COLLOQUIUM MATHEMATICUM

VOL. 114 2009 NO. 1

BALL REMOTAL SUBSPACES OF BANACH SPACES

BY

PRADIPTA BANDYOPADHYAY (Kolkata), BOR-LUH LIN (Iowa City, IA) and
T. S. S. R. K. RAO (Bangalore)

Abstract. We study Banach spaces X with subspaces Y whose unit ball is densely remotal in X. We show that for several classes of Banach spaces, the unit ball of the space of compact operators is densely remotal in the space of bounded operators. We also show that for several classical Banach spaces, the unit ball is densely remotal in the duals of higher even order. We show that for a separable remotal set $E \subseteq X$, the set of Bochner integrable functions with values in E is a remotal set in $L^1(\mu, X)$.

1. Introduction. For a closed and bounded set K in a real Banach space X, the *farthest distance map* ϱ is defined as $\varrho(x, K) = \sup\{\|z - x\| : z \in K\}$, $x \in X$. For $x \in X$, we define the *farthest point map* as $F_K(x) = \{z \in K : \|z - x\| = \varrho(x, K)\}$, i.e., the set of points of K farthest from x. Note that this set may be empty. Let $R(K, X) = \{x \in X : F_K(x) \neq \emptyset\}$. We will write simply $R(K)$ if the space X is understood. Call a closed and bounded set K *remotal* if $R(K, X) = X$, and *densely remotal* if $R(K, X)$ is norm dense in X. A sequence $\{z_n\} \subseteq K$ such that $\lim \|z_n - x\| = \varrho(x, K)$ is called a *maximizing sequence* for x.

Let $Y \subseteq X$ be a closed subspace. In this paper, we study when the closed unit ball B_Y of Y is remotal or densely remotal in X. It is known that:

(a) If $Y \subseteq X$ is finite-dimensional, then B_Y is remotal in X.
(b) If Y is a reflexive subspace of any Banach space, then B_Y is densely remotal [6].
(c) If X^* is an Asplund space with a LUR dual norm, then for any closed subspace Y of X^*, B_Y is densely remotal [13]. In particular, any space X with X^{**} separable has an equivalent renorming with the above property. Note that such a space need not be reflexive.
(d) If X has the Radon–Nikodým property (RNP), then for any w^*-closed subspace Y of X^*, B_Y is densely remotal [2, Proposition 3].

2000 *Mathematics Subject Classification*: Primary 46B20, 47L05; Secondary 46E40.
Key words and phrases: farthest points, densely remotal sets.
Research partially supported by a DST-NSF project grant DST/INT/US(NSF-RPO-0141)/2003 and NSF/OISE-03-52523.

Our main emphasis in this paper is to study these concepts in spaces of operators and in function spaces. We exhibit several spaces of operators on classical Banach spaces and function spaces whose unit balls are densely remotal in an appropriate superspace. Among other things, we show:

(1) For any closed subspace Y of a Hilbert space X, B_Y is remotal in X.
(2) B_{ℓ_1} is remotal in ℓ_1^{**}.
(3) B_{c_0} is densely remotal, but not remotal, in ℓ_∞.
(4) Suppose X is strictly convex and has the RNP. Then $B_{\mathcal{K}(L^1(\mu),X)}$ is densely remotal in $\mathcal{L}(L^1(\mu),X)$, where μ is the Lebesgue measure on $[0,1]$.
(5) For a large class of Banach spaces that includes all reflexive spaces, $C(K)$ spaces and $L^1(\mu)$ spaces, $B_{\mathcal{K}(X)}$ is densely remotal in $\mathcal{L}(X)$.
(6) For any reflexive Banach space X, $B_{C(K,X)}$ is densely remotal in $WC(K,X)$.

We also study the dense remotality of X in X^{**}. This leads us to the study of the largeness of the set of vectors in X^{**} which attain their norm at a norm attaining element of X^*. In some cases, we also give an explicit description of the set $R(B_Y,X)$. Another interesting geometric property that we use in this study is that all the extreme points in B_{X^*} attain their norm.

We also obtain several stability results. In particular, we show that for any separable remotal set $E \subseteq X$, the set $L^1(\mu,E)$ of Bochner integrable functions is remotal in $L^1(\mu,X)$, where μ is the Lebesgue measure on $[0,1]$, producing a correct proof, under weaker assumptions, of [5, Theorem 2.3]. We also prove that if $E \subseteq X$ is densely remotal and K is a compact totally disconnected space, then $C(K,E)$ is densely remotal in $C(K,X)$.

The closed unit ball and the unit sphere of X will be denoted by B_X and S_X respectively. We will identify any $x \in X$ with its canonical image in X^{**}. We will denote by $\mathrm{NA}(X)$ the set of all $x^* \in X^*$ which attain their norm on B_X. For a closed bounded convex set C, we let $\mathrm{ext}(C)$ denote the set of extreme points of C. Any unexplained terminology can be found in [4].

2. $(*)$-subspaces

Definition 2.1. We say that $A \subseteq B_{X^*}$ is a *norming set* for X if $\|x\| = \sup\{|x^*(x)| : x^* \in A\}$ for all $x \in X$.

Let us say that a subspace Y of X is a $(*)$-*subspace* of X if the set

$$A_Y = \{x^* \in S_{X^*} : \|x^*|_Y\| = 1\}$$

is a norming set for X.

Definition 2.2. A subspace Y of a Banach space X is said to be an *ideal* in X if there is a projection P of norm 1 on X^* with $\ker(P) = Y^\perp$.

Y is said to be an *M-ideal* in X if there is a projection P on X^* with $\ker(P) = Y^\perp$ and, for all $x^* \in X^*$, $\|x^*\| = \|Px^*\| + \|x^* - Px^*\|$.

A Banach space X is said to be *M-embedded* if X is an M-ideal in X^{**} under the canonical embedding.

See [4, Chapters III and VI] for several important examples of M-embedded spaces and their geometric and topological properties.

Here are some natural examples of $(*)$-subspaces.

EXAMPLE 2.3.

