ON THE IRREDUCIBILITY OF
0,1-POLYNOMIALS OF THE FORM f(x)x^n + g(x)

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Abstract. If f(x) and g(x) are relatively prime polynomials in \( \mathbb{Z}[x] \) satisfying certain conditions arising from a theorem of Capelli and if \( n \) is an integer > \( N \) for some sufficiently large \( N \), then the non-reciprocal part of \( f(x)x^n + g(x) \) is either identically ±1 or is irreducible over the rationals. This result follows from work of Schinzel in 1965. We show here that under the conditions that f(x) and g(x) are relatively prime 0,1-polynomials (so each coefficient is either 0 or 1) and \( f(0) = g(0) = 1 \), one can take \( N = \deg g + 2 \max\{\deg f, \deg g\} \).

1. Introduction. For \( f(x) \in \mathbb{C}[x] \) with \( f(x) \neq 0 \), we define \( \tilde{f}(x) = x^{\deg f}(1/x) \). The polynomial \( \tilde{f} \) is called the reciprocal of \( f(x) \). The constant term of \( \tilde{f} \) is always non-zero. If the constant term of \( f \) is non-zero, then \( \deg \tilde{f} = \deg f \) and the reciprocal of \( \tilde{f} \) is \( f \). If \( \alpha \neq 0 \) is a root of \( f \), then \( 1/\alpha \) is a root of \( f \). If \( f(x) = g(x)h(x) \) with \( g(x) \) and \( h(x) \) in \( \mathbb{C}[x] \), then \( \tilde{f} = \tilde{g}\tilde{h} \). If \( f = \pm \tilde{f} \), then \( f \) is called reciprocal. If \( f \) is not reciprocal, we say that \( f \) is non-reciprocal. If \( f \) is reciprocal and \( \alpha \) is a root of \( f \), then \( 1/\alpha \) is a root of \( f \).

The product of reciprocal polynomials is reciprocal so that a non-reciprocal polynomial must have a non-reciprocal irreducible factor. For \( f(x) \in \mathbb{Z}[x] \), we refer to the non-reciprocal part of \( f(x) \) as the polynomial \( f(x) \) removed of its irreducible reciprocal factors in \( \mathbb{Z}[x] \) having a positive leading coefficient.

For example, the non-reciprocal part of \( 3(-x+1)x(x^2+2) \) is \( -x(x^2+2) \) (the irreducible reciprocal factors 3 and \( x-1 \) have been removed from the polynomial \( 3(-x+1)x(x^2+2) \)).

In [2], Filaseta, Ford, and Konyagin established the following result.

THEOREM 1. Let \( f(x) \) and \( g(x) \) be in \( \mathbb{Z}[x] \) with \( f(0) \neq 0 \), \( g(0) \neq 0 \), and \( \gcd_Z(f(x), g(x)) = 1 \). Let \( r_1 \) and \( r_2 \) denote the number of non-zero terms in \( f(x) \) and \( g(x) \), respectively. If \( n \geq n_0 \), where
\[
n_0 = n_0(f, g) = \max\{2 \times 5^{2N-1}, 2 \max\{\deg f, \deg g\}(5^{N-1} + 1/4)\}
\]

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and

\[ N = 2 \|f\|^2 + 2 \|g\|^2 + 2r_1 + 2r_2 - 7, \]

then the non-reciprocal part of \( f(x)x^n + g(x) \) is irreducible or identically 1 or \(-1\) unless one of the following holds:

(i) The polynomial \(-f(x)g(x)\) is a \(p\)th power for some prime \(p\) dividing \(n\).

(ii) For either \(\varepsilon = 1\) or \(\varepsilon = -1\), one of \(\varepsilon f(x)\) and \(\varepsilon g(x)\) is a 4th power, the other is 4 times a 4th power, and \(n\) is divisible by 4.

The work in [2] was motivated by work of Schinzel [3, 4] where a similar result is obtained without an explicit estimate on \(n_0\) (though the methods there do allow for such an estimate).

