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GOOD-λ INEQUALITIES FOR WAVELETS OF COMPACT SUPPORT

BY

SARAH V. COOK (Topeka, KS)

Abstract. For a wavelet ψ of compact support, we define a square function $S_{\rm w}$ and a maximal function $N\Lambda$. We then obtain the L_p equivalence of these functions for $0 < p < \infty$. We show this equivalence by using good-λ inequalities.

1. Introduction. In 1970, Burkholder and Gundy [4] showed, among other results, that if for a martingale $f = (f_n)$ the square function Sf and maximal function f^* are given by

$$Sf(x) = \Big(\sum_{n=1}^{\infty}(f_n(x) - f_{n-1}(x))^2\Big)^{1/2}, \qquad f^*(x) = \sup_n |f_n(x)|,$$

then $\|Sf\|_p \approx \|f^*\|_p$ for $0 < p < \infty$. Previously this result was known only for $1 < p < \infty$.

Burkholder and Gundy proved their results by first showing that Sf controls f^* and conversely f^* controls Sf by what is now commonly known as a good-λ inequality.

DEFINITION 1. For positive measurable functions f and g, we say that *g controls f by a good-λ inequality* if there exist constants $K > 1$, $0 < \varepsilon_0 \le 1$ and a function $C(\varepsilon)$, with $C(\varepsilon) \to 0$ as $\varepsilon \to 0$, such that for all $\lambda > 0$ and $0 < \varepsilon < \varepsilon_0$ we have

$$|\{x \in \mathbb{R} : f(x) > K\lambda,\ g(x) < \varepsilon\lambda\}| \le C(\varepsilon)|\{x \in \mathbb{R} : f(x) > \lambda\}|.$$

Burkholder ([2], [3]) later refined these results and in particular gave in [2] the following lemma, which demonstrates the usefulness of a good-λ inequality.

LEMMA 1. *Consider a non-decreasing continuous function Φ defined on $[0,\infty)$ such that $\Phi(0) = 0$ and Φ is not identically zero. Suppose Φ satisfies $\Phi(2\lambda) \le C\Phi(\lambda)$ for all $\lambda > 0$ and for some fixed constant C depending only on Φ. Suppose also that g controls f by a good-λ inequality. For a fixed ε, $0 < \varepsilon < \varepsilon_0$, there exist real numbers ϱ and ν which satisfy*

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$\Phi(K\lambda) \le \varrho\Phi(\lambda)$ and $\Phi(\varepsilon^{-1}\lambda) \le \nu\Phi(\lambda)$ for every $\lambda > 0$. (The growth condition on Φ ensures the existence of ϱ and ν.) Finally, suppose $\varrho C(\varepsilon) < 1$ and $\int_{\mathbb{R}} \Phi(\min\{1, f(x)\})\, dx < \infty$. Then

$$\int_{\mathbb{R}} \Phi(f(x))\, dx \le \frac{\varrho\nu}{1 - \varrho C(\varepsilon)} \int_{\mathbb{R}} \Phi(g(x))\, dx.$$

Burkholder and Gundy's results along with the fact that a martingale is essentially a Haar wavelet provide us with a reason to believe that good-λ inequalities exist for a more generalized wavelet. As further justification, we have the following theorem from Meyer [8] which shows the L_p equivalence of two square functions for an r-regular wavelet. In the statement of the theorem and throughout the paper, for a dyadic interval $Q = [k/2^n, (k+1)/2^n)$, let $\psi_Q(x) := \psi_{n,k}(x) = 2^{n/2}\psi(2^n x - k)$ be the standard dilation and translation of a wavelet ψ. Also, for each dyadic Q, let a_Q denote the corresponding wavelet coefficient.