(a) X is a $(*)$-subspace of X^{**}.
(b) If $Y \subseteq Z \subseteq X$ and Y is a $(*)$-subspace of X, then Z is a $(*)$-subspace of X, and Y is a $(*)$-subspace of Z.
(c) For any two Banach spaces X and Y, the space $\mathcal{K}(X, Y)$ of all compact operators from X to Y is a $(*)$-subspace of $\mathcal{L}(X, Y)$, the space of all bounded operators from X to Y.
(d) For a compact Hausdorff space K, the space $C(K, X)$ of continuous functions from K to X is a $(*)$-subspace of $WC(K, X)$, the space of continuous functions when X has the weak topology, with the sup norm.
(e) More generally, if Y is an ideal in X and satisfies the conditions of [8, Lemma 1(i)], then Y is a $(*)$-subspace of X. See [8] for details.

PROPOSITION 2.4. *If Y is a $(*)$-subspace of X, then*

$$\varrho(x, B_Y) = \|x\| + 1 \quad \text{for any } x \in X. \tag{1}$$

Proof. Clearly, $\varrho(x, B_Y) \le \|x\| + 1$. Now, let $\varepsilon > 0$. Since A_Y is norming for X, there exists $x^* \in A_Y$ such that $x^*(x) > \|x\| - \varepsilon/2$.

Since $\|x^*|_Y\| = 1$, there exists $y \in S_Y$ such that $-x^*(y) > 1 - \varepsilon/2$. Therefore, $\|x - y\| \ge x^*(x - y) > \|x\| + 1 - \varepsilon$. ■

We next recall the following definitions from [1].

DEFINITION 2.5. We say that $x \in S_X$ is

(a) a *rotund point* of B_X if $\|y\| = \|(x + y)/2\| = 1$ implies $x = y$;
(b) an *almost LUR* (ALUR) (resp. *weakly almost LUR* (wALUR)) point of B_X if for any $\{x_n\} \subseteq B_X$ and $\{x_m^*\} \subseteq B_{X^*}$, the condition

$$\lim_m \lim_n x_m^* \left(\frac{x_n + x}{2} \right) = 1$$

implies $x_n \to x$ in norm (resp. in the weak topology).

We say that a Banach space X has one of the above properties if every point of S_X has the same property.

REMARK 2.6. It follows that if every point of S_Y is a rotund point of B_X, then no point of $X \setminus Y$ has a farthest point in B_Y. This happens in particular when X is strictly convex.

In view of [1, Corollary 8], in the special case of X in X^{**}, we have the following result:

If X is wALUR, in particular, if X^* is smooth, then no point of $X^{**} \setminus X$ has a farthest point in B_X.

À propos Example 2.3(b) above, for $x \in X$ we have $1 + \|x\| = \varrho(x, B_Z) = \varrho(x, B_Y)$. Thus if B_Y is remotal in X, then B_Y is remotal in Z and B_Z is remotal in X. Similarly, if B_Y is densely remotal in X, then B_Z is densely remotal in X. But we do not know in general if B_Y is densely remotal in Z.

It is immediate from Proposition 2.4 that if Y is a $(*)$-subspace of X, then any farthest point in B_Y comes from S_Y. We now observe that this is actually always true.

PROPOSITION 2.7. *Let Y be a subspace of a Banach space X. Let $x_0 \in X$. Then either $F_{B_Y}(x_0) \subseteq S_Y$ or $F_{B_Y}(x_0) = B_Y$.*

Proof. Suppose there exists $y_0 \in F_{B_Y}(x_0)$ such that $\|y_0\| < 1$. Then there exists $\varepsilon > 0$ such that $y_0 + \varepsilon y \in B_Y$ for all $y \in B_Y$.

Let $x^* \in S_{X^*}$ be such that $x^*(x_0 - y_0) = \|x_0 - y_0\| = \varrho(x_0, B_Y)$. Then for any $y \in B_Y$,

$$\begin{aligned}\|x_0 - y_0\| &\geq \|x_0 - y_0 - \varepsilon y\| \geq x^*(x_0 - y_0 - \varepsilon y)\\ &= x^*(x_0 - y_0) - \varepsilon x^*(y) = \|x_0 - y_0\| - \varepsilon x^*(y).\end{aligned}$$

Thus $x^*(y) \geq 0$ for all $y \in B_Y$, whence $x^*|_Y \equiv 0$. It follows that for any $z \in B_Y$,

$$\|x_0 - z\| \geq x^*(x_0 - z) = x^*(x_0) = x^*(x_0 - y_0) = \|x_0 - y_0\| = \varrho(x_0, B_Y).$$

Therefore, $z \in F_{B_Y}(x_0)$. ∎

PROPOSITION 2.8. *Suppose $Y \subseteq X$ is a subspace of a wLUR Banach space X. If $x \in X$ is such that $\varrho(x, B_Y) = \|x\| + 1$, then $x \in Y$.*

Proof. If $x \in X$ is such that $\varrho(x, B_Y) = \|x\| + 1$, then for any maximizing sequence $\{y_n\} \subseteq B_Y$, $\|x - y_n\| \to \|x\| + 1$. Since X is wLUR, $y_n \to -x/\|x\|$ weakly and thus $x \in Y$. ∎

It follows that a wLUR Banach space has no proper $(*)$-subspace. This produces one of the easiest proofs of the following well-known fact.

COROLLARY 2.9. *If X^{**} is wLUR, then X is reflexive.*

PROPOSITION 2.10. *If $Y \subseteq X$ is a subspace such that the farthest distance formula* (1) *holds, in particular, if Y is a $(*)$-subspace of X, then a* *point $x_0 \in X$ has a farthest point in B_Y if and only if there exists $x^* \in A_Y$ and $y \in S_Y$ such that*

$$x^*(x_0) = \|x_0\| \quad \text{and} \quad x^*(y) = 1.$$

In other words, $R(B_Y) = \{x \in X :$ there exist $x^* \in A_Y$ and $y \in S_Y$ such that $x^*(x) = \|x\|$ and $x^*(y) = 1\}$.

Proof. If such x^* and y exist then

$$\|x_0 + y\| \geq x^*(x_0 + y) = \|x_0\| + 1.$$

Hence $-y$ is farthest from x_0.

Conversely, if $-y$ is farthest from x_0, then $\|x_0 + y\| = \|x_0\| + 1$. Now, there exists $x^* \in S_{X^*}$ such that

$$x^*(x_0 + y) = \|x_0 + y\| = \|x_0\| + 1.$$

Since $x^*(x_0) \leq \|x_0\|$ and $x^*(y) \leq 1$, the result follows. ∎

REMARK 2.11. It follows that in the case of $(*)$-subspaces, $R(B_Y)$ is closed under scalar multiplications. Is this generally true?

We now specialize to some of the cases listed above.