Theorem 1 is an assertion about the irreducibility of the non-reciprocal part of \(F(x) = f(x)x^n + g(x)\). If the non-reciprocal part of \(F(x)\) is irreducible and \(\gcd(F, \tilde{F}) = 1\), then \(F(x)\) is irreducible. Thus, the above result can be combined with an analysis of \(\gcd(F, \tilde{F})\) to determine information about the irreducibility of \(F(x)\).

We remark that the bound \(n_0\) cannot be replaced by a bound that is independent of the size of the coefficients of \(f\) and \(g\). To see this, consider an arbitrary integer \(k > 1\) and observe that \(f(x) = 1\) and \(g(x) = x - 2^k - 2\) imply that the non-reciprocal part of \(F(x) = f(x)x^n + g(x)\) is reducible for \(n = k\) (since \(x - 2\) is a factor of \(F(x)\) and the quotient \(F(x)/(x - 2)\) is non-reciprocal). Since \(k\) is arbitrary, the remark follows.

In this paper, we obtain a result similar to Theorem 1 but restricted to 0,1-polynomials \(f(x)\) and \(g(x)\), that is, polynomials \(f(x)\) and \(g(x)\) with each coefficient either 0 or 1. In this case, it is not difficult to check that neither (i) nor (ii) can hold.

**Theorem 2.** Let \(f(x)\) and \(g(x)\) be relatively prime 0,1-polynomials with \(f(0) = g(0) = 1\). If

\[ n > \deg g + 2 \max\{\deg f, \deg g\}, \]

then the non-reciprocal part of \(f(x)x^n + g(x)\) is irreducible or identically 1.

An interesting aspect of the proof given here is that Theorem 1, even without an explicit value for \(n_0\), will play a crucial role in establishing the bound given in Theorem 2.

2. **Proof of Theorem 2.** To prove Theorem 2, we make use of the following result that can be found in [1].

**Lemma 1.** Let \(f(x)\) be a 0,1-polynomial with \(f(0) = 1\). Then the non-reciprocal part of \(f(x)\) is reducible if and only if there exists \(w(x)\) satisfying
If \( w(x) \neq f(x), \ w(x) \neq \tilde{f}(x), \ w\tilde{w} = f\tilde{f}, \) and \( w(x) \) is a 0,1-polynomial with the same number of non-zero terms as \( f(x) \).

Assume (1) holds for some integer \( n \) and that the non-reciprocal part of \( f(x)x^n + g(x) \) is reducible. Let \( w(x) \) be the 0,1-polynomial that exists by Lemma 1 with \( f(x) \) replaced there by \( f(x)x^n + g(x) \). In particular,
\[
(2) \quad w(x) \neq f(x)x^n + g(x) \quad \text{and} \quad w(x) \neq \tilde{g}(x)x^{n+\deg f - \deg g} + \tilde{f}(x)
\]
and
\[
(3) \quad w(x)\tilde{w}(x) = (f(x)x^n + g(x))\tilde{g}(x)x^{n+\deg f - \deg g} + \tilde{f}(x)
\]
First, consider the case that \( \deg b \geq \deg g \). Write \( w(x) \) in the form \( a(x)x^n + b(x) \) where \( a(x) \) and \( b(x) \) are 0,1-polynomials with \( b(0) = 1 \) (by (3)) and \( \deg b(x) < n \). Also, (3) implies that \( \deg a(x) = \deg f(x) \) (so that \( w(x) \) and \( f(x)x^n + g(x) \) have the same degree). Applying (3) again, we obtain
\[
f(x)\tilde{g}(x)x^{2n+\deg f - \deg g} + f(x)\tilde{f}(x)x^n + g(x)\tilde{g}(x)x^{n+\deg f - \deg g} + \tilde{f}(x)g(x)
\]
\[
= (f(x)x^n + g(x))\tilde{g}(x)x^{n+\deg f - \deg g} + \tilde{f}(x)
\]
\[
= (a(x)x^n + b(x))\tilde{b}(x)x^{n+\deg a - \deg b} + \tilde{a}(x)
\]
\[
= a(x)\tilde{b}(x)x^{2n+\deg a - \deg b} + a(x)\tilde{a}(x)x^n + b(x)\tilde{b}(x)x^{n+\deg a - \deg b} + \tilde{a}(x)b(x).
\]
The significance of working with 0,1-polynomials here is that there is no cancellation of terms above. In particular, the expression \( \tilde{a}(x)b(x) \) on the right contains a term with degree equal to \( \deg b(x) \), which is \( < n \), and every term of degree \( < n \) on the left also has degree \( \leq \deg f + \deg g \). Hence, \( \deg b(x) \leq \deg f + \deg g \).