THEOREM 1. *For ψ an r-regular wavelet and $1 < p < \infty$, the norms $\|(\sum_J |a_J|^2 |\psi_J(x)|^2)^{1/2}\|_p$ and $\|(\sum_J |a_J|^2 |J|^{-1} \mathcal{X}_J(x))^{1/2}\|_p$ are equivalent.*

We now make some definitions which will be used throughout the paper. Fix ψ to be a wavelet with compact support. Then there exists $M \in \mathbb{Z}$ such that if $\overline{Q}$ is the unique interval that has the same center as Q and length $|\overline{Q}| = 2^M|Q|$, then $\operatorname{supp}(\psi_Q) \subseteq \overline{Q}$. Fix such an M and for a dyadic Q, define $\overline{Q}$ in this manner. Similarly, define $\widehat{Q}$ to be the interval of $\mathbb{R}$ that has the same center as Q and length $|\widehat{Q}| = 2^{M+3}|Q|$. Further, let $Q_n(x)$ be the unique dyadic interval that contains x and has length $|Q_n(x)| = 2^{-n}$.

In this paper, we show the L_p equivalence, $0 < p < \infty$, of a maximal function and a square function for our wavelet by using good-λ inequalities. We define our maximal function, $N\Lambda$, by

$$N\Lambda(x) := \sup_n \sup_{y \in Q_n(x)} |\Lambda_n(y)|, \quad \text{where } \Lambda_n(x) := \sum_{|J| > 2^{-n}} a_J \psi_J(x).$$

We also define the square function, S_{w}, by

$$S_{\mathrm{w}}(x) := \left(\sum_{Q \subseteq \mathbb{R}} a_Q^2 \, \frac{1}{|Q|} \, \mathcal{X}_{\widehat{Q}}(x) \right)^{1/2}.$$

In Section 2, we show S_{w} controls $N\Lambda$ by a good-λ inequality. The proof of this inequality roughly follows the proof of the martingale case. In fact, we shall make use of the following theorem, which is a variation of that found in [4].

THEOREM 2. *There exists a constant $K > 1$, and constants C and c possibly depending on K, such that for $0 < \varepsilon < 1$, $\lambda > 0$ we have*

$$|\{x \in Q_0 : f^*(x) > K\lambda,\ Sf < \varepsilon\lambda\}| \le C \exp\left(\frac{-c}{\varepsilon^2}\right) |\{x \in Q_0 : f^*(x) > \lambda\}|.$$

In the third section, we will define a new maximal function $N_\alpha \Lambda$ and show $N_\alpha \Lambda$ controls S_{w}. We shall then estimate $N\Lambda$ by $N_\alpha \Lambda$. In that section we will use the theory of dyadic bounded mean oscillation to obtain our good-λ inequality. In particular, we will use the following corollary to the John–Nirenberg Theorem [9].

COROLLARY 1. *Suppose* $g \in \mathrm{BMO_d}$, $g \not\equiv 0$, *with* $\|g\|_{\mathrm{d}} \le C(\varepsilon\lambda)^2$ *for some* $0 < \varepsilon < 1$ *and* $\lambda > 0$. *Then there exist constants* $K > 1$, $c_1 > 0$, *and* $c_2 > 0$ *independent of* ε *and* λ *such that*

$$|\{x \in \mathbb{R} : |g(x)| > K\lambda^2\}| \le c_1 \exp\left(-\frac{c_2}{\varepsilon^2}\right)|\{x \in \mathbb{R} : |g(x)| > \lambda^2\}|.$$

2. Control of $N\Lambda$ by S_{w}. Our goal in this section is to prove the following good-λ inequality:

THEOREM 3. *There exist* $k > 1$, $0 < \varepsilon_0 \le 1$ *and constants* C *and* c *such that for* $0 < \varepsilon < \varepsilon_0$, $\lambda > 0$ *we have*

$$|\{x \in \mathbb{R} : N\Lambda(x) > k\lambda,\ S_{\mathrm{w}}(x) < \varepsilon\lambda\}| \le C \exp\left(-\frac{c}{\varepsilon^2}\right)|\{x \in \mathbb{R} : N\Lambda(x) > \lambda\}|.$$

To prove Theorem 3, we shall divide the dyadic intervals of $\mathbb{R}$ into a finite number of sets and examine the square function indexed over these sets. To this end, we make use of the following which is a slight variation of a lemma found in [6].