DEFINITION 2.12. X is said to be *weakly Hahn–Banach smooth* if every $x^* \in \mathrm{NA}(X)$ has a unique norm preserving extension to all of X^{**}, i.e., whenever $x^{***} \in X^{***}$ is such that $x^{***}|_X = x^*$ and $\|x^{***}\| = \|x^*\|$, then $x^{***} = x^*$.

By [4, Lemma III.2.14], $x^* \in S_{X^*}$ has a unique norm preserving extension to all of X^{**} if and only if x^* is a w^*-weak point of continuity of B_{X^*}, that is, the relative weak and w^* topologies on B_{X^*} agree at x^*.

Let us denote by $\mathrm{NA}_2(X)$ the set $\{x^{**} \in X^{**} : x^{**}(x^*) = \|x^{**}\|$ for some $x^* \in \mathrm{NA}(X) \cap S_{X^*}\}$. Clearly, $X \subseteq \mathrm{NA}_2(X) \subseteq \mathrm{NA}(X^*)$. The relevance of this set in our context follows from our next result.

PROPOSITION 2.13. *For any Banach space X, $R(B_X, X^{**}) \supseteq \mathrm{NA}_2(X)$. If B_{X^*} is w^*-sequentially compact and X is weakly Hahn–Banach smooth, then equality holds.*

Proof. If $x_0^{**} \in \mathrm{NA}_2(X)$, there exist $x^* \in S_{X^*}$ and $x \in S_X$ such that $x_0^{**}(x^*) = \|x_0^{**}\|$ and $x^*(x) = 1$; then $-x \in B_X$ is farthest from x_0^{**}.

Conversely, suppose $x_0^{**} \in S_{X^{**}}$ has a farthest point $-x \in B_X$, so that $\|x_0^{**} + x\| = 2$. Let $\{x_n^*\} \subseteq B_{X^*}$ be such that $\lim_n (x_0^{**} + x)(x_n^*) = 2$. It follows that

$$\lim_n x_0^{**}(x_n^*) = 1 = \lim_n x_n^*(x).$$

Now since B_{X^*} is w^*-sequentially compact, there is a subsequence $\{x_{n_k}^*\}$ w^*-converging to some $x_0^* \in B_{X^*}$. It then follows that $x_0^*(x) = 1$ and therefore $x_0^* \in \mathrm{NA}(X)$. Since X is weakly Hahn–Banach smooth, x_0^* is a

w^*-weak point of continuity. Therefore, $x^*_{n_k} \to x^*_0$ weakly. It follows that $x^{**}_0(x^*_0) = 1$, and hence $x^{**}_0 \in \mathrm{NA}_2(X)$. ∎

REMARK 2.14. In the above result, instead of assuming that B_{X^*} is w^*-sequentially compact, we may assume that X is smooth. In that case, any w^*-cluster point $x^* \in B_{X^*}$ of $\{x^*_n\}$ satisfies $x^* \in S_{X^*}$ and $x^*(x) = 1$. Since X is smooth, $\{x^*_n\}$ has a unique w^*-cluster point, i.e., $\{x^*_n\}$ is w^*-convergent. The rest of the argument remains as before.

If X is M-embedded, then by [4, Proposition I.1.12 & Corollary III.4.7], it satisfies the hypothesis of the above proposition.

COROLLARY 2.15. *B_{c_0} is densely remotal, but not remotal, in ℓ_∞.*

Proof. Observe that

$$R(B_{c_0}, \ell_\infty) = \{(\alpha_n) \in \ell_\infty : \exists n_0 \geq 1 \text{ such that } |\alpha_{n_0}| = \|(\alpha_n)\|_\infty\}.$$

Indeed, if (α_n) is in the set on the RHS, then $-e_{n_0}$ is easily seen to be farthest from it. For the converse, apply the above result, noting that c_0 is M-embedded [4, p. 4]. Also, if $(a_n) \in \mathrm{NA}(c_0)$, then only finitely many a_n's are non-zero.

But these are precisely the points of ℓ_∞ that belong to $\mathrm{NA}(\ell_1)$, and hence, by the Bishop–Phelps theorem, $R(B_{c_0}, \ell_\infty)$ is dense in ℓ_∞.

Clearly, the sequence $(1 - 1/n)$ is not in $R(B_{c_0}, \ell_\infty)$. ∎

REMARK 2.16. The above argument can be adapted to show that for any family $\{X_\alpha\}$ of Banach spaces, the unit ball of $\bigoplus_{c_0} X^*_\alpha$ is densely remotal in $\bigoplus_{\ell_\infty} X^*_\alpha$. Indeed, it is easy to see that $R(B_{\bigoplus_{c_0} X^*_\alpha}) \supseteq \{(x^*_\alpha) \in \bigoplus_{\ell_\infty} X^*_\alpha$: there exists α_0 such that $\|x^*_{\alpha_0}\| = \|(x^*_\alpha)\|_\infty\}$; note that $\bigoplus_{\ell_\infty} X^*_\alpha = (\bigoplus_{\ell_1} X_\alpha)^*$ and again appeal to the Bishop–Phelps theorem.

QUESTION 2.17. *For any family $\{X_\alpha\}$ of Banach spaces, is $B_{\oplus_{c_0} X_\alpha}$ densely remotal in $\bigoplus_{\ell_\infty} X_\alpha$?*

We now give an example of an M-embedded space X for which $R(B_X, X^{**}) = X$.

EXAMPLE 2.18. Let A be the disc algebra on the unit circle $\mathbb{T} \subseteq \mathbb{C}$. Then it is known that the quotient space $X = C(\mathbb{T})/A$ is an M-embedded space and its dual H^1_0 is a smooth space. Thus, by Remark 2.6, no point of $X^{**} \setminus X$ has a farthest point in B_X.

We now come to the case of $\mathcal{K}(X, Y)$ as a subspace of $\mathcal{L}(X, Y)$. For simplicity, we will work with $X = Y$, but most of our results extend, under natural assumptions, to the general case as well.

LEMMA 2.19.

(a) $R(B_{\mathcal{K}(X)}) \supseteq \{T \in \mathcal{L}(X)$: *there exists $x \in S_X$ such that* $\|Tx\| = \|T\|\}$.

(b) $R(B_{\mathcal{K}(X)}) \supseteq \{T \in \mathcal{L}(X) :$ *there exists* $x^* \in \mathrm{NA}(X) \cap S_{X^*}$ *such that* $\|T^*x^*\| = \|T^*\|\}$.

