to equation (3),

$$\begin{aligned} f(x)f(y) &= f(x + M(f(x))y) \\ &= f((M(f(x)) - 1)x_0 + z_1 + M(f(x))((M(f(y)) - 1)x_0 + z_2)) \\ &= f((M(f(x))M(f(y)) - 1)x_0 + z_1 + M(f(x))z_2). \end{aligned}$$

Since A is linear, $z_1 + M(f(x))z_2 \in A$. Thus, in view of Lemma 2(i),

$$f(x)f(y) = f((M(f(x))M(f(y)) - 1)x_0) \neq 0.$$

Next, by (6),

$$(M(f(x))M(f(y)) - 1)x_0 \in (M(f(x)f(y)) - 1)x_0 + A$$

and hence

$$[M(f(x))M(f(y)) - M(f(x)f(y))]x_0 \in A.$$

Consequently, since $x_0 \notin A$, we have $M(f(x))M(f(y)) = M(f(x)f(y))$, which completes the proof. ∎

Now we are in a position to prove our main result.

THEOREM 1 (cf. [4, Theorem 1]). *Let $f : X \to \mathbb{R}$ and $M : \mathbb{R} \to \mathbb{R}$ have the Darboux property. Then f, M satisfy* (3) *if and only if one of the following conditions holds*:

(i) $f = 0$ *or* $f = 1$;
(ii) $M|_{(0,\infty)} = 1$ *and there is an additive surjection $a : X \to \mathbb{R}$ such that* $f = \exp a$;
(iii) *there are a nontrivial continuous linear functional $h : X \to \mathbb{R}$ and some $c > 0$ such that either*

$$\begin{aligned} f(x) &= |h(x) + 1|^{1/c} \operatorname{sgn}(h(x) + 1) && \text{for } x \in X, \\ M(y) &= |y|^c \operatorname{sgn} y && \text{for } y \in \mathbb{R}, \end{aligned} \tag{7}$$

or

$$\begin{aligned} f(x) &= (\max\{0, h(x) + 1\})^{1/c} && \text{for } x \in X, \\ M(y) &= y^c && \text{for } y \in [0, \infty). \end{aligned} \tag{8}$$

Proof. Let f, M satisfy (3). If $f = \text{const}$, then, according to (3), $f = 1$ or $f = 0$. So assume that $f \neq \text{const}$. Then $1 \in f(X)$ by Lemma 2(ii). Since f has the Darboux property and X is connected, so is $f(X)$. Hence $\operatorname{int} f(X) \neq \emptyset$ and, in view of Lemma 2(ii), $W = f(X) \setminus \{0\} \in \{(0, \infty), \mathbb{R} \setminus \{0\}\}$. Moreover, by Lemma 2(vi), f with $\widetilde{M}$ given by (5) also fulfills (3) and $\widetilde{M}(1) = 1$.

If $M \circ f = c$ for some $c \in \mathbb{R}$, then $\widetilde{M} \circ f = 1$. Using equation (3) we find that f is a nonconstant exponential function and hence, according to [10, Theorem 1, p. 308], $f = \exp a$, where $a : X \to \mathbb{R}$, $a \neq 0$, is an additive function. Thus $f(X) = (0, \infty)$. Putting $x = 0$ in (3), we have $f((c-1)y) = 1$ for each $y \in X$. Consequently, $c = 1$ and so (ii) holds.

Now assume that $M \circ f \neq$ const. Then $\widetilde{M} \circ f \neq 1$ and $\widetilde{M}(1) = 1$. Since $W \in \{(0, \infty), \mathbb{R}\setminus\{0\}\}$, according to Lemma 2(v) we have $f(X) \in \{[0, \infty), \mathbb{R}\}$. Moreover, since $\widetilde{M}$ has the Darboux property and $f(X)$ is connected, so is $\widetilde{M}(f(X))$. In view of Lemma 2(iv) the inclusion $\{0, 1\} \subset f(X)$ implies $\{0, 1\} \subset \widetilde{M}(f(X))$. Hence $[0, 1] \subset \widetilde{M}(f(X))$.

Let W_0 be the multiplicative group generated by $\widetilde{M}(W)$. By Lemma 2(iv), $(0, 1] \subset \widetilde{M}(W) \subset W_0$ and thus $W_0 \in \{(0, \infty), \mathbb{R} \setminus \{0\}\}$. Moreover, in view of Lemma 2(iii), $W_0 A \subset A$. Consequently, Lemma 2(i) implies that $\mathbb{R}A \subset A$ and A is a linear subspace of X. Then, according to Lemma 3, there is some $x_0 \in X \setminus A$ such that (6) holds and $\widetilde{M}|_{f(X)}$ is a multiplicative injection. Hence $\widetilde{M}(W) = W_0 \in \{(0, \infty), \mathbb{R} \setminus \{0\}\}$ by Lemma 2(ii).

