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# EQUIVALENT METRICS AND THE SPANS OF GRAPHS

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**Abstract.** We present a result which affords the existence of equivalent metrics on a space having distances between certain pairs of points predetermined, with some restrictions. This result is then applied to obtain metric spaces which have interesting properties pertaining to the span, semispan, and symmetric span of metric continua. In particular, we show that no two of these variants of span agree for all simple closed curves or for all simple triods.

**1. Introduction.** The span of a continuum was defined by A. Lelek in [4]. Since then, several variants of his definition have been given. The most prevalent of these are the semispan (see [6]), the symmetric span (see [3]), and, for simple closed curves, the essential span (see [1]).

It has been asked (see, for instance, [1] and [2]) whether some of these different quantities always agree for certain classes of continua, particularly for simple triods and simple closed curves. In this paper, we demonstrate that no two of these versions of span agree for all simple triods or for all simple closed curves. We also include an example which violates a conjectured bound between two versions of span.

A natural way to construct examples of metric spaces is to look at subsets of $\mathbb{R}^3$ with the Euclidean metric (see, for instance, [5], [6], and Section 7 of this paper). In Section 3 we develop an alternative approach which allows one to construct a metric for a space with certain distances predetermined. Related results have been obtained in [7] and [8]. This enables us to prove in Section 6 the existence of spaces with interesting span properties without producing subsets of $\mathbb{R}^3$.

**2. Notation.** If $(X, d)$ is a metric space, $x \in X$, and $A, B \subset X$, then define

$$d(x, B) = \inf\{d(x, b) : b \in B\},$$
$$d(A, B) = \inf\{d(a, b) : a \in A,\ b \in B\}.$$

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If $\varepsilon > 0$, then define the *$\varepsilon$-neighborhood* of $A$ by

$$A_\varepsilon = \{x \in X : d(x, A) < \varepsilon\}.$$

Define also the *$\varepsilon$-ball* about $x$ with respect to the metric $d$ by

$$S_d(x, \varepsilon) = \{y \in X : d(x, y) < \varepsilon\}.$$

If $\varrho$ is another metric on $X$, then $\varrho$ is *equivalent* to $d$ if the topologies generated by $\varrho$ and $d$ are identical.

Define the metric $d \times d$ on $X \times X$ by

$$(d \times d)((x_1, y_1), (x_2, y_2)) = d(x_1, x_2) + d(y_1, y_2).$$

The *diagonal* of $X$ is $\Delta X = \{(x, x) : x \in X\} \subset X \times X$.

An *arc* is a space which is homeomorphic to the closed unit interval $[0, 1]$. A *simple closed curve* is a space which is homeomorphic to the unit circle $\mathbb{S}^1$.

A *simple triod* with *legs* $A_1$, $A_2$, and $A_3$ is a space of the form $T = A_1 \cup A_2 \cup A_3$, where $A_1$, $A_2$, and $A_3$ are arcs which have a common endpoint $o$, and which are otherwise pairwise disjoint. The point $o$ is called the *branch point* of $T$.

## 3. Equivalent metrics with given distances

THEOREM 3.1. *Let $(X, d)$ be a metric space. Suppose $k > 0$, $0 < \varepsilon \leq 2/k$, and $f : X \times X \to \mathbb{R}$ is a function such that:*

(1) *$f$ is Lipschitz continuous with Lipschitz constant $k$ (with respect to the metric $d \times d$ on $X \times X$),*
(2) *$f(x, y) = f(y, x)$ for all $x, y \in X$,*
(3) *$f(X \times X) \subseteq [1, 2]$,*
(4) *$f((\Delta X)_\varepsilon) = \{1\}$, where $(\Delta X)_\varepsilon$ is the $\varepsilon$-neighborhood of the diagonal $\Delta X$.*

*Then there exists an equivalent metric $\varrho$ on $X$ such that $\varrho(x, y) = f(x, y)$ whenever $d(x, y) \geq \varepsilon/2$, and $\varrho(x, y) < 1$ whenever $d(x, y) < \varepsilon/2$.*

*Proof.* Define the function $\varrho : X \times X \to \mathbb{R}$ by

$$\varrho(x, y) = \begin{cases} f(x, y) & \text{if } d(x, y) \geq \varepsilon/2, \\ (2/\varepsilon)d(x, y) & \text{if } d(x, y) < \varepsilon/2. \end{cases}$$

It is clear from the definition of $\varrho$, and from conditions (2) and (3) on $f$, that $\varrho(x, y) = 0$ if and only if $x = y$, and $\varrho(x, y) = \varrho(y, x)$ for all $x, y \in X$. Hence to prove that $\varrho$ is a metric on $X$, we need only check that $\varrho$ satisfies the triangle inequality. Let $x, y, z \in X$, and consider the following cases:

CASE 1: $d(x, y), d(y, z), d(x, z) \geq \varepsilon/2$. In this case $\varrho(x, y) = f(x, y) \geq 1$, $\varrho(y, z) = f(y, z) \geq 1$, and $\varrho(x, z) = f(x, z) \leq 2 \leq \varrho(x, y) + \varrho(y, z)$.

CASE 2: $d(x,y) < \varepsilon/2$ and $d(y,z), d(x,z) \geq \varepsilon/2$. Here $\varrho(y,z) = f(y,z)$ and $\varrho(x,z) = f(x,z)$, so

$$\begin{aligned} |\varrho(y,z) - \varrho(x,z)| &= |f(y,z) - f(x,z)| \\ &\leq k \cdot (d \times d)((y,z),(x,z)) \quad \text{(by condition (1))} \\ &= k \cdot (d(y,x) + d(z,z)) = k \cdot d(x,y) \\ &\leq (2/\varepsilon) d(x,y) \quad (\text{since } \varepsilon \leq 2/k) \\ &= \varrho(x,y) \end{aligned}$$

It follows that $\varrho(x,y) + \varrho(y,z) \geq \varrho(x,z)$.

CASE 3: $d(x,y), d(y,z) < \varepsilon/2$ and $d(x,z) \geq \varepsilon/2$. Note that $d(x,z) \leq d(x,y) + d(y,z) < \varepsilon/2 + \varepsilon/2 = \varepsilon$, so $(x,z)$ is in the $\varepsilon$-neighborhood of $\Delta X$. This implies by condition (4) that $\varrho(x,z) = f(x,z) = 1$. Also, since $\varepsilon/2 \leq d(x,z) \leq d(x,y) + d(y,z)$, we have $1 \leq (2/\varepsilon)d(x,y) + (2/\varepsilon)d(y,z) = \varrho(x,y) + \varrho(y,z)$. Thus $\varrho(x,y) + \varrho(y,z) \geq \varrho(x,z)$.

CASE 4: $d(x,y), d(y,z), d(x,z) < \varepsilon/2$. Here $\varrho(x,y) + \varrho(y,z) = (2/\varepsilon)d(x,y) + (2/\varepsilon)d(y,z) \geq (2/\varepsilon)d(x,z) = \varrho(x,z)$.

CASE 5: $d(x,z) < \varepsilon/2$ and $d(x,y) \geq \varepsilon/2$. In this case, $\varrho(x,z) = (2/\varepsilon)d(x,z) < 1$ and $\varrho(x,y) = f(x,y) \geq 1$. Therefore $\varrho(x,y) + \varrho(y,z) \geq 1 > \varrho(x,z)$.

Because of the symmetry between $\varrho(x,y)$ and $\varrho(y,z)$ in the inequality $\varrho(x,y) + \varrho(y,z) \geq \varrho(x,z)$, we need not consider the remaining cases, as they have already been dealt with above (specifically in Cases 2 and 5) but with the pairs $(x,y)$ and $(y,z)$ interchanged. Thus $\varrho$ satisfies the triangle inequality, and so it is a metric.