Proof. (a) Let $T \in S_{\mathcal{L}(X)}$ be such that there exists $x \in S_X$ with $\|Tx\| = 1$. Let $x^* \in S_{X^*}$ be such that $x^*(x) = 1$. Define $S = x^* \otimes Tx$. Then $S \in S_{\mathcal{K}(X)}$ and $Sx = x^*(x) \cdot Tx = Tx$. It follows that $\|T+S\| \geq \|Tx+Sx\| = 2\|Tx\| = 2$, and therefore $-S$ is farthest from T.

(b) Let $T \in S_{\mathcal{L}(X)}$ be such that there exists $x^* \in \mathrm{NA}(X)$ with $\|T^*x^*\| = \|T^*\| = 1$. Let $x \in S_X$ be such that $x^*(x) = 1$. Then $S = -T^*x^* \otimes x$ works. ■

THEOREM 2.20. *Let μ be the Lebesgue measure on* $[0,1]$. *Suppose X is strictly convex and has the RNP. Then $B_{\mathcal{K}(L^1(\mu),X)}$ is densely remotal in* $\mathcal{L}(L^1(\mu), X)$.

Proof. It follows from arguments similar to (a) above that $R(B_{\mathcal{K}(L^1(\mu),X)}) \supseteq \{T \in \mathcal{L}(L^1(\mu), X) :$ there exists $f \in S_{L^1(\mu)}$ such that $\|Tf\| = \|T\|\}$. By [12], this set is dense in $\mathcal{L}(L^1(\mu), X)$. ■

THEOREM 2.21. *Suppose X is a reflexive Banach space. Then $B_{\mathcal{K}(X)}$ is densely remotal in* $\mathcal{L}(X)$.

Proof. Since for a reflexive space, $\mathrm{NA}(X) = X^*$, by Lemma 2.19(b) we have $R(B_{\mathcal{K}(X)}) \supseteq \{T \in \mathcal{L}(X) :$ there exists $x^* \in S_{X^*}$ such that $\|T^*x^*\| = \|T^*\|\}$.

Now by [13, Proposition 4], $\{T \in \mathcal{L}(X) : T^*$ attains its norm$\}$ is dense in $\mathcal{L}(X)$. Thus the conclusion follows. ■

REMARK 2.22. It follows that for any measure space (Ω, Σ, μ), $B_{\mathcal{K}(L_p(\mu))}$ is densely remotal in $\mathcal{L}(L_p(\mu))$ for $1 < p < \infty$. It is well-known that in this case $\mathcal{K}(L_p(\mu))^{**} = \mathcal{L}(L_p(\mu))$. Thus these are also examples of spaces whose unit ball is densely remotal in the bidual.

Also, it is clear from our arguments above that several of the results go through for other subspaces of operators that contain finite rank operators. From the results above, we also see that for $I \in \mathcal{L}(X)$ farthest points in $B_{\mathcal{K}(X)}$ (or any other operator ideal) are related to the so-called Daugavet equation $\|I + T\| = 1 + \|T\|$.

PROPOSITION 2.23. *Suppose $\mathcal{K}(X)$ is an M-ideal in $\mathcal{L}(X)$. Then* $R(B_{\mathcal{K}(X)}) \subseteq \{T \in \mathcal{L}(X)$: *there exists* $x^* \in S_{X^*}$ *such that* $\|T^*x^*\| = \|T^*\|\}$. *In particular, if X is reflexive, then equality holds.*

Proof. Let $T \in R(B_{\mathcal{K}(X)})$ be such that $\|T\| = 1$ and let $-S$ be farthest from it. Then $\|T + S\| = 2$ and there exists $\tau \in \mathrm{ext}(B_{\mathcal{L}(X)^*})$ such that

$$\tau(T) = 1 \quad \text{and} \quad \tau(S) = 1.$$

Since $\mathcal{K}(X)$ is an M-ideal in $\mathcal{L}(X)$,

$$\mathcal{L}(X)^* = \mathcal{K}(X)^* \oplus_1 \mathcal{K}(X)^{\perp}.$$

Since τ is extreme, and $\tau(S) \neq 0$, we have $\tau \in \mathrm{ext}(B_{\mathcal{K}(X)^*})$. That is, $\tau = x^{**} \otimes x^*$ for some $x^{**} \in \mathrm{ext}(B_{X^{**}})$ and $x^* \in \mathrm{ext}(B_{X^*})$. It follows that

$$x^{**}(T^*x^*) = x^{**}(S^*x^*) = 1,$$

therefore,

$$\|T^*x^*\| = \|S^*x^*\| = 1.$$

If X is reflexive, combining with Lemma 2.19(b), we get the result. ∎

It is known that $X = \ell_p$ for $1 < p < \infty$ satisfies the above hypothesis. See [4, Chapter VI] for more examples of such spaces.

Another class of $(*)$-subspaces that can be treated as a special case of $\mathcal{K}(X,Y) \subseteq \mathcal{L}(X,Y)$ is $C(K,X) \subseteq WC(K,X)$. Note that for any $f \in C(K,X)$, $T : X^* \to C(K)$ defined by $T(x^*)(k) = x^*(f(k))$ is a linear map with $\|f\| = \|T\|$ and is a compact operator. Thus $C(K,X) \subseteq \mathcal{K}(X^*, C(K))$ and similarly one can see that $WC(K,X) \subseteq \mathcal{F}(X^*, C(K))$, the space of weakly compact operators. It can be seen that in the case of a dual space this map is onto so that $C(K,X^*)$ is isometric to $\mathcal{K}(X, C(K))$ and $WC(K,X^*)$ is isometric to $\mathcal{F}(X, C(K))$.

To discuss the dense remotality in this case we need the following lemma, which is of independent interest.