We now consider the case that \( \deg f < \deg g \). The somewhat disguised idea will be to work instead with the reciprocal of \( f(x)x^n + g(x) \) and proceed as in the case of \( \deg f \geq \deg g \). For this purpose, we define \( k = n + \deg f - \deg g \) and write \( w(x) \) in the form \( a(x)x^k + b(x) \) where now \( a(x) \) and \( b(x) \) are 0,1-polynomials with \( b(0) = 1 \), \( \deg b(x) < k \), and \( \deg a(x) = n + \deg f - k = \deg g \). Instead of the equations above, we use
\[
f(x)\tilde{g}(x)x^{2k+\deg g - \deg f} + f(x)\tilde{f}(x)x^k + g(x)\tilde{g}(x)x^k + \tilde{f}(x)g(x)
\]
\[
= f(x)\tilde{g}(x)x^{2n+\deg f - \deg g} + f(x)\tilde{f}(x)x^n + g(x)\tilde{g}(x)x^{n+\deg f - \deg g} + \tilde{f}(x)g(x)
\]
\[
= (f(x)x^n + g(x))\tilde{g}(x)x^{n+\deg f - \deg g} + \tilde{f}(x)
\]
\[
= (a(x)x^k + b(x))\tilde{b}(x)x^{k+\deg a - \deg b} + \tilde{a}(x)
\]
\[
= a(x)\tilde{b}(x)x^{2k+\deg a - \deg b} + a(x)\tilde{a}(x)x^k + b(x)\tilde{b}(x)x^{k+\deg a - \deg b} + \tilde{a}(x)b(x).
\]
Arguing as before, a term of degree \( \deg b(x) \) appears on the right and the only terms of degree \( < k \) on the left have degree \( \leq \deg f + \deg g \), so \( \deg b(x) \leq \deg f + \deg g \).
Thus, in both of the cases $\deg f \geq \deg g$ and $\deg f < \deg g$, we deduce that $w(x)$ is of the form $a(x)x^n + b(x)$ where $\deg b(x) \leq \deg f + \deg g$ and where either $m = n$ and $\deg a = \deg f$ or $m = n + \deg f - \deg g$ and $\deg a = \deg g$. In both cases, $m + \deg a = n + \deg f$. Inequality (1) implies that the product $\tilde{a}(x)b(x)$ consists of terms of degree $< m$ (for either choice of $m$) and, hence, corresponds to terms in $f(x)g(x)$ on the left-hand sides above of degree $\leq \deg f + \deg g$. Therefore, $\tilde{a}(x)b(x)$ has degree $\leq \deg f + \deg g$. From (1), we deduce that each of the exponents $m$ and $m + \deg a - \deg b$ is $> \deg f + \deg g$. It follows that

$$\tilde{f}(x)g(x) = \tilde{a}(x)b(x).$$

The possibility that $a(0) = 0$ exists. We consider a non-negative integer $l$ such that $a(x) = a_0(x)x^l$ where $a_0(x)$ is a 0, 1-polynomial with $a_0(0) = 1$. Then $\tilde{a} = \tilde{a}_0$ and $\deg a = l + \deg \tilde{a}$. Since $\tilde{a}(x)b(x)$ has degree $\deg f + \deg g$, we have $\deg a - l + \deg b = \deg f + \deg g$ so that $\deg b = l - \deg a + \deg f + \deg g$. We use this to make further comparisons of exponents. For example, to see that the terms in $a(x)\tilde{b}(x)x^{2m+\deg a-\deg b}$ have degrees exceeding the degrees of the terms in $b(x)\tilde{b}(x)x^{m+\deg a-\deg b}$, we can justify instead that