Let $x \in X$ and let $\alpha > 0$. Let $\alpha' = \min\{\alpha, \varepsilon/2\}$. Then $\varrho(x,y) < (2/\varepsilon)\alpha' \leq 1$ implies $\varrho(x,y) = (2/\varepsilon)d(x,y)$, so $d(x,y) < \alpha' \leq \alpha$, hence $S_\varrho(x, (2/\varepsilon)\alpha') \subseteq S_d(x,\alpha)$. Conversely, let $\alpha'' = \min\{\alpha, 1\}$. Then $d(x,y) < (\varepsilon/2)\alpha'' \leq \varepsilon/2$ implies $\varrho(x,y) = (2/\varepsilon)d(x,y) < \alpha'' \leq \alpha$, so $S_d(x, (\varepsilon/2)\alpha'') \subseteq S_\varrho(x,\alpha)$. This implies that $\varrho$ is equivalent to $d$. ■

THEOREM 3.2. *Let $(X,d)$ be a metric space. Suppose $F, N \subset X \times X$ are nonempty subsets such that:*

(1) *for some $\varepsilon > 0$, $(d \times d)(F, N \cup (\Delta X)_\varepsilon) > 0$,*
(2) *$F = F^{-1}$, $N = N^{-1}$.*

*Then there exists an equivalent metric $\varrho$ on $X$ such that $\varrho(X \times X) \subseteq [0,2]$, $\varrho(F) = \{2\}$, and $\varrho(N) \subseteq [0,1]$.*

*Proof.* Let $N' = N \cup (\Delta X)_\varepsilon$, and let $\delta = (d \times d)(F, N') > 0$. Then $F_{\delta/2} \cap N'_{\delta/2} = \emptyset$.

Define the function $f : X \times X \to \mathbb{R}$ by

$$f(x,y) = \begin{cases} 2 - (1/\delta)\cdot(d\times d)((x,y),F) & \text{if } (x,y)\in F_{\delta/2},\\ 1 + (1/\delta)\cdot(d\times d)((x,y),N') & \text{if } (x,y)\in N'_{\delta/2},\\ 3/2 & \text{otherwise.}\end{cases}$$

It follows from the fact that $F = F^{-1}$ and $N' = (N')^{-1}$ that $f(x,y) = f(y,x)$ for all $x, y \in X$. Also, it is clear that $1 \leq f(x,y) \leq 2$ for all $x, y \in X$. Furthermore, we claim that $f$ is Lipschitz continuous, with Lipschitz constant $1/\delta$. To see this, we will argue for two particular cases; the rest follow similarly.

Case 1: $(x_1,y_1) \in F_{\delta/2}$ and $(x_2,y_2) \in F_{\delta/2}$. Assume $(d\times d)((x_1,y_1),F) \leq (d\times d)((x_2,y_2),F)$. For any $\alpha > 0$, there is some $z \in F$ such that $(d\times d)((x_1,y_1),z) < (d\times d)((x_1,y_1),F) + \alpha$. Then

$$\begin{aligned}(d\times d)((x_2,y_2),F) &\leq (d\times d)((x_2,y_2),z)\\ &\leq (d\times d)((x_1,y_1),(x_2,y_2)) + (d\times d)((x_1,y_1),z)\\ &< (d\times d)((x_1,y_1),(x_2,y_2)) + (d\times d)((x_1,y_1),F) + \alpha\end{aligned}$$

and it follows that

$$\begin{aligned}&|f(x_1,y_1) - f(x_2,y_2)|\\ &\quad= \left|\left[2 - \frac{1}{\delta}\cdot(d\times d)((x_1,y_1),F)\right] - \left[2 - \frac{1}{\delta}\cdot(d\times d)((x_2,y_2),F)\right]\right|\\ &\quad= \frac{1}{\delta}\cdot(d\times d)((x_2,y_2),F) - \frac{1}{\delta}\cdot(d\times d)((x_1,y_1),F)\\ &\quad< \frac{1}{\delta}\cdot(d\times d)((x_1,y_1),(x_2,y_2)) + \frac{\alpha}{\delta}.\end{aligned}$$

This holds for any $\alpha > 0$, so the claim follows.

Case 2: $(x_1,y_1) \in F_{\delta/2}$ and $(x_2,y_2) \in N'_{\delta/2}$. Given any $\alpha > 0$, there is some $z \in F$ and some $w \in N'$ which satisfy

$$\begin{aligned}(d\times d)((x_1,y_1),z) &< (d\times d)((x_1,y_1),F) + \alpha,\\ (d\times d)((x_2,y_2),w) &< (d\times d)((x_2,y_2),N') + \alpha.\end{aligned}$$

Then

$$\begin{aligned}&(d\times d)((x_1,y_1),(x_2,y_2))\\ &\qquad\geq (d\times d)(z,w) - (d\times d)((x_1,y_1),z) - (d\times d)((x_2,y_2),w)\\ &\qquad> \delta - (d\times d)((x_1,y_1),F) - (d\times d)((x_2,y_2),N') - 2\alpha\end{aligned}$$

and so

$$
\begin{aligned}
&|f(x_1,y_1)-f(x_2,y_2)|\\
&\quad=\left[2-\frac{1}{\delta}\cdot(d\times d)((x_1,y_1),F)\right]-\left[1+\frac{1}{\delta}\cdot(d\times d)((x_2,y_2),N')\right]\\
&\quad=1-\frac{1}{\delta}\cdot(d\times d)((x_1,y_1),F)-\frac{1}{\delta}\cdot(d\times d)((x_2,y_2),N')\\
&\quad<\frac{1}{\delta}\cdot(d\times d)((x_1,y_1),(x_2,y_2))+\frac{2\alpha}{\delta}.
\end{aligned}
$$

This holds for any $\alpha > 0$, so again the claim follows.

Finally, we have $f((\Delta X)_{\varepsilon'}) = \{1\}$ for any $\varepsilon' \le \varepsilon$, since $(\Delta X)_\varepsilon \subseteq N'$. Therefore $f$ satisfies conditions (1) through (4) of Theorem 3.1. Thus we may apply this theorem to obtain an equivalent metric $\varrho$ on $X$ such that $\varrho$ agrees with $f$ outside of some small neighborhood of $\Delta X$ (which is contained in $(\Delta X)_\varepsilon$, hence is disjoint from $F$), and $\varrho < 1$ within this neighborhood. It follows that $\varrho(X \times X) \subseteq [0, 2]$, $\varrho(F) = \{2\}$, and $\varrho(N) \subseteq [0, 1]$. ■

**4. Visualizing the set $T \times T$ for $T$ a simple triod.** Given a space $X$, it will be useful to have a practical but accurate way of visualizing the product $X \times X$. The goal of this section is to explain the pictures that will be used in later sections to describe the sets $F$ and $N$ required by Theorem 3.2.

If $X$ is the unit interval $[0, 1]$ or the circle $\mathbb{S}^1$, then we can easily visualize $X \times X$ in the plane as the square (with opposite sides identified in the case of the circle).

We now extend this idea to the case of $X$ a simple triod. Let $J = [0, 1] \cup [2, 4] \subset \mathbb{R}$. Observe that $X$ is homeomorphic to $J/\{1, 3\}$, i.e. the space $J$ with the points 1 and 3 identified. Then we can view $X \times X$ as a

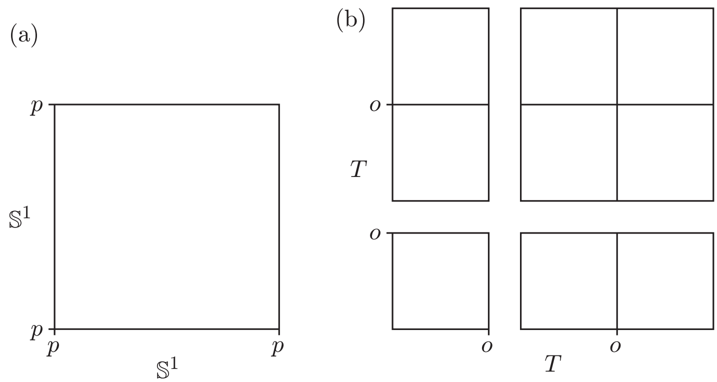


Fig. 1. (a) Standard visualization of the torus $\mathbb{S}^1 \times \mathbb{S}^1$ in the plane. (b) Analogous visualization of $T \times T$ as a quotient of a subset of the plane, for $T$ a simple triod.

subset of the plane consisting of four rectangles as shown in Figure 1, where each point on the right edges of the two rectangles on the left is identified with the point at the same height in the middle of the rectangles on the right, and each point on the top edges of the two rectangles at the bottom is identified with the point in the same vertical line in the middle of the rectangles at the top.