Lemma 2.24. *Let X,Y be Banach spaces. If $T \in \mathcal{L}(X,Y)$ with $\|T^*\| = \|T^*(y^*)\| = \|y^*\| = 1$, there is an extreme point in $y_0^* \in \mathrm{ext}(B_{Y^*})$ such that $\|T^*(y_0^*)\| = 1$.*

Proof. Let $F = \{y^* \in B_{Y^*} : \|T^*(y^*)\| = 1\}$. It is easy to see that F is an extreme set. We recall that a set is *extreme* if and only if it is a union of faces (convex, extreme sets). Also, $B_{Y^*} \setminus F = \bigcup \{y^* \in B_{Y^*} : \|T^*(y^*)\| \leq 1 - 1/n\}$. As these sets are w^*-compact and convex, it follows from [7, Theorem 5.8] that $F \cap \mathrm{ext}(B_{Y^*}) \neq \emptyset$. ∎

Theorem 2.25. *Let K be a compact set and let X be a reflexive Banach space. Then $B_{C(K,X)}$ is densely remotal in $WC(K,X)$.*

Proof. By Proposition 2.4, $\varrho(f, B_{C(K,X)}) = 1 + \|f\|$ for $f \in WC(K,X)$. It is easy to see that $R(B_{C(K,X)}) \supseteq \{f \in WC(K,X) :$ there exists $k \in K$ such that $\|f(k)\| = \|f\|\}$.

Since X is reflexive, we have $C(K,X) = \mathcal{K}(X^*, C(K))$ and $WC(K,X) = \mathcal{L}(X^*, C(K))$. We further note that if $T \in \mathcal{L}(X^*, C(K))$ is such that $T^* : C(K)^* \to X$ attains its norm, then as $\mathrm{ext}(B_{C(K)^*}) = \mathbb{T}K$ (via the canonical embedding of K), by Lemma 2.24, it attains its norm on K.

By [13, Proposition 4] again, $B_{C(K,X)}$ is densely remotal in $WC(K,X)$. ∎

In the following result, we assume that all elements of $\mathrm{ext}(B_{X^*})$ attain their norm. Apart from the space $C(K)$, as noted above, it is easy to see that any reflexive space, any subspace of $C(K)$ which contains constants and any $L^1(\mu)$ space have this property.

Corollary 2.26. *Let X be a Banach space such that $\operatorname{ext}(B_{X^*}) \subseteq \mathrm{NA}(X)$. Then $B_{\mathcal{K}(X)}$ is densely remotal in $\mathcal{L}(X)$.*

Proof. By Lemma 2.24, a $T \in \mathcal{L}(X)$ with T^* attaining its norm actually attains it at an extreme point x^*. Now by our assumption $x^* \in \mathrm{NA}(X)$. Hence the conclusion follows from Lemma 2.19 and [13, Proposition 4] as before. ∎

It is easy to see that the property $\operatorname{ext}(B_{X^*}) \subseteq \mathrm{NA}(X)$ is preserved under c_0-direct sums. Also, if X has this property then since any extreme point of $B_{C(K,X)^*}$ is of the form $\delta(k) \otimes x^*$ for some $k \in K$ and $x^* \in \operatorname{ext}(B_{X^*})$, we see that $C(K,X)$ also has this property. Our next two propositions show that under some natural conditions, this property is preserved by subspaces and quotients.

Proposition 2.27. *Let X be a Banach space such that $\operatorname{ext}(B_{X^*}) \subseteq \mathrm{NA}(X)$. Let Y be the range of a projection $P : X \to X$ of norm one. Suppose each $y^* \in Y^*$ has a unique norm preserving extension to X. Then $\operatorname{ext}(B_{Y^*}) \subseteq \mathrm{NA}(Y)$.*

Proof. Let $y^* \in \operatorname{ext}(B_{Y^*})$. It is well-known that there is always an $x^* \in \operatorname{ext}(B_{X^*})$ extending y^*. By our hypothesis, there exists an $x \in S_X$ with $x^*(x) = 1$. Now $P^*(x^*)|_Y = x^*|_Y = y^*$ and thus $\|P^*(x^*)\| = 1$, so that it is a norm preserving extension of y^*. Thus by the uniqueness hypothesis, $P^*(x^*) = x^*$. Now $y^*(P(x)) = x^*(P(x)) = P^*(x^*)(x) = x^*(x) = 1$. Hence $\operatorname{ext}(B_{Y^*}) \subseteq \mathrm{NA}(Y^*)$. ∎

Proposition 2.28. *Let $\operatorname{ext}(B_{X^*}) \subseteq \mathrm{NA}(X)$. Let $Y \subseteq X$ be an M-ideal. Then $\operatorname{ext}(B_{Y^\perp}) \subseteq \mathrm{NA}(X/Y)$.*

Proof. Let π be the quotient map. Let $y^* \in \operatorname{ext}(B_{Y^\perp})$. As Y is an M-ideal we have $X^* = Y^\perp \oplus_1 Y^*$. Thus $y^* \in \operatorname{ext}(B_{X^*})$. Therefore by hypothesis there exists an $x_0 \in S_X$ such that $y^*(x_0) = 1$. Now $y^*(x_0) = 1 = y^*(\pi(x_0)) \le \|\pi(x_0)\| \le 1$. So $y^* \in \mathrm{NA}(X/Y)$. ∎

We next consider the "3-space" property for norm attaining extreme points.

Proposition 2.29. *Let $Y \subseteq X$ be an M-ideal. If $\operatorname{ext}(B_{Y^*}) \subseteq \mathrm{NA}(Y)$ and $\operatorname{ext}(B_{Y^\perp}) \subseteq \mathrm{NA}(X/Y)$ then $\operatorname{ext}(B_{X^*}) \subseteq \mathrm{NA}(X)$.*

Proof. As before we have $X^* = Y^* \otimes_1 Y^\perp$. Now if $x^* \in \operatorname{ext}(B_{X^*})$ then either $x^* \in \operatorname{ext}(B_{Y^*})$ or $x^* \in \operatorname{ext}(B_{Y^\perp})$. If $x^* \in \operatorname{ext}(B_{Y^*})$ $x^* \in \mathrm{NA}(X)$. If $x^* \in \operatorname{ext}(B_{Y^\perp})$ then by our assumption there exists an $x \in X$ such that $x^*(x) = 1 = \|\pi(x)\|$. Now since Y is a proximinal subspace of X

[4, Proposition II.1.1] there exists a $y \in Y$ such that $x^*(x - y) = 1 = \|\pi(x)\| = \|x - y\|$. Thus $x^* \in \mathrm{NA}(X)$. ∎

Our next results show that density of $\mathrm{NA}_2(X)$ in X^{**} is preserved under some natural direct sums.

THEOREM 2.30. *Let $\{X_\alpha : \alpha \in \Gamma\}$ be a family of Banach spaces and let $X = \bigoplus_{c_0} X_\alpha$.*

(a) *If for every $\alpha \in \Gamma$, $\mathrm{NA}_2(X_\alpha) = \mathrm{NA}(X_\alpha^*)$, then $\mathrm{NA}_2(X) = \mathrm{NA}(X^*)$.*
(b) *If for every $\alpha \in \Gamma$, $\mathrm{NA}_2(X_\alpha)$ is dense in X_α^{**}, then $\mathrm{NA}_2(X)$ is dense in X^{**}.*

Proof. For simplicity, we work with $\Gamma = \mathbb{N}$.