LEMMA 2. *Let* F *denote the set of all dyadic intervals of* $\mathbb{R}$ *and for* $m \in \mathbb{Z}$ *let* $F_m = \{Q \in F : |Q| = 2^{-m}\}$. *For* $x \in \mathbb{R}$, *set* $F^x = \{x + Q : Q \in F\}$ *and* $F_m^x = \{x + Q : Q \in F_m\}$. *For a dyadic interval* Q_0, *there exist* $N \in \mathbb{N}$, $\{x_j\}_{j=1}^N \subseteq \mathbb{R}$, *and disjoint subsets* $(B^j)_{j=1}^N$ *of* F *such that*

$$\{Q \in F : Q \subseteq Q_0\} = \bigcup_{j=1}^N B^j.$$

Furthermore, if $Q \in B^j$, *then we have* $\overline{Q} \subseteq Q'$ *for a unique* $Q' \in F^{x_j}$ *with* $|Q'| = 2^{M+2}|Q|$. *Also, if* $Q_1, Q_2 \in B^j$ *and* $Q_1 \ne Q_2$, *then* $Q_1' \ne Q_2{}'$.

Of importance here is the fact that for any dyadic Q_0, we have subsets B^j, $j \in \{1, \ldots, N\}$, where N depends only on M. For simplification, we shall assume in what follows that Q_0 is the unit dyadic interval $[0, 1)$. Similar results for an arbitrary dyadic Q_0 also hold and we shall be free to use this later on.

Fix $j \in \{1, \ldots, N\}$, where N is as in Lemma 2. Continuing with the notation from the lemma, we re-index the wavelet coefficients and functions by

$$c_{Q'}^{(j)} := a_Q, \qquad \omega_{Q'}^{(j)}(x) := \psi_Q(x).$$

Similarly, we also subdivide the function Λ_n by defining

$$\Lambda^{(j)}_{Q_0,m}(x) := \sum_{\substack{Q\in B^j\\ |Q|>2^{-m}}} a_Q\psi_Q(x) = \sum_{\substack{Q'\in G^{x_j}\\ |Q'|>2^{-m+M+2}}} c^{(j)}_{Q'}\omega^{(j)}_{Q'}(x),$$

for $m \geq 0$ and where $G^{x_j} := \{Q' \in F^{x_j} : Q \subseteq Q_0,\, Q \in B^j\}$.

Note that we now have $\operatorname{supp}\omega^{(j)}_{Q'} \subseteq Q'$ and $\int_{Q'} \omega^{(j)}_{Q'}(x)\,dx = 0$. Thus, if we set $f^{(j)}_{Q_0,m} := E(\Lambda^{(j)}_{Q_0,m} \mid \mathcal{G}^{x_j}_m)$ where $\mathcal{G}^{x_j}_m$ is the σ-field generated by intervals in $F^{x_j}_{m-M+2}$, then $f^{(j)}_{Q_0} = (f^{(j)}_{Q_0,m})$ is a martingale with $f^{(j)}_{Q_0,0} \equiv 0$. For each martingale in this indexed collection, we denote the martingale maximal function and square function as $(f^{(j)}_{Q_0})^*(x)$ and $Sf^{(j)}_{Q_0}(x)$ respectively.

We wish to estimate the wavelet square function by the martingale square functions and apply Theorem 2. Since we have a collection of indexed martingales, it is necessary to define an indexed set of functions similar to the wavelet square function by

$$S^{(j)}_{Q_0,\mathrm{w}}(x) := \Big(\sum_{Q'\in G^{x_j}} (c^{(j)}_{Q'})^2\,\frac{1}{|Q'|}\,\mathcal{X}_{Q'}(x)\Big)^{1/2}.$$

The following lemmas relate the wavelet square function to the martingale. Lemma 3 is due to Bañuelos and Moore [1].