In fact, for any space $X$ that can be written as a quotient of a bounded subset of the real line, we can visualize $X \times X$ as a quotient of a bounded subset of the real plane in a straightforward way. In particular, any graph admits such a visualization. See Section 7 for another example.

**5. Identifying components of a subset of $T \times T$ for $T$ a simple triod.** When working with the spans of a space $X$ (defined below), it is helpful to be able to identify components of a subset of $X \times X$. We will suppose that the reader is comfortable working with connected sets in the "square with opposite sides identified" representation of the torus $\mathbb{S}^1 \times \mathbb{S}^1$; one need only keep in mind that connected sets can "wrap around" from the left edge of the square to the right, and from the top to bottom.

We note the following sufficient condition for finding a component of a subset of the torus:

*Given nonempty sets $K \subseteq Y \subset \mathbb{S}^1 \times \mathbb{S}^1$, consider the corresponding subsets $K' \subseteq Y'$ of the square $I^2 = [0,1] \times [0,1]$. If $K'$ is clopen in $Y'$ and is such that its intersections with the edges of the square $I^2$ match up in pairs on opposite sides, then $K$ is clopen in $Y$. In particular, if $K$ is connected, then it is a component of $Y$.*

A similar approach is available for finding components of subsets of $T \times T$, where $T$ is a simple triod, using the visualization described in the previous section. The following technical result validates the intuitive idea that we can identify a component of a set $Y \subset T \times T$ by finding a collection of clopen (in $Y$) sets in the squares of our picture whose union is connected, and whose boundaries match up in triples in the appropriate places.

We emphasize that the following provides merely a sufficient condition for $K$ to be a component of $Y \subset T \times T$; for less well-behaved subsets $Y$ than what we will consider below, there may be components which do not arise this way.

PROPOSITION 5.1. *Let $T$ be a simple triod with legs $A_1$, $A_2$, $A_3$ and branch point $o$. Suppose $Y \subset T \times T$, and let $Y_{ij} = Y \cap (A_i \times A_j)$ for $i, j = 1, 2, 3$. Suppose $\{K_{ij} : i, j = 1, 2, 3\}$ is a collection of nine sets such that $K_{ij} \subseteq Y_{ij}$. Let $K := \bigcup_{i,j=1}^{3} K_{ij}$, and assume the following*:

(1) *$K_{ij}$ is clopen in $Y_{ij}$ for each $i, j = 1, 2, 3$,*
(2) *if $(o, y) \in K_{ij}$ then $(o, y) \in K_{1j} \cap K_{2j} \cap K_{3j}$,*
(3) *if $(x, o) \in K_{ij}$ then $(x, o) \in K_{i1} \cap K_{i2} \cap K_{i3}$.*

*Then $K$ is clopen in $Y$. In particular, if $K$ is also connected (and nonempty), then it is a component of $Y$.*

*Proof.* Observe that $A_i \times A_j$ is closed in $T \times T$, hence $Y_{ij}$ is closed in $Y$. Furthermore, distinct sets $Y_{i_0 j_0}$ and $Y_{i_1 j_1}$ may intersect only in points of the form $(x, o)$ or $(o, y)$.

Since each set $K_{ij}$ is a closed subset of $Y_{ij}$ (by condition (1)), which is a closed subset of $Y$, we see that $K_{ij}$ is closed in $Y$. This implies $K = \bigcup_{i,j=1}^{3} K_{ij}$ is closed in $Y$.

Also, note that each set of the form $Y_{ij} \setminus K_{ij}$ is closed in $Y_{ij}$ (since $K_{ij}$ is open in $Y_{ij}$, again by condition (1)), so likewise we find that $\bigcup_{i,j=1}^{3}(Y_{ij} \setminus K_{ij})$ is closed in $Y$.

Since $Y = \bigcup_{i,j=1}^{3} Y_{ij}$, it is clear that $Y \setminus K \subseteq \bigcup_{i,j=1}^{3}(Y_{ij} \setminus K_{ij})$. For the reverse inclusion, suppose that $(x, y) \in \bigcup_{i,j=1}^{3}(Y_{ij} \setminus K_{ij})$, but $(x, y) \notin Y \setminus K$, that is, $(x, y) \in K$.

If $x \neq o$ and $y \neq o$, then there is a unique pair $(i, j)$ such that $(x, y) \in Y_{ij}$, so we must have $(x, y) \in Y_{ij} \setminus K_{ij}$. But since $(x, y) \in K$, we must also have $(x, y) \in K_{ij}$, which is a contradiction.

If $x = o$ and $y \neq o$, then there is exactly one $j_0$ such that $(o, y) \in Y_{1j_0} \cap Y_{2j_0} \cap Y_{3j_0}$, and $(o, y) \notin Y_{ij}$ whenever $j \neq j_0$. Since $(o, y) \in K$, we must have $(o, y) \in K_{ij_0}$ for some $i$, which implies by condition (2) that $(o, y) \in K_{1j_0} \cap K_{2j_0} \cap K_{3j_0}$. This contradicts the fact that $(x, y) = (o, y) \in \bigcup_{i,j=1}^{3}(Y_{ij} \setminus K_{ij})$.

Likewise, if $y = o$ and $x \neq o$, we arrive at a contradiction using condition (3). If $x = y = o$, then it follows from condition (2) (applied once) and then condition (3) (applied three times) that $(o, o) \in K_{ij}$ for each $i, j$, which again contradicts the fact that $(x, y) = (o, o) \in \bigcup_{i,j=1}^{3}(Y_{ij} \setminus K_{ij})$.

Therefore, $Y \setminus K = \bigcup_{i,j=1}^{3}(Y_{ij} \setminus K_{ij})$, which, as noted above, is closed in $Y$. Hence $K$ is open in $Y$. ∎

**6. Application to span theory.** Let $\pi_1$ and $\pi_2$ denote the first and second coordinate projections, respectively, of $X \times X$ onto $X$; that is, $\pi_1(x_1, x_2) = x_1$ and $\pi_2(x_1, x_2) = x_2$.

If $(X, d)$ is a connected metric space, then define the *surjective semispan of $X$*, $\sigma_0^*(X)$ (see [6]), to be

$$\sigma_0^*(X) = \sup_{Z \in \mathcal{Z}} \inf_{(x_1, x_2) \in Z} d(x_1, x_2),$$

where $\mathcal{Z}$ is the family of subsets $Z$ of $X \times X$ with the following properties:

(1) $Z$ is connected,
(2) $\pi_1(Z) = X$.

If we strengthen condition (2) to

(2′) $\pi_1(Z) = \pi_2(Z) = X$

then the value we obtain is called the *surjective span* of $X$, $\sigma^*(X)$. If on top of this we require the condition

(3) $Z = Z^{-1}$

on $\mathcal{Z}$, then the value we obtain is called the *surjective symmetric span* of $X$, $s^*(X)$ (see [3]).

Now we define the *semispan*, *span*, and *symmetric span* of $X$ to be, respectively,

$$\sigma_0(X) = \sup\{\sigma_0^*(Y) : Y \text{ is a connected subset of } X\},$$
$$\sigma(X) = \sup\{\sigma^*(Y) : Y \text{ is a connected subset of } X\},$$
$$s(X) = \sup\{s^*(Y) : Y \text{ is a connected subset of } X\}.$$

Notice that the only connected proper subsets of a simple closed curve are arcs. Since arcs have surjective semispan, span, and symmetric span all equal to zero (see [6]), we have $\sigma_0 = \sigma_0^*$, $\sigma = \sigma^*$, and $s = s^*$ for simple closed curves.

Suppose $\Gamma$ is a simple closed curve with metric $d$. Define the *essential span* of $\Gamma$, $\sigma_e(\Gamma)$ (see [1]), to be

$$\sigma_e(\Gamma) = \sup_{f,g} \inf_{\theta\in\mathbb{S}^1} d(f(\theta), g(\theta)),$$

where $f$ and $g$ are degree one maps from $\mathbb{S}^1$ to $\Gamma$.