(a) It is well-known that $\mathrm{NA}(X) = \{\Lambda \in \bigoplus_1 X_n^* : \Lambda$ has only finitely many non-zero coordinates and they attain their norm$\}$. It is also easy to see that $\tau \in \mathrm{NA}(\bigoplus_1 X_n^*)$ if and only if $\tau(\Lambda) = \|\tau\|$ for $\|\Lambda\| = 1$ and $\|\tau\| = \tau(n)(\Lambda(n)/\|\Lambda(n)\|) = \|\tau(n)\|$ whenever $\Lambda(n) \neq 0$. Now for $\tau \in \mathrm{NA}(\bigoplus_1 X_n^*)$, fix such an n_0. Then by our hypothesis, there exists an $f_{n_0} \in \mathrm{NA}(X_{n_0})$ such that $\tau(n_0)(f_{n_0}) = \|\tau(n_0)\| = \|\tau\|$. Let $f \in \bigoplus_1 X_n^*$ have $f(n_0) = f_{n_0}$ and 0 at other coordinates. Then from our observation in the first part of the proof, $f \in \mathrm{NA}(X)$ and $\tau(f) = \|\tau\|$, and thus $\tau \in \mathrm{NA}_2(X)$.

To prove (b), consider $A = \{n \in \mathbb{N} : \Lambda(n) \neq 0\}$. For any $\varepsilon > 0$ and for $n \in A$, let $\tau'(n) \in \mathrm{NA}_2(X_n)$ with $\|\tau(n) - \tau'(n)\| \leq \varepsilon$. Now define $\tau' \in \bigoplus_{\ell^\infty} X_n^{**}$ by taking $\tau'(n)$ for $n \in A$ and $\tau(n)$ at other coordinates. It then follows from the arguments given above that $\tau' \in \mathrm{NA}_2(X)$ and $\|\tau - \tau'\| \leq \varepsilon$. ∎

THEOREM 2.31. *Let $\{X_\alpha : \alpha \in \Gamma\}$ be a family of Banach spaces and let $X = \bigoplus_{\ell_p} X_\alpha$, $1 < p < \infty$. Then $\mathrm{NA}_2(X)$ is dense in X^{**} if and only if for every $\alpha \in \Gamma$, $\mathrm{NA}_2(X_\alpha)$ is dense in X_α^{**}.*

Proof. It clearly suffices to show that

$$\mathrm{NA}_2(X) = \Big\{(F_\alpha) \in \bigoplus_{\ell_p} X_\alpha^{**} : \text{either } F_\alpha = 0 \text{ or } F_\alpha \in \mathrm{NA}_2(X_\alpha)\Big\}.$$

Let $1/p + 1/q = 1$. Suppose $F = (F_\alpha) \in \mathrm{RHS}$ and $\|F\|_p = 1$. If $F_\alpha \neq 0$, then $F_\alpha/\|F_\alpha\| \in \mathrm{NA}_2(X_\alpha)$, that is, there exist $f_\alpha \in S_{X_\alpha^*}$ and $x_\alpha \in S_{X_\alpha}$ such that $F_\alpha(f_\alpha) = \|F_\alpha\|$ and $f_\alpha(x_\alpha) = 1$. Define $y = (y_\alpha)$ and $g = (g_\alpha)$ by

$$y_\alpha = \|F_\alpha\| x_\alpha, \quad g_\alpha = \|F_\alpha\|^{p-1} f_\alpha \quad \text{for all } \alpha \in \Gamma.$$

It is easy to check that $\|y\|_p = \|g\|_q = 1$ and $g(y) = F(g) = 1$. Thus, $g \in \mathrm{NA}(X)$ and $F \in \mathrm{NA}_2(X)$.

To see the converse, note that if $f = (f_\alpha) \in S_{X^*}$ attains its norm at $x = (x_\alpha) \in S_X$, then

$$1 = f(x) = \sum f_a(x_a) \leq \sum |f_a(x_a)| \leq \sum \|f_a\| \, \|x_a\| \leq \|f\|_q \|x\|_p = 1.$$

By the conditions of equality in Hölder's inequality, it follows that $\|f_\alpha\|^q = \|x_\alpha\|^p$ and $f_a(x_a) = \|f_a\|\,\|x_a\|$ for all $\alpha \in \Gamma$.

Now if $F = (F_\alpha) \in \mathrm{NA}_2(X)$ and $\|F\|_p = 1$, then there exist $f = (f_\alpha) \in S_{X^*}$ and $x \in S_X$ such that $F(f) = f(x) = 1$. By the above observation, $\|f_\alpha\|^q = \|F_\alpha\|^p = \|x_\alpha\|^p$, $F_a(f_a) = \|F_a\|\,\|f_a\|$ and $f_a(x_a) = \|f_a\|\,\|x_a\|$ for all $\alpha \in \Gamma$. It follows that whenever F_α is zero, so are f_α and x_α. And if $F_\alpha \neq 0$, then F_α attains its norm at $f_\alpha/\|f_\alpha\|$, which, in turn, attains its norm at $x_\alpha/\|x_\alpha\|$. That is, $F_\alpha \in \mathrm{NA}_2(X_\alpha)$. ∎

3. Stability results. This section deals with stability aspects of dense remotality and remotality under various natural operations like direct sums and in spaces of vector-valued functions. Our first result is about certain natural summands.

LEMMA 3.1. *Suppose $X = Y \oplus Z$ and there exists a monotone map $\varphi : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ such that if $x = y + z$, then $\|x\| = \varphi(\|y\|, \|z\|)$. Let $E \subseteq Y$ and $F \subseteq Z$ be remotal sets. Then $E + F$ is remotal in X. In particular, B_Y and B_Z are remotal in X.*

Proof. Let $x_0 = y_0 + z_0 \in X$. Let e_0 be farthest from y_0 in E and f_0 be farthest from z_0 in F. Then for any $e \in E$ and $f \in F$,

$$\begin{aligned}\|y_0 + z_0 - e - f\| &= \varphi(\|y_0 - e\|, \|z_0 - f\|)\\ &\le \varphi(\|y_0 - e_0\|, \|z_0 - f_0\|) = \|y_0 + z_0 - e_0 - f_0\|\end{aligned}$$

by the monotonicity of φ. ∎

We note that the above lemma holds with "remotal" replaced by "densely remotal" when convergence in X is equivalent to componentwise convergence.