Lemma 3. *There exist c_1 and c_2 depending only on ψ such that*

$$(Sf^{(j)}_{Q_0}(x))^2 \leq c_1(S^{(j)}_{Q_0,\mathrm{w}}(x))^2 \tag{1}$$

$$|f^{(j)}_{Q_0,k}(x) - \Lambda^{(j)}_{Q_0,k}(x)|^2 \leq c_2(S^{(j)}_{Q_0,\mathrm{w}}(x))^2. \tag{2}$$

Lemma 4. *Fix Q_n to be a dyadic interval with length 2^{-n}. There exists a constant c_3, depending only on ψ, such that if $x, y \in Q_n$ then*

$$\Big|\sum_{j=1}^{N}(\Lambda^{(j)}_{Q_0,n}(y) - \Lambda^{(j)}_{Q_0,n}(x))\Big| \leq c_3 S_{\mathrm{w}}(x). \tag{3}$$

Proof. We have

$$\Big|\sum_{j=1}^{N}(\Lambda^{(j)}_{Q_0,n}(y) - \Lambda^{(j)}_{Q_0,n}(x))\Big| = \Big|\sum_{\substack{J\subseteq Q_0\\ |J|>2^{-n}}} a_J(\psi_J(y) - \psi_J(x))\Big|.$$

Since $x, y \in Q_n$ it follows that if either x or y is in $\operatorname{supp}\psi_J$ for some dyadic interval J with $|J| > 2^{-n}$, then both $x, y \in \overline{J}$. Hence we may approximate the above by

$$\Big|\sum_{\substack{J\subseteq Q_0\\ |J|>2^{-n}}} a_J(\psi_J(y) - \psi_J(x))\Big| \leq \sum_{\substack{J\subseteq Q_0\\ |J|>2^{-n}}} |a_J| 2^{-n}\|\psi'_J\|_\infty \mathcal{X}_{\overline{J}}(x)$$

$$\leq \sum_{|J|>2^{-n}} |a_J| 2^{-n} \|\psi'\|_\infty |J|^{-3/2} \mathcal{X}_{\bar{J}}(x)$$

$$\leq \|\psi'\|_\infty \Big(\sum_{|J|>2^{-n}} |a_J|^2 |J|^{-1} \mathcal{X}_{\bar{J}}(x) \Big)^{1/2} \Big(\sum_{|J|>2^{-n}} 2^{-2n} |J|^{-2} \mathcal{X}_{\bar{J}}(x) \Big)^{1/2}$$

$$\leq 2\|\psi'\|_\infty S_{\mathrm{w}}(x).$$

To prove Theorem 3, we must obtain constants k and ε_0. We shall explain how these constants are obtained later. In what follows, we simply assume $k > 1$, $\varepsilon_0 \leq 1$, and $\varepsilon < \varepsilon_0$.

Choose a maximal dyadic $Q \subseteq \mathbb{R}$ with $|\{x \in Q : N\Lambda(x) > \lambda\}| > \frac{1}{4}|Q|$. By maximality of Q, there exists $x_0 \in Q$ such that $N\Lambda(x_0) \leq \lambda$. In particular, for $y \in Q$ we have

$$\Big| \sum_{|J|>|Q|} a_J \psi_J(y) \Big| \leq \lambda. \tag{4}$$

Fix $x_Q \in Q$ such that $N\Lambda(x_Q) > k\lambda$ and $S_{\mathrm{w}}(x_Q) < \varepsilon\lambda$. Say Q has length $|Q| = 2^{-m}$. Note that since $x_0, x_Q \in Q$, it follows that for every n with $n < m$, both x_0 and x_Q must be in the same dyadic interval of length 2^{-n}. Thus, if $\sup_{y \in Q_n(x_Q)} |\Lambda_n(y)| > k\lambda$, then $\sup_{y \in Q_n(x_0)} |\Lambda_n(y)| > k\lambda$, which implies $N\Lambda(x_0) > k\lambda$. But by choice of x_0, $N\Lambda(x_0) \leq \lambda < k\lambda$. Hence, for $N\Lambda(x_Q)$, it suffices to take the supremum over only those n where $n \geq m$. Thus