Remark. The original definition of the essential span given in [1] is restricted to simple closed curves in the plane $\mathbb{R}^2$. There is no problem in extending the definition to arbitrary simple closed curves; however, the examples we consider below are not planar, and so it remains an open question whether essential span can differ from the other versions of span in the plane. In fact, it is still unknown for most pairs of spans whether they can differ among continua in the plane.

For each of the examples below, the metric is constructed by using Theorem 3.2. This means that each space has diameter equal to 2, and hence each version of span takes a value $\leq 2$.

Example 6.1. *There exists a simple closed curve $\Gamma$ with $\sigma(\Gamma) = \sigma_e(\Gamma) = 2$ and $s(\Gamma) = 1$.*

*Proof.* Let $F$ and $N$ be subsets of the torus $\mathbb{S}^1 \times \mathbb{S}^1$ as shown in Fig. 2, where $F$ is depicted by the thick solid lines, and $N$ is depicted by the dashed

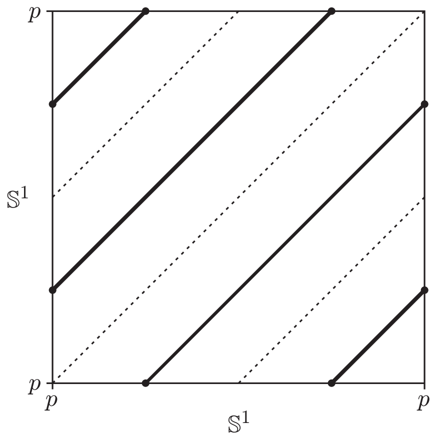


Fig. 2. Subsets $F$ (thick solid lines) and $N$ (dashed lines) of $\mathbb{S}^1 \times \mathbb{S}^1$ for constructing a metric $\varrho$ on $\mathbb{S}^1$ for which essential span and symmetric span differ.

lines. It is clear that $F = F^{-1}$ and $N = N^{-1}$, so we can apply Theorem 3.2 to obtain a metric $\varrho$ on $\mathbb{S}^1$ such that $\varrho(x, y) = 2$ if $(x, y) \in F$, and $\varrho(x, y) \leq 1$ if $(x, y) \in N$. Denote the metric space $(\mathbb{S}^1, \varrho)$ by $\Gamma$.

Notice that $F$ consists of two essential loops in $\Gamma \times \Gamma$, and this implies that $\sigma(\Gamma) = \sigma_e(\Gamma) = 2$.

Note also that $(\Gamma \times \Gamma) \setminus N$ consists of two components, say $K_1$ and $K_2$, and $K_1^{-1} = K_2$. In particular, $K_1 \cap K_1^{-1} = K_2 \cap K_2^{-1} = \emptyset$. Thus if $Z$ is a connected subset of $\Gamma \times \Gamma$ such that $Z = Z^{-1}$, then $Z$ must meet $N$. It follows that $s(\Gamma) \leq 1$. It is clear from the construction of the metric $\varrho$ in the proof of Theorem 3.2 that in fact $s(T) = 1$ (since $\varrho(x, y) \geq 1$ for all points $(x, y) \in \Gamma \times \Gamma$ except those in a small neighborhood of the diagonal). ∎

Remark. We can in fact find a simple closed curve $\Gamma$ in $\mathbb{R}^3$ with the Euclidean metric which (nearly) satisfies the properties of Example 6.1. Take $\Gamma$ to be the boundary circle of the usual embedding of the Möbius strip in $\mathbb{R}^3$. Then if we graph the Euclidean metric $d : \Gamma \times \Gamma \to \mathbb{R}$, the resulting picture will have the form of Figure 2, where the thick solid lines represent pairs of points that are far apart (say at distance 1), and the dashed lines represent pairs of points which are closer than $\varepsilon$ to one another, where $\varepsilon$ is the width of the strip. Thus for any $\varepsilon > 0$, we can find a simple closed curve $\Gamma \subset \mathbb{R}^3$ with the Euclidean metric such that $\sigma(\Gamma) = \sigma_e(\Gamma) = 1$ and $s(\Gamma) < \varepsilon$.

Example 6.2. *There exists a simple closed curve $\Gamma$ with $\sigma(\Gamma) = 2$ and $\sigma_e(\Gamma) = 1$.*

*Proof.* Let $F$ and $N$ be subsets of the torus $\mathbb{S}^1 \times \mathbb{S}^1$ as shown in Fig. 3, where $F$ is depicted by the thick solid lines, and $N$ is depicted by the dashed lines. Again, we have $F = F^{-1}$ and $N = N^{-1}$, so we can apply Theorem 3.2

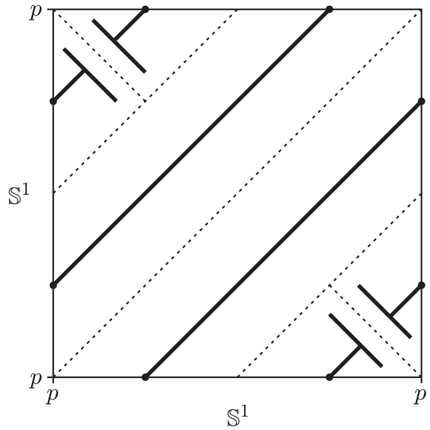


Fig. 3. Subsets $F$ (thick solid lines) and $N$ (dashed lines) of $\mathbb{S}^1 \times \mathbb{S}^1$ for constructing a metric $\varrho$ on $\mathbb{S}^1$ for which span and essential span differ.

to obtain a metric $\varrho$ on $\mathbb{S}^1$ such that $\varrho(x, y) = 2$ if $(x, y) \in F$, and $\varrho(x, y) \leq 1$ if $(x, y) \in N$. Denote the metric space $(\mathbb{S}^1, \varrho)$ by $\Gamma$.

Note that $F$ consists of two components, both of which have both first and second coordinate projections equal to $\Gamma$ (since the "handles" in the corners overlap horizontally and vertically). It follows that $\sigma(\Gamma) = 2$.

Note also that $(\Gamma \times \Gamma) \setminus N$ consists of two components, both of which are simply connected. This implies that any essential loop in $\Gamma \times \Gamma$ must meet $N$. It follows that $\sigma_{\mathrm{e}}(\Gamma) \leq 1$. Again, it is clear from the construction of the metric $\varrho$ in the proof of Theorem 3.2 that in fact $\sigma_{\mathrm{e}}(\Gamma) = 1$. ■

EXAMPLE 6.3. *There exists a simple closed curve $\Gamma$ with $\sigma_0(\Gamma) = 2$ and $\sigma(\Gamma) = 1$.*

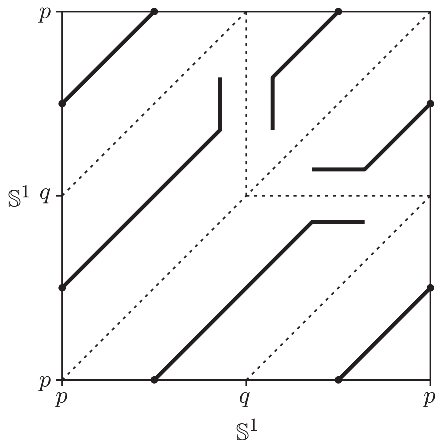


Fig. 4. Subsets $F$ (thick solid lines) and $N$ (dashed lines) of $\mathbb{S}^1 \times \mathbb{S}^1$ for constructing a metric $\varrho$ on $\mathbb{S}^1$ for which span and semispan differ.

*Proof.* Let $F$ and $N$ be subsets of the torus $\mathbb{S}^1 \times \mathbb{S}^1$ as shown in Fig. 4, where $F$ is depicted by the solid black lines, and $N$ is depicted by the dashed lines. Again, we have $F = F^{-1}$ and $N = N^{-1}$, so we can apply Theorem 3.2 to obtain a metric $\varrho$ on $\mathbb{S}^1$ such that $\varrho(x, y) = 2$ if $(x, y) \in F$, and $\varrho(x, y) \leq 1$ if $(x, y) \in N$. Denote the metric space $(\mathbb{S}^1, \varrho)$ by $\Gamma$.