Since any subspace of a Hilbert space is an ℓ^2-summand, we get

COROLLARY 3.2. *Let M be a subspace of a Hilbert space H. Then B_M is remotal in H.*

Since for any measure space (Ω, Σ, μ), $L^1(\mu)$ is an L-summand in its bidual, we also have

COROLLARY 3.3. *$B_{L^1(\mu)}$ is remotal in $L^1(\mu)^{**}$.*

It follows from Lemma 3.1 that for any M-summand Y in X, B_Y is remotal in X. But Example 2.18 shows that this is no longer true for M-ideals. Indeed, one may actually have $R(B_Y, X) = Y$. This lemma also leads to an easy example of an ideal with a densely remotal unit ball but such that formula (1) for ϱ in Proposition 2.4 fails to hold:

EXAMPLE 3.4. Let $X = Y \oplus_\infty Z$ be a non-trivial direct sum. Then Y is an ideal, B_Y is remotal in X. For an appropriate choice of $x = y + z$, we see that $\varrho(x, B_Y) < 1 + \|x\|$.

REMARK 3.5. For an M-embedded non-reflexive Banach space X, it was shown in [9] that X is also an M-ideal, under the canonical embedding, in all the even order duals of X. This was achieved by showing that X is an M-ideal in a component of some ℓ^∞-decomposition of any higher even order dual. Now let X be an M-embedded space with B_X densely remotal in X^{**}. It thus follows from the above remark that B_X continues to be a densely remotal set in all the higher even order duals of X.

Our next result exhibits densely ball remotal subspaces of c_0-direct sum of reflexive spaces. We recall that a subspace $Y \subseteq X$ is *proximinal* if for every $x \in X$, there exists a $y \in Y$ such that $d(x, Y) = \|x - y\|$, and *factor reflexive* if the quotient space X/Y is reflexive.

THEOREM 3.6. *Let $\{X_i\}_{i \in I}$ be any family of reflexive spaces. Let $X = \bigoplus_{c_0} X_i$. For any factor reflexive proximinal subspace $Y \subseteq X$, B_Y is densely remotal.*

Proof. Since Y is a proximinal factor reflexive subspace it follows from [10, Lemma 1.1, p. 292] that $Y^\perp \subseteq \mathrm{NA}(X)$. It follows from our computations in the proof of Theorem 2.30 above that any $\Lambda \in Y^\perp$ has only finitely many non-zero components. As $Y^\perp$ is a Banach space, a simple Baire category argument shows that there is a finite set $F \subseteq I$ such that $Y^\perp \subseteq \bigoplus_1 \{X_i^* : i \in F\}$. Let $X_1 = \bigoplus_\infty \{X_i : i \in F\}$ and $X_2 = \bigoplus_{c_0} \{X_i : I \notin F\}$, so that $X = X_1 \oplus_\infty X_2$. Since $X^* = X_1^* \oplus_1 X_2^*$, by duality we have $Y = (Y \cap X_1) \oplus_\infty X_2 = Y_1 \oplus X_2$ (say). As only finitely many reflexive spaces occur in X_1, it is a reflexive space and thus B_{Y_1} is densely remotal in X_1. Further, $B_Y = B_{Y_1} \oplus_\infty B_{X_2}$. Thus, from Lemma 3.1 we conclude that B_Y is densely remotal in X. ∎

We see in particular that for any proximinal finite-codimensional subspace $Y \subseteq c_0$, B_Y is densely remotal in c_0.

Coming to the spaces of Bochner integrable functions, we note that

LEMMA 3.7. *Let μ be the Lebesgue measure on $[0, 1]$. For any densely remotal set $E \subseteq X$, $L^1(\mu, E)$ is densely remotal in $L^1(\mu, X)$.*

Proof. Let $s = \sum_{i=1}^n \chi_{A_i} x_i$ be a simple function in $L^1(\mu, X)$. Since $R(E)$ is dense, given $\varepsilon > 0$, we can get a simple function $s' = \sum_{i=1}^n \chi_{A_i} x_i'$ with $x_i' \in R(E)$ and $\|s - s'\|_1 < \varepsilon$. For each i, let $e_i \in E$ be farthest from x_i'. Then $s_0 = \sum_{i=1}^n \chi_{A_i} e_i$ is pointwise farthest from s' and hence, by [5, Theorem 1.1], it is farthest from s' in $L^1(\mu, E)$ as well. ∎

In general, for a remotal set $E \subseteq X$, we do not expect $L^1(\mu, E)$ to be remotal in $L^1(\mu, X)$, but if E is separable, the result is given below.

We would like to point out that the proof of [5, Theorem 2.3]—which states that if E is a closed and bounded set spanning a finite-dimensional space, then $L^1(\mu, E)$ is remotal in $L^1(\mu, X)$—is incomplete, since the claim that weakly convergent sequences in $L^1(\mu, X)$ have pointwise convergent subsequences is incorrect. For example, the well-known Rademacher sequence $\{r_n\} \subseteq L^1(\mu)$ weakly converges to 0, and $\|r_n\| = 1$ for all $n \geq 1$, so that there is no pointwise convergent subsequence. Since their hypothesis implies that E is separable and remotal, our result below gives a correct proof of their result under a weaker hypothesis.

Theorem 3.8. *Let μ be the Lebesgue measure on $[0,1]$. For any separable remotal set $E \subseteq X$, $L^1(\mu, E)$ is remotal in $L^1(\mu, X)$.*

Proof. Let $f \in L^1(\mu, X)$. Since the measure space is complete, we may assume without loss of generality that all the functions involved are defined everywhere. In view of [5, Theorem 1.1], we only need to get a $g \in L^1(\mu, E)$ such that $g(t) \in E$ is a farthest point for $f(t)$ for all $t \in [0,1]$.

Let $\{e_n\}_{n\geq 1} \subseteq E$ be a dense sequence. Define $F : [0,1] \to 2^E$ by $F(t) = F_E(f(t)) = \{e \in E : \varrho(f(t), E) = \|f(t) - e\|\}$. By hypothesis, F is a non-empty, closed set-valued map. We now show the set $\mathrm{Gr}(F) = \{(t, e) : e \in F(t)\}$ is a measurable set. Note that $\mathrm{Gr}(F) = \{(t, e) : \varrho(f(t), E) = \|f(t) - e\|\} = \bigcap\{(t, e) : \|f(t) - e_n\| \leq \|f(t) - e\|\}$. Since f is measurable and $\|\cdot\|$ is continuous we see that $\mathrm{Gr}(F)$ is measurable.