$$\begin{aligned} N\Lambda(x_Q) &= \sup_{m \leq n} \sup_{y \in Q_n(x_Q)} |\Lambda_n(y)| \\ &\leq \sup_{m \leq n} \sup_{y \in Q_n(x_Q)} \Big| \sum_{|Q| \geq |J| > 2^{-n}} a_J \psi_J(y) \Big| + \Big| \sum_{|J|>|Q|} a_J \psi_J(y) \Big| \\ &\leq \lambda + \sup_{m \leq n} \sup_{y \in Q_n(x_Q)} \Big| \sum_{|Q| \geq |J| > 2^{-n}} a_J \psi_J(y) \Big| \end{aligned}$$

where the last inequality comes from (4).

Since $N\Lambda(x_Q) > k\lambda$, we then have

$$(k-1)\lambda \leq \sup_{m \leq n} \sup_{y \in Q_n(x_Q)} \Big| \sum_{|Q| \geq |J| > 2^{-n}} a_J \psi_J(y) \Big|.$$

By applying Lemma 2 to Q, there must exist $n_0 > m$ and $y_{n_0} \in Q_{n_0}(x_Q)$ where

$$(k-1)\lambda \leq \Big| \sum_{j=1}^{N} \Lambda^{(j)}_{Q,n_0}(y_{n_0}) \Big|$$

$$\leq \Big| \sum_{j=1}^{N} \Lambda^{(j)}_{Q,n_0}(x_Q) \Big| + \Big| \sum_{j=1}^{N} (\Lambda^{(j)}_{Q,n_0}(y_{n_0}) - \Lambda^{(j)}_{Q,n_0}(x_Q)) \Big|. \tag{5}$$

It is clear that $\sum_{j=1}^N (S_{Q,w}^{(j)}(x_Q))^2 \le 2^{-M}(S_{\mathrm{w}}(x_Q))^2$. Using this with (2) and $S_{\mathrm{w}}(x_Q) < \varepsilon\lambda$, we may estimate the first summation in (5) by

$$\sum_{j=1}^{N} |\Lambda_{Q,k_0}^{(j)}(x_Q)| \le c_\psi \varepsilon\lambda + \sum_{j=1}^{N} f_{Q,k_0}^{(j)}(x_Q). \tag{6}$$

Using (3) and (6) in inequality (5) we obtain

$$(k-1)\lambda \le c_\psi\varepsilon\lambda + \sum_{j=1}^{N} f_{Q,k_0}^{(j)}(x_Q) + c_3\varepsilon\lambda,$$

which implies $k^*\lambda \le \sum_{j=1}^N f_{Q,k_0}^{(j)}(x_Q) \le \sum_{j=1}^N (f_Q^{(j)})^*(x_Q)$. We then have

$$\begin{aligned} &|\{x\in Q : N\Lambda(x) > k\Lambda,\ S_{\mathrm{w}}(x) < \varepsilon\lambda\}| \\ &\qquad \le \sum_{j=1}^{N} \left|\left\{x \in Q : (f_Q^{(j)})^*(x) > \frac{k^*\lambda}{N},\ S_{Q,w}^{(j)}(x) < \frac{\varepsilon\lambda}{\sqrt{2^{M+2}}}\right\}\right| \\ &\qquad \le \sum_{j=1}^{N} \left|\left\{x \in Q : (f_Q^{(j)})^*(x) > \frac{k^*\lambda}{N},\ Sf_Q^{(j)}(x) < c_1\frac{\varepsilon\lambda}{\sqrt{2^{M+2}}}\right\}\right| \end{aligned}$$

where the last estimate is from (1). Now set k large enough and ε_0 small enough so that we may apply Theorem 2 to the above to obtain

$$|\{x\in Q : N\Lambda(x) > k\Lambda,\ S_{\mathrm{w}}(x) < \varepsilon\lambda\}| \le C\exp\left(-\frac{c}{\varepsilon^2}\right)|Q|.$$

Note that our constants will depend on N. However, recall that for all dyadic Q, N depends only on M. Summing over all such maximal dyadic Q yields Theorem 3.