Note that $F$ consists of two components, say $F_1$ and $F_2$, with $F_1^{-1} = F_2$, where $\pi_1(F_1) = \Gamma$ (and $\pi_2(F_2) = \Gamma$). It follows that $\sigma_0(\Gamma) = 2$.

Note also that $(\Gamma \times \Gamma) \setminus N$ consists of two components, say $K_1$ and $K_2$, with $K_1^{-1} = K_2$, where $F_1 \subset K_1$, $F_2 \subset K_2$. We have $\pi_2(K_1) = \Gamma \setminus \{q\}$ and $\pi_1(K_2) = \Gamma \setminus \{q\}$ (where $q$ is as shown in Figure 4). Hence if $Z \subset \Gamma \times \Gamma$ is a connected set with $\pi_1(Z) = \pi_2(Z) = \Gamma$, then we must have $Z \cap N \neq \emptyset$. This implies $\sigma^*(\Gamma) \leq 1$, so $\sigma(\Gamma) \leq 1$. Again, it is clear from the construction of the metric $\varrho$ in the proof of Theorem 3.2 that in fact $\sigma(\Gamma) = 1$. ∎

EXAMPLE 6.4. *There exists a simple triod $T$ with $\sigma(T) = 2$ and $s(T) = 1$.*

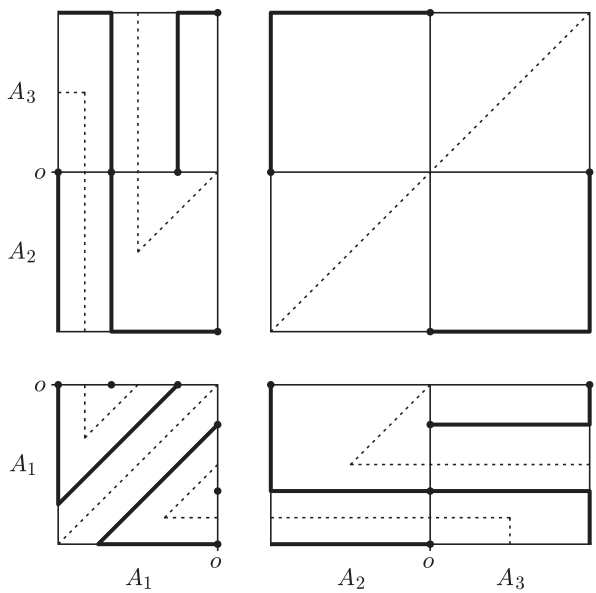


Fig. 5. Subsets $F$ (thick solid lines) and $N$ (dashed lines) of $T \times T$ for constructing a metric $\varrho$ on the simple triod $T$ for which span and symmetric span differ.

*Proof.* Let $F$ and $N$ be subsets of $T \times T$ as shown in Figure 5, where $F$ is depicted by the thick black lines, and $N$ is depicted by the dashed lines. It is clear that $F = F^{-1}$ and $N = N^{-1}$, so we can apply Theorem 3.2 to obtain a metric $\varrho$ on $T$ such that $\varrho(x, y) = 2$ if $(x, y) \in F$, and $\varrho(x, y) \leq 1$ if $(x, y) \in N$. We shall refer to the metric space $(T, \varrho)$ simply as $T$ for the rest of this example.

It can readily be seen that $F$ consists of two components, each of which has both first and second coordinate projections equal to $T$. It follows that $\sigma^*(T) = 2$, hence $\sigma(T) = 2$ (since the diameter of $T$ is 2).

Using Proposition 5.1, one can verify that $(T \times T) \setminus N$ consists of two components, say $K_1$ and $K_2$, with $K_1^{-1} = K_2$. In particular, $K_1 \cap K_1^{-1} = K_2 \cap K_2^{-1} = \emptyset$. Thus if $Z$ is a connected subset of $T \times T$ such that $Z = Z^{-1}$, then $Z$ must meet $N$. It follows that $s(T) \le 1$. Once again, it is clear from the construction of the metric $\varrho$ in the proof of Theorem 3.2 that in fact $s(T) = 1$. ∎

EXAMPLE 6.5. *There exists a simple triod $T$ with $\sigma_0(T) = 2$ and $\sigma^*(T) = 1$.*

*Proof.* Let $F$ and $N$ be subsets of $T \times T$ as shown in Figure 6, where $F$ is depicted by the thick solid lines, and $N$ is depicted by the dashed lines. It is clear that $F = F^{-1}$ and $N = N^{-1}$, so we can apply Theorem 3.2 to obtain a metric $\varrho$ on $T$ such that $\varrho(x, y) = 2$ if $(x, y) \in F$, and $\varrho(x, y) \le 1$ if $(x, y) \in N$. We shall refer to the metric space $(T, \varrho)$ simply as $T$ for the rest of this example.

It can readily be seen that $F$ consists of two components, say $F_1$ and $F_2$, with $F_1^{-1} = F_2$, where $\pi_1(F_1) = T$ (and $\pi_2(F_2) = T$). It follows that $\sigma_0^*(T) = 2$, hence $\sigma_0(T) = 2$.

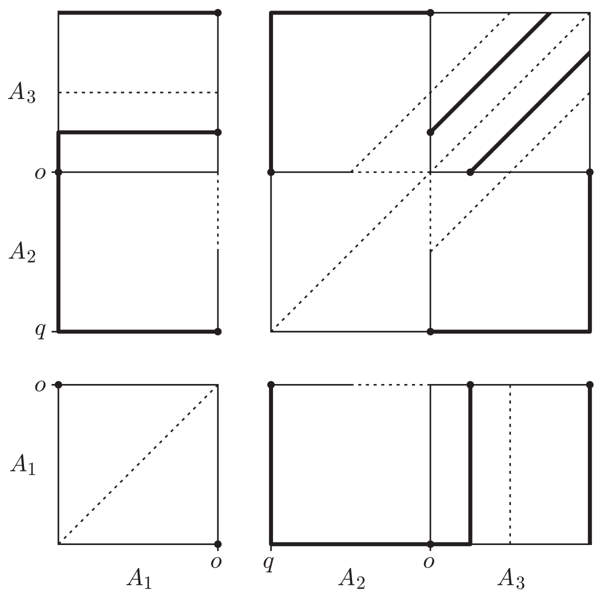


Fig. 6. Subsets $F$ (thick solid lines) and $N$ (dashed lines) of $T \times T$ for constructing a metric $\varrho$ on the simple triod $T$ for which surjective span and semispan differ.

Using Proposition 5.1, one can verify that $(T \times T) \setminus N$ consists of two components, say $K_1$ and $K_2$, with $K_1^{-1} = K_2$, where $F_1 \subset K_1$, $F_2 \subset K_2$. Once these components have been ascertained, it can easily be seen that $\pi_2(K_1) = \pi_1(K_2) = T \setminus \{q\}$ (where $q$ is the endpoint of $A_2$ which is distinct from $o$; see Figure 6). Hence if $Z \subset T \times T$ is a connected set with $\pi_1(Z) = \pi_2(Z) = T$, then we must have $Z \cap N \neq \emptyset$. This implies $\sigma^*(T) \leq 1$. Again, it is clear from the construction of the metric $\varrho$ in the proof of Theorem 3.2 that in fact $\sigma^*(T) = 1$. ∎

Remark. By looking at subtriods of the triod $T$ of Example 6.5, it is not difficult to see that in fact $\sigma(T) = 1$.

**7. Further span examples in $\mathbb{R}^3$.** We begin by remarking that there exists a simple triod $T$ in $\mathbb{R}^3$ with the property that $\sigma_0(T) = \sigma_0^*(T) = 1$ and $\sigma(T) = \sigma^*(T) = 1/2$, that is, with the same ratio as in Example 6.5. The specifics of the construction of this triod may be gleaned from Example 7.1 below, but for now we will omit the details, and simply refer to Figure 7 to get some sense of the shape of it. In this picture, the positive $x$-axis points to the right, the positive $y$-axis points away from the viewer, and the positive $z$-axis points upward. Note that the lower leg and the right upper leg both lie entirely in the $xz$-plane, and the third leg spirals around in the $x$ and $y$ directions while simultaneously rising in the $z$ direction.