$$m + l > 2(l - \deg a + \deg f + \deg g).$$

For the latter, we want $m > l + 2(\deg f + \deg g - \deg a)$, which follows from (1). By comparing coefficients in this manner, we deduce

$$f(x)\tilde{g}(x)x^{2n+\deg f-\deg g} = a(x)\tilde{b}(x)x^{2m+\deg a-\deg b}$$

and, consequently,

$$f(x)\tilde{f}(x)x^n + g(x)\tilde{g}(x)x^{n+\deg f-\deg g} = a(x)\tilde{a}(x)x^m + b(x)\tilde{b}(x)x^{m+\deg a-\deg b}.$$

Recall that $n$ is a fixed integer satisfying (1) for which the non-reciprocal part of $f(x)x^n + g(x)$ is reducible. We now consider an arbitrary positive integer $n'$ satisfying (1) and set $m' = n'$ if $\deg f \geq \deg g$ and $m' = n' + \deg f - \deg g$ if $\deg f < \deg g$. Thus, if $n' = n$, then $m' = m$. We use the polynomials $a(x)$ and $b(x)$ constructed above (corresponding to the case $n' = n$). Multiplying both sides of the equations above by a suitable power of $x$, we obtain

$$f(x)\tilde{g}(x)x^{2n'+\deg f-\deg g} = a(x)\tilde{b}(x)x^{2m'+\deg a-\deg b}$$

and

$$f(x)\tilde{f}(x)x^{n'} + g(x)\tilde{g}(x)x^{n'+\deg f-\deg g} = a(x)\tilde{a}(x)x^m + b(x)\tilde{b}(x)x^{m'+\deg a-\deg b}.$$

Hence,

$$(f(x)x^{n'} + g(x))\tilde{g}(x)x^{n'+\deg f-\deg g} + \tilde{f}(x))$$

$$= f(x)\tilde{g}(x)x^{2n'+\deg f-\deg g} + f(x)\tilde{f}(x)x^{n'} + g(x)\tilde{g}(x)x^{n'+\deg f-\deg g} + \tilde{f}(x)g(x).$$
We consider \( n' \) sufficiently large with at least \( n_0(f, g) \), where \( n_0(f, g) \) is defined in Theorem 1. Since \( F(x) = f(x)x^{n'} + g(x) \) is a 0, 1-polynomial, we deduce from Theorem 1 that the non-reciprocal part of \( F(x) \) is irreducible or identically 1. On the other hand, the polynomial \( W(x) = a(x)x^{m'} + b(x) \) satisfies \( WW = FF \) and \( W(x) \) is a 0, 1-polynomial containing the same number of non-zero terms as \( F(x) \). By Lemma 1, either \( W(x) = F(x) \) or \( W(x) = \tilde{F}(x) \). If \( W(x) = F(x) \), then
\[
a(x)x^{m'} + b(x) = f(x)x^{n'} + g(x).
\]
If \( m' = n' \), then \( a(x) = f(x) \) and \( b(x) = g(x) \), contradicting (2). If \( m' \neq n' \), then \( m' = n' + \deg f - \deg g, \deg g > \deg f, a(x) = f(x)x^{\deg g - \deg f} \), and \( b(x) = g(x) \), contradicting (2). Similarly, \( W(x) = \tilde{F}(x) \) leads to a contradiction to (2). It follows that our assumption that \( n \) exists satisfying (1) and such that the non-reciprocal part of \( f(x)x^n + g(x) \) is reducible is incorrect. The theorem follows.

REFERENCES


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