It thus follows from the von Neumann measurable selection theorem [11, Corollary 5.5.8] that there exists a $g : [0,1] \to E$ measurable such that $g(t) \in F(t)$. As E is separable by the Pettis measurability theorem [3, Chapter 2, Theorem 2], we deduce that g is strongly measurable. As g is bounded, we have $g \in L^1(\mu, E)$. ∎

It is not clear how these arguments can be extended to spaces which are "rich" in separable subspaces like the weakly compactly generated Banach spaces (for example reflexive spaces). The usual separable reduction argument based on the fact that Bochner integrable functions are almost everywhere separably-valued does not work as remotality is in general not a hereditary property. Thus in the following corollary we make an appropriate assumption to get remotality in spaces of Bochner integrable functions. Hilbert spaces satisfy the hypothesis assumed below.

Corollary 3.9. *Let μ be the Lebesgue measure on $[0,1]$. Let E be a remotal set such that every maximizing sequence is contained in a separable remotal set. Then $L^1(\mu, E)$ is remotal in $L^1(\mu, X)$.*

Remark 3.10. Now let X be a reflexive Banach space. It is known that $L^1(\mu, X)$ is an L-summand in its bidual $L^1(\mu, X)^{**}$ (see [4, p. 200]). By Lemma 3.7, $L^1(\mu, B_X)$ is densely remotal in $L^1(\mu, X)$, since B_X is always a remotal subset of X. Thus by Lemma 3.1, $L^1(\mu, B_X)$ is also densely remotal in $L^1(\mu, X)^{**}$. In particular, when X is also separable we even find that $L^1(\mu, B_X)$ is remotal in $L^1(\mu, X)^{**}$.

We now turn to the question of dense remotality in $C(K, X)$. Let $E \subseteq X$ be a densely remotal set. For any $f \in C(K, X)$, if $g \in C(K, E)$ is such that $g(k)$ is a farthest point from $f(k)$ for all $k \in K$, then it is easy to see that g is a farthest point from f.

THEOREM 3.11. *Let $E \subseteq X$ be a densely remotal set. Let K be a compact totally disconnected space. Then $C(K, E)$ is densely remotal in $C(K, X)$.*

Proof. Since E is densely remotal, $R(E)$ is dense. Let $s = \sum_{i=1}^n \chi_{A_i} x_i$ be a simple function, where $x_i \in R(E)$ and the A_i's are clopen sets. We may assume that the A_i's are disjoint and that their union is K. Now let $s_0 = \sum_{i=1}^n \chi_{A_i} e_i$, where $e_i \in E$ are the farthest points from x_i. Since s_0 is pointwise farthest from s, it follows that $s \in R(C(K, E))$. It is well-known that $C(K) \otimes X$ is dense in $C(K, X)$. Since K is totally disconnected, it is easy to see that simple functions of the form $s' = \sum_{i=1}^n \chi_{A_i} x_i$, where $x_i \in X$, are dense in $C(K, X)$. Finally, since $R(E)$ is dense in X we conclude that s' can be approximated by a simple function of the form s. ∎

Acknowledgements. Most of this work was done during the visit of the first and third authors to the University of Iowa in August 2007. They would like to thank the second author and the UI for their warm hospitality. The third author thanks the Commission for Development and Exchange of the IMU and the National Board for Higher Mathematics, India, for travel support.

Note. Some of the results of this paper have since been improved by P. Bandyopadhyay and Tanmoy Paul. In particular, they have proved that any M-ideal in a $C(K)$ space is densely ball remotal and any function space is ball remotal.

REFERENCES

[1] P. Bandyopadhyay, D. Huang, B. L. Lin and S. L. Troyanski, *Some generalizations of locally uniform rotundity*, J. Math. Anal. Appl. 252 (2000), 906–916.

[2] R. Deville and V. E. Zizler, *Farthest points in w^*-compact sets*, Bull. Austral. Math. Soc. 38 (1988), 433–439.

[3] J. Diestel and J. J. Uhl, Jr., *Vector Measures*, Math. Surveys 15, Amer. Math. Soc., Providence, RI, 1977.

[4] P. Harmand, D. Werner and W. Werner, *M-Ideals in Banach Spaces and Banach Algebras*, Lecture Notes in Math. 1547, Springer, Berlin, 1993.

[5] R. Khalil and Sh. Al-Sharif, *Remotal sets in vector valued function spaces*, Sci. Math. Japon. 63 (2006), 433–441.

[6] K.-S. Lau, *Farthest points in weakly compact sets*, Israel J. Math. 22 (1975), 168–174.

[7] Å. Lima, *Intersection properties of balls in spaces of compact operators*, Ann. Inst. Fourier (Grenoble) 28 (1978), 35–65.
[8] T. S. S. R. K. Rao, *On ideals in Banach spaces*, Rocky Mountain J. Math. 31 (2001), 595–609.
[9] —, *On the geometry of higher duals of a Banach space*, Illinois J. Math. 45 (2001), 1389–1392.
[10] I. Singer, *Best Approximation in Normed Linear Spaces by Elements of Linear Subspaces*, Grundlehren Math. Wiss. 171, Springer, New York, 1970.
[11] S. M. Srivastava, *A Course on Borel Sets*, Grad. Texts in Math. 180, Springer, New York, 1998.
[12] J. J. Uhl, Jr., *Norm attaining operators on $L^1[0,1]$ and the Radon–Nikodým property*, Pacific J. Math. 63 (1976), 293–300.
[13] V. Zizler, *On some extremal problems in Banach spaces*, Math. Scand. 32 (1973), 214–224.

Stat–Math Division
Indian Statistical Institute
202, B. T. Road
Kolkata 700108, India
E-mail: pradipta@isical.ac.in

Department of Mathematics
University of Iowa
Iowa City IA 52242, U.S.A.
E-mail: bllin@math.uiowa.edu

Stat–Math Unit
Indian Statistical Institute
R. V. College P.O.
Bangalore 560059, India
E-mail: tss@isibang.ac.in

Received 27 December 2007;
revised 2 June 2008 (5001)