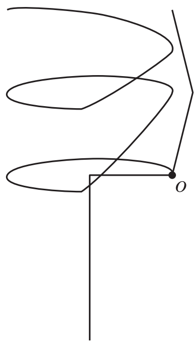


Fig. 7. A simple triod in $\mathbb{R}^3$ for which surjective span and semispan differ.

We will not prove the above statements about the spans of this triod $T$, but the same approach taken in Example 6.5 above works here as well; one needs to find appropriate subsets $F$ and $N$ of $T \times T$ so that pairs in $F$ are at distance $\geq 1$ and points in $N$ are at distance $\leq 1/2$ (in the Euclidean metric), then proceed as above.

Next, given any $\delta > 0$, we can embed a simple closed curve in $\mathbb{R}^3$ as follows: start near (i.e. within $\delta/2$ of) $o$, then travel along one leg of $T$ to its tip, then back near $o$, then along the next leg of $T$ to its tip, then back near $o$, then along the third leg of $T$ to its tip, then back to the original starting point, all that while staying within $\delta/2$ of $T$ and avoiding any unwanted self-intersections. One can verify that the resulting simple closed curve $\Gamma$ is such that $\sigma_0(\Gamma) > 1-\delta$ and $\sigma(\Gamma) < 1/2+\delta$; hence it nearly attains the same ratio as in Example 6.3.

Now, it turns out that one can add an arc to this space $\Gamma$ to obtain the following example, which answers negatively the question of whether $\sigma_0(X) \le 2\sigma_0^*(X)$ for all continua $X$ (see [6]).

EXAMPLE 7.1. *For any $\delta > 0$, there exists a graph $\Gamma$ in $\mathbb{R}^3$ (with the Euclidean metric) such that $\sigma_0(\Gamma) > 1-\delta$ and $\sigma_0^*(\Gamma) \le 1/4+\delta$.*

*Proof.* Define the maps $\alpha_i : [0,1] \to \mathbb{R}^3$ as follows:

$$\alpha_1(t) = (1-t)\big(-\tfrac{1}{2},0,-1\big) + t\big(-\tfrac{1}{2},0,0\big),$$
$$\alpha_2(t) = (1-t)\big(-\tfrac{1}{2},0,0\big) + t(0,0,0),$$
$$\alpha_3(t) = (1-t)(0,0,0) + t\big(\tfrac{1}{8},0,\tfrac{1}{2}\big),$$
$$\alpha_4(t) = (1-t)\big(\tfrac{1}{8},0,\tfrac{1}{2}\big) + t(0,0,1),$$
$$\alpha_5(t) = \alpha_4(1-t),$$
$$\alpha_6(t) = \alpha_3(1-t),$$
$$\alpha_7(t) = \begin{cases} \big(-\tfrac{1}{2}+\tfrac{1}{2}\cos(5t\pi), \tfrac{1}{2}\sin(5t\pi), 0\big) & \text{if } t \in \big[0,\tfrac{3}{10}\big], \\ \big(-\tfrac{1}{2}+\tfrac{1}{2}\cos(5t\pi), \tfrac{1}{2}\sin(5t\pi), \tfrac{1}{2}\cos(5t\pi)\big) & \text{if } t \in \big[\tfrac{3}{10},\tfrac{4}{10}\big], \\ \big(-\tfrac{1}{2}+\tfrac{1}{2}\cos(5t\pi), \tfrac{1}{2}\sin(5t\pi), \tfrac{1}{2}\big) & \text{if } t \in \big[\tfrac{4}{10},\tfrac{7}{10}\big], \\ \big(-\tfrac{1}{2}+\tfrac{1}{2}\cos(5t\pi), \tfrac{1}{2}\sin(5t\pi), \tfrac{1}{2}+\tfrac{1}{4}\cos(5t\pi)\big) & \text{if } t \in \big[\tfrac{7}{10},\tfrac{8}{10}\big], \\ \big(-\tfrac{1}{2}+\tfrac{1}{2}\cos(5t\pi), \tfrac{1}{2}\sin(5t\pi), \tfrac{5}{4}t-\tfrac{1}{4}\big) & \text{if } t \in \big[\tfrac{8}{10},1\big], \end{cases}$$
$$\alpha_8(t) = \alpha_7(1-t),$$
$$\alpha_9(t) = \alpha_2(1-t),$$
$$\alpha_{10}(t) = \alpha_1(1-t),$$

and define $\phi^* : [0,1] \to \mathbb{R}^3$ by

$$\phi^* = \alpha_1 * \cdots * \alpha_{10},$$

where $*$ denotes the usual homotopy product of paths.

Note that each $\alpha_i$ is one-to-one, and $\phi^*(0) = \phi^*(1) = \big(-\tfrac{1}{2},0,-1\big)$, so we may regard $\phi^*$ as a piecewise homeomorphism of the circle $\mathbb{S}^1$. Given any $\delta > 0$, there exists a continuous function $\phi_\delta : [0,1] \to \mathbb{R}^3$ such that $\phi_\delta$ is one-to-one on $[0,1)$, $\phi_\delta(0) = \phi_\delta(1)$ (hence $\phi_\delta([0,1])$ is a simple closed curve), and $d(\phi^*(t), \phi_\delta(t)) < \delta/2$ for all $t \in [0,1]$. In other words, we can perturb the path in $\mathbb{R}^3$ defined by $\phi^*$ by an arbitrarily small amount to obtain an embedding of the circle $\mathbb{S}^1$. Let $\Gamma_\delta$ be the resulting simple closed curve, that is, $\Gamma_\delta = \phi_\delta([0,1])$.

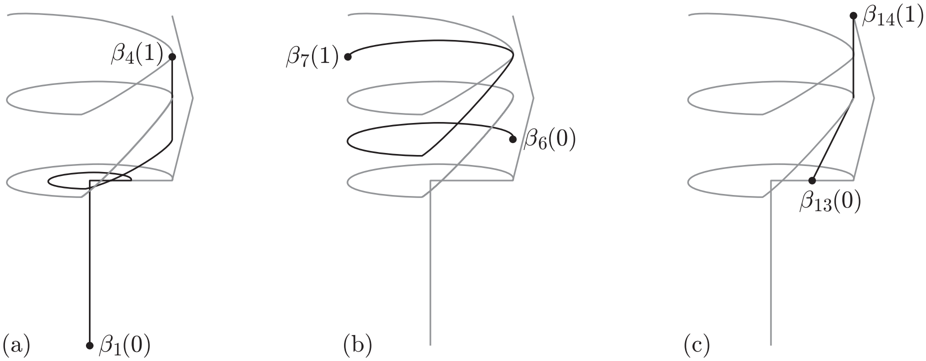


Fig. 8. Images of the maps $\beta_i$ against $\phi^*([0,1])$ (in gray), for (a) $i = 1,2,3,4$; (b) $i = 6,7$; (c) $i = 13,14$.

Next, define the maps $\beta_i : [0,1] \to \mathbb{R}^3$ as follows (see Figure 8):