Put $Y = \mathbb{R}x_0 + A$ and define a linear functional $h : Y \to \mathbb{R}$ as follows: $h(ax_0 + y) = a$ for every $a \in \mathbb{R}$ and $y \in A$. Then

$$h(x) = \widetilde{M}(f(x)) - 1 \quad \text{for each } x \in (\widetilde{M}(W) - 1)x_0 + A.$$

Consequently, if $\widetilde{M}(W) = \mathbb{R}\setminus\{0\}$, then $\widetilde{M}(f(x)) = h(x) + 1$ for $x \in Y$, while if $\widetilde{M}(W) = (0, \infty)$, then $\widetilde{M}(f(x)) = \max\{h(x) + 1, 0\}$ for $x \in Y$. Moreover, in view of (6) and Lemma 2(iv), $\widetilde{M}(f(x)) = 0$ for each $x \in X \setminus Y$.

Suppose that $X \neq Y$ and pick an $x_1 \in X \setminus Y$. By the linearity of Y, $rx_1 \notin Y$ for each $r \in \mathbb{R} \setminus \{0\}$. Since $\widetilde{M}(1) = 1$ and $f(0) = 1$, we obtain $\widetilde{M}(f(\mathbb{R}x_1)) = \{0, 1\}$. But $\mathbb{R}x_1$ is connected and f, $\widetilde{M}$ have the Darboux property, so $\widetilde{M}(f(\mathbb{R}x_1))$ is connected. This contradiction proves $X = Y$. Hence either

$$\widetilde{M}(f(x)) = h(x) + 1 \quad \text{for } x \in X, \tag{9}$$

or

$$\widetilde{M}(f(x)) = \max\{0, h(x) + 1\} \quad \text{for } x \in X. \tag{10}$$

Since $\widetilde{M} \circ f$ has the Darboux property, so does the linear functional h. Thus, by Lemma 1, h is continuous. Moreover, in view of the injectivity and multiplicativity of $\widetilde{M}|_{f(X)}$, and (5), we see that $M|_{f(X)}$ is injective,

$$M(1)M(ab) = M(a)M(b) \quad \text{for every } a, b \in f(X),$$

and either

$$M(f(x)) = M(1)(h(x) + 1) \quad \text{for } x \in X,$$

or

$$M(f(x)) = M(1)\max\{0, h(x)+1\} \quad \text{for } x \in X.$$

Since f, M satisfy (3), it is easy to check that $M(1) = 1$ (see the proof of [8, Theorem 1]). Hence $M = \widetilde{M}$, once again by (5).

Let $G = f(X) \in \{[0,\infty), \mathbb{R}\}$. Note that every injection $M : G \to \mathbb{R}$ having the Darboux property is monotonic. Indeed, suppose otherwise and, without loss of generality, take $x_1, x_2, x_3 \in G$ such that $x_1 < x_2 < x_3$ and $M(x_2) < M(x_3) < M(x_1)$. Then, by the Darboux property of M, there is $x_0 \in (x_1, x_2)$ with $M(x_0) = M(x_3)$, which contradicts the injectivity of M. Since $M(0) = 0$ and $M(1) = 1$, M is increasing. Hence, by the multiplicativity of $M|_G$, according to [10, Theorem 3, p. 310], there is an additive function $a : \mathbb{R} \to \mathbb{R}$ such that M is given by

$$M(z) = \begin{cases} \exp(a(\ln|z|))\operatorname{sgn} z & \text{for } z \in G \setminus \{0\}, \\ 0 & \text{for } z = 0. \end{cases}$$

But $M|_G$ is increasing, hence is a. Therefore, from [2, Corollary 5, p. 15], $a(z) = cz$ with some $c > 0$. Thus $M(z) = |z|^c \operatorname{sgn} z$ for $z \in G$. So, condition (iii) holds. This ends the first part of the proof. The converse is easy to check. ■

From the above theorem we obtain

COROLLARY 1 (cf. [4, Corollary 2]). *If $f : X \to \mathbb{R}$ and $M : \mathbb{R} \to \mathbb{R}$ having the Darboux property satisfy* (3), *then either f is continuous*, *or f is a nontrivial exponential function.*

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Department of Mathematics
Rzeszów University of Technology
W. Pola 2
35-959 Rzeszów, Poland
E-mail: elizapie@prz.rzeszow.pl

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