$$\beta_1(t) = (1-t)\big(-\tfrac{1}{2}, 0, -1\big) + t\big(-\tfrac{1}{2}, 0, 0\big),$$
$$\beta_2(t) = (1-t)\big(-\tfrac{1}{2}, 0, 0\big) + t\big(-\tfrac{1}{4}, 0, 0\big),$$
$$\beta_3(t) = \begin{cases} \big(-\tfrac{1}{2} + \tfrac{1}{4}\cos(2t\pi), \tfrac{1}{4}\sin(2t\pi), 0\big) & \text{if } t \in \big[0, \tfrac{3}{4}\big], \\ \big(-\tfrac{1}{2} + \tfrac{1}{2}\cos(2t\pi), \tfrac{1}{4}\sin(2t\pi), \tfrac{1}{4}\cos(2t\pi)\big) & \text{if } t \in \big[\tfrac{3}{4}, 1\big], \end{cases}$$
$$\beta_4(t) = (1-t)\big(0, 0, \tfrac{1}{4}\big) + t\big(0, 0, \tfrac{3}{4}\big),$$
$$\beta_5(t) = \beta_4(1-t),$$
$$\beta_6(t) = \begin{cases} \big(-\tfrac{1}{2} + \tfrac{1}{2}\cos(2t\pi), \tfrac{1}{2}\sin(2t\pi), \tfrac{1}{4}\big) & \text{if } t \in \big[0, \tfrac{3}{4}\big], \\ \big(-\tfrac{1}{2} + \tfrac{1}{2}\cos(2t\pi), \tfrac{1}{2}\sin(2t\pi), \tfrac{1}{4} + \tfrac{1}{2}\cos(2t\pi)\big) & \text{if } t \in \big[\tfrac{3}{4}, 1\big], \end{cases}$$
$$\beta_7(t) = \big(-\tfrac{1}{2} + \tfrac{1}{2}\cos(t\pi), \tfrac{1}{2}\sin(t\pi), \tfrac{3}{4}\big),$$
$$\beta_8(t) = \beta_7(1-t),$$
$$\beta_9(t) = \beta_6(1-t),$$
$$\beta_{10}(t) = \beta_5(1-t),$$
$$\beta_{11}(t) = \beta_4(1-t),$$
$$\beta_{12}(t) = \beta_3(1-t),$$
$$\beta_{13}(t) = (1-t)\big(-\tfrac{1}{4}, 0, 0\big) + t\big(0, 0, \tfrac{1}{2}\big),$$
$$\beta_{14}(t) = (1-t)\big(0, 0, \tfrac{1}{2}\big) + t(0, 0, 1),$$

and define $\psi^* : [0,1] \to \mathbb{R}^3$ by

$$\psi^* = \beta_1 * \cdots * \beta_{14}.$$

Let $P$ be the "lollipop" figure obtained by taking the union of the unit circle $\mathbb{S}^1$ in the plane with the line segment joining the points $(1,0)$ and $(2,0)$. Denote the point $(1,0) \in P$ by $v$.

Note that each function $\beta_i$ above is one-to-one, and $\psi^*(0) = \phi^*(0) = \phi^*(1) = \left(-\frac{1}{2}, 0, -1\right) =: v'$, so the combination of $\phi^*$ and $\psi^*$ may be regarded as a piecewise homeomorphism $\theta^*$ of the space $P$ into $\mathbb{R}^3$, taking $v$ to $v'$. Given any $\delta > 0$, there exists an embedding $\theta_\delta : P \to \mathbb{R}^3$ such that $d(\theta^*(x), \theta_\delta(x)) < \delta/2$ for all $x \in P$. Let $\Gamma_\delta = \theta_\delta(P)$. We can ensure that $\theta_\delta$ restricted to the circle in $P$ agrees with the embedding $\phi_\delta$, so that $\Gamma_\delta$ contains the simple closed curve $\Gamma_\delta$ as a subset.

Let the functions $\alpha_i'$ and $\beta_i'$ be approximations of the maps $\alpha_i$ and $\beta_i$, respectively, such that we can regard the function $\theta_\delta$ restricted to the circle $\mathbb{S}^1 \subset P$ as $\alpha_1' * \cdots * \alpha_{10}'$, and $\theta_\delta$ restricted to the arc $[1, 2] \subset P$ as $\beta_1' * \cdots * \beta_{14}'$.

We can show, using the following subsets $F_1, \ldots, F_6$ of $\Gamma_\delta \times \Gamma_\delta$, that $\sigma_0^*(\Gamma_\delta) > 1 - \delta$. This then implies that $\sigma_0(\Gamma_\delta) > 1 - \delta$. The subsets are:

$$\begin{aligned}
F_1 &= \left(\phi_\delta\left(\left[\tfrac{58}{60}, 1\right]\right) \cup \phi_\delta\left(\left[0, \tfrac{3}{6}\right]\right)\right) \times \left\{\phi_\delta\left(\tfrac{5}{6}\right)\right\},\\
F_2 &= \left\{\phi_\delta\left(\tfrac{3}{6}\right)\right\} \times \left(\phi_\delta\left(\left[\tfrac{5}{6}, 1\right]\right) \cup \phi_\delta\left(\left[0, \tfrac{1}{6}\right]\right)\right),\\
F_3 &= \phi_\delta\left(\left[\tfrac{3}{6}, \tfrac{42}{60}\right]\right) \times \left\{\phi_\delta\left(\tfrac{1}{6}\right)\right\},\\
F_4 &= \left\{\phi_\delta\left(\tfrac{42}{60}\right)\right\} \times \phi_\delta\left(\left[0, \tfrac{1}{6}\right]\right),\\
F_5 &= \left\{\left(\phi_\delta\left(\tfrac{42+t}{60}\right), \phi_\delta\left(1 - \tfrac{t}{60}\right)\right) : t \in [0, 8]\right\},\\
F_6 &= \left\{\left(\phi_\delta\left(\tfrac{50+t}{60}\right), \phi_\delta\left(\tfrac{52+t}{60}\right)\right) : t \in [0, 8]\right\}.
\end{aligned}$$

Let $F = \bigcup_{i=1}^{6} F_i$. One can verify that the set $F$ is connected, $\pi_1(F) = \Gamma_\delta$, and $d(x_1, x_2) > 1 - \delta$ for all $(x_1, x_2) \in F$.

To show that $\sigma_0^*(\Gamma_\delta) < 1/4 + \delta$, we define the following subsets of $\Gamma_\delta \times \Gamma_\delta$:

$$\begin{aligned}
N_1 &= \{(\beta_1'(t), \alpha_1'(t)) : t \in [0, 1]\},\\
N_2 &= \beta_2'([0, 1]) \times \{\alpha_2'(0)\},\\
N_3 &= \beta_3'\left(\left[0, \tfrac{3}{4}\right]\right) \times \{\alpha_2'(0)\},\\
N_4 &= \left\{\left(\beta_3'\left(\tfrac{3}{4} + \tfrac{1}{4}t\right), \alpha_2'\left(\cos\left[\left(\tfrac{3}{2} + \tfrac{1}{2}\right)t\pi\right]\right)\right) : t \in [0, 1]\right\},\\
N_5 &= \left\{\left(\beta_4'\left(\tfrac{1}{2}t\right), \alpha_3'(t)\right) : t \in [0, 1]\right\},\\
N_6 &= \left\{\left(\beta_4'\left(\tfrac{1}{2} + \tfrac{1}{2}t\right), \alpha_4'(t)\right) : t \in [0, 1]\right\},\\
N_7 &= \left\{\left(\beta_5'\left(\tfrac{1}{2}t\right), \alpha_5'(t)\right) : t \in [0, 1]\right\},\\
N_8 &= \left\{\left(\beta_5'\left(\tfrac{1}{2} + \tfrac{1}{2}t\right), \alpha_6'(t)\right) : t \in [0, 1]\right\},\\
N_9 &= \left\{\left(\beta_6'(t), \alpha_7'\left(\tfrac{4}{10}t\right)\right) : t \in [0, 1]\right\},\\
N_{10} &= \left\{\left(\beta_7'(t), \alpha_7'\left(\tfrac{4}{10} + \tfrac{2}{10}t\right)\right) : t \in [0, 1]\right\},\\
N_{11} &= \left\{\left(\beta_8'(t), \alpha_7'\left(\tfrac{6}{10} - \tfrac{2}{10}t\right)\right) : t \in [0, 1]\right\},\\
N_{12} &= \left\{\left(\beta_9'(t), \alpha_7'\left(\tfrac{4}{10} - \tfrac{4}{10}t\right)\right) : t \in [0, 1]\right\},
\end{aligned}$$

$$N_{13} = \big\{\big(\beta'_{10}\big(\tfrac{1}{2}t\big), \alpha'_6(1-t)\big) : t \in [0,1]\big\},$$
$$N_{14} = \big\{\big(\beta'_{10}\big(\tfrac{1}{2}+\tfrac{1}{2}t\big), \alpha'_5(1-t)\big) : t \in [0,1]\big\},$$
$$N_{15} = \big\{\big(\beta'_{11}\big(\tfrac{1}{2}t\big), \alpha'_4(1-t)\big) : t \in [0,1]\big\},$$
$$N_{16} = \big\{\big(\beta'_{11}\big(\tfrac{1}{2}+\tfrac{1}{2}t\big), \alpha'_3(1-t)\big) : t \in [0,1]\big\},$$
$$N_{17} = \big\{\big(\beta'_{12}\big(\tfrac{1}{4}t\big), \alpha'_2\big(\cos\big[\big(\tfrac{3}{2}+\tfrac{1}{2}\big)(1-t)\pi\big]\big)\big) : t \in [0,1]\big\},$$
$$N_{18} = \beta'_{12}\big(\big[\tfrac{1}{4},1\big]\big) \times \{\alpha'_2(0)\},$$
$$N_{19} = \big\{\beta'_{12}(1)\big\} \times \alpha'_2([0,1]),$$
$$N_{20} = \{(\beta'_{13}(t), \alpha'_3(t)) : t \in [0,1]\},$$
$$N_{21} = \{(\beta'_{14}(t), \alpha'_4(t)) : t \in [0,1]\},$$
$$N_{22} = \{(\beta'_{14}(1-t), \alpha'_5(t)) : t \in [0,1]\},$$
$$N_{23} = \{(\beta'_{13}(1-t), \alpha'_6(t)) : t \in [0,1]\},$$
$$N_{24} = \big\{\big(\beta'_{12}(1-t), \alpha'_7\big(\tfrac{4}{10}t\big)\big) : t \in [0,1]\big\},$$
$$N_{25} = \beta'_{11}([0,1]) \times \big\{\alpha'_7\big(\tfrac{4}{10}\big)\big\},$$
$$N_{26} = \beta'_{10}([0,1]) \times \big\{\alpha'_7\big(\tfrac{4}{10}\big)\big\},$$
$$N_{27} = \big\{\big(\beta'_9(1-t), \alpha'_7\big(\tfrac{4}{10}+\tfrac{4}{10}t\big)\big) : t \in [0,1]\big\},$$
$$N_{28} = \big\{\big(\beta'_8(1-t), \alpha'_7\big(\tfrac{8}{10}+\tfrac{2}{10}t\big)\big) : t \in [0,1]\big\},$$
$$N_{29} = \big\{\big(\beta'_7(1-t), \alpha'_8\big(\tfrac{2}{10}t\big)\big) : t \in [0,1]\big\},$$
$$N_{30} = \big\{\big(\beta'_6(1-t), \alpha'_8\big(\tfrac{2}{10}+\tfrac{4}{10}t\big)\big) : t \in [0,1]\big\},$$
$$N_{31} = \beta'_5([0,1]) \times \big\{\alpha'_8\big(\tfrac{6}{10}\big)\big\},$$
$$N_{32} = \beta'_4([0,1]) \times \big\{\alpha'_8\big(\tfrac{6}{10}\big)\big\},$$
$$N_{33} = \big\{\big(\beta'_3(1-t), \alpha'_8\big(\tfrac{6}{10}+\tfrac{4}{10}t\big)\big) : t \in [0,1]\big\},$$
$$N_{34} = \{(\beta'_2(1-t), \alpha'_9(t)) : t \in [0,1]\},$$
$$N_{35} = \{(\beta'_1(1-t), \alpha'_{10}(t)) : t \in [0,1]\},$$
$$N_{36} = \{(\beta'_i(t), \beta'_i(t)) : t \in [0,1],\ i = 1,\dots,14\}.$$

Let $N = \bigcup_{i=1}^{36} N_i$. Figure 9 depicts the set $\Gamma_\delta \times \Gamma_\delta$ as a subset of the plane in a manner similar to that described in Section 4, with the set $N$ drawn in with dashed line. Essentially, this picture is obtained as follows: starting with the set $P$ (which is homeomorphic to $\Gamma_\delta$), we pluck one of the ends of the circle away from the point $v$, so that we are left with an arc. We then "unroll and flatten" this arc to view it as a subset of the real line $\mathbb{R}$. This allows us to view $P \times P$, and hence $\Gamma_\delta \times \Gamma_\delta$, as a subset of the plane $\mathbb{R}^2$.

One can check that $d(x_1, x_2) < 1/4 + \delta$ for all $(x_1, x_2) \in N$. One may also verify (see Figure 9) that $(\Gamma_\delta \times \Gamma_\delta) \setminus N$ consists of two components, neither

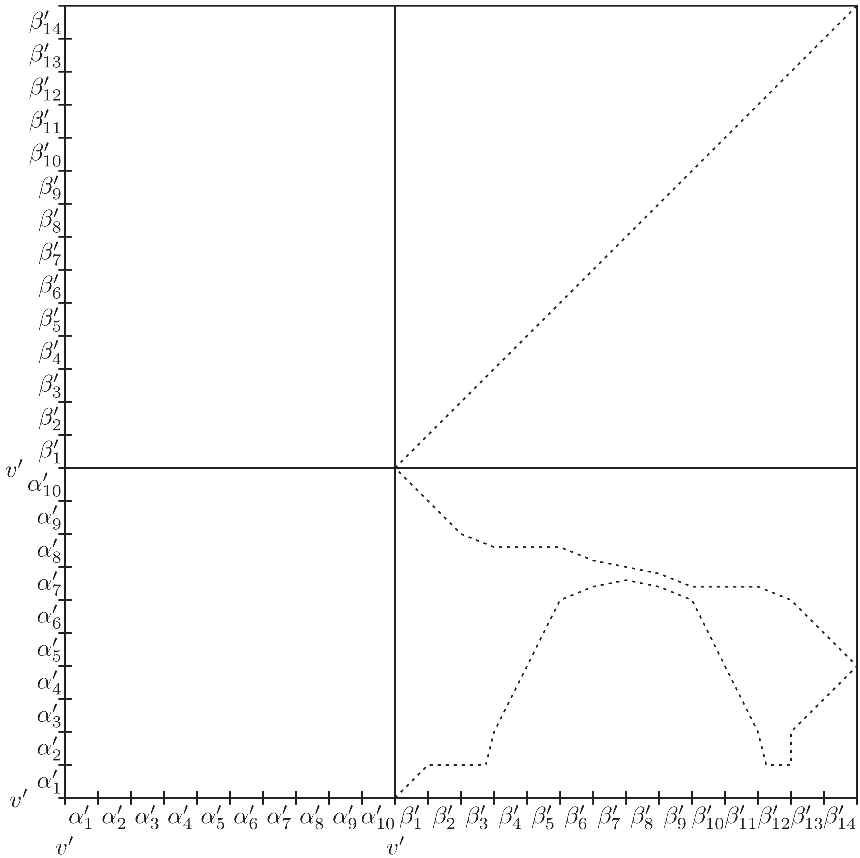


Fig. 9. The space $\Gamma_\delta \times \Gamma_\delta$ with the subset $N$ depicted with dashed line.

of which has first coordinate projection equal to $\Gamma_\delta$. This implies that if $Z$ is a connected subset of $\Gamma_\delta \times \Gamma_\delta$ with $\pi_1(Z) = \Gamma_\delta$, then $Z$ must meet $N$. It follows that $\sigma_0^*(\Gamma_\delta) \leq 1/4 + \delta$. ∎

## *REFERENCES*


[1] M. Barge and R. Swanson, *The essential span of a closed curve*, in: Continua (Cincinnati, OH, 1994), with the Houston Problem Book, H. Cook *et al.* (eds.), Lecture Notes in Pure Appl. Math. 170, Dekker, New York, 1995, 183–192.

[2] H. Cook, W. T. Ingram, and A. Lelek, *A list of problems known as Houston Problem Book*, ibid., 365–398.

[3] J. F. Davis, *The equivalence of zero span and zero semispan*, Proc. Amer. Math. Soc. 90 (1984), 133–138.

[4] A. Lelek, *Disjoint mappings and the span of spaces*, Fund. Math. 55 (1964), 199–214.

[5] —, *An example of a simple triod with surjective span smaller than span*, Pacific J. Math. 64 (1976), 207–215.

[6] A. Lelek, *On the surjective span and semispan of connected metric spaces*, Colloq. Math. 37 (1977), 35–45.
[7] J. Luukkainen, *Extension of locally uniformly equivalent metrics*, ibid. 46 (1982), 205–207.
[8] A. I. Vasil'ev, *Introducing an equivalent metric in a linear space by a family of its subsets*, Proc. Steklov Inst. Math. 2001, Suppl. 1, S235–S242.

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