3. Control of S_{w} by $N\Lambda$. Ideally, in this third section we would prove a version of Theorem 3 with the roles of $N\Lambda$ and S_{w} reversed. Unfortunately, we have not been able to obtain this inequality. However, we will obtain our desired L_p equivalence of S_{w} and $N\Lambda$ by showing S_{w} is controlled by a new maximal function, $N_\alpha\Lambda$, where we shall take the supremum over non-dyadic intervals.

Before we specifically define the maximal function $N_\alpha\Lambda$, we require a few definitions. Since ψ is of compact support, there is a smallest integer L such that for a dyadic interval $J = [k/2^j, (k+1)/2^j)$, we have $\widehat{J} \subseteq [(k-L)/2^j, (k+L)/2^j)$. Fix this L and for a dyadic interval J, define $\overline{\overline{J}} = [(k-L-1)/2^j, (k+L+2)/2^j)$. Thus $\operatorname{supp}\psi_J \subseteq \widehat{J} \subseteq \overline{\overline{J}}$.

For $\alpha > 0$, $n \in \mathbb{Z}$, and $x\in\mathbb{R}$, define $\Gamma_{n,\alpha}(x) := \{t\in\mathbb{R} : |t-x| < \alpha 2^{-n}\}$, the open interval of length $2^{-n+1}\alpha$ that has its center at x. We now define the maximal function $N_\alpha\Lambda$ by

$$N_\alpha \Lambda(x) := \sup_n \sup_{t \in \Gamma_{n,\alpha}(x)} |\Lambda_n(t)|.$$

As mentioned before, our intermediate goal is to prove $N_\alpha \Lambda$ controls S_{w} by a good-λ inequality. We record this as Theorem 4.

THEOREM 4. *There exist* $\alpha_0 > 0$, $K > 1$ *and constants* C, c *such that*

$$|\{x : S_{\mathrm{w}}(x) > K\lambda,\ N_\alpha \Lambda(x) < \varepsilon\lambda\}| \le C \exp\left(\frac{-c}{\varepsilon^2}\right) |\{x : S_{\mathrm{w}}(x) > \lambda\}|$$

for $0 < \varepsilon < 1$, $\lambda > 0$, $\alpha > \alpha_0$.

To prove this theorem, we shall define a function similar to S_{w} that is of dyadic bounded mean oscillation. To obtain our desired $\mathrm{BMO_d}$ function, set $E := \{x \in \mathbb{R} : N_\alpha \Lambda(x) \le \varepsilon\lambda\}$ for fixed $\lambda > 0$, $0 < \varepsilon < 1$, and define

$$\widetilde{S}_{\mathrm{w}}(x) := \Big(\sum_{\substack{J \subseteq \mathbb{R} \\ \widehat{J} \cap E \neq \emptyset}} a_J^2 \frac{1}{|J|} \mathcal{X}_{\widehat{J}}(x) \Big)^{1/2}.$$

Note that $S_{\mathrm{w}}(x) \ge \widetilde{S}_{\mathrm{w}}(x)$ for all $x \in \mathbb{R}$, and $S_{\mathrm{w}}(x) = \widetilde{S}_{\mathrm{w}}(x)$ for all $x \in E$. We shall show

PROPOSITION 1. $(\widetilde{S}_{\mathrm{w}})^2 \in \mathrm{BMO_d}$ *and* $\|(\widetilde{S}_{\mathrm{w}})^2\|_{\mathrm{d}} \le c_4(\varepsilon\lambda)^2$, *where* c_4 *depends only on* $\operatorname{supp}\psi$.

Fix an arbitrary dyadic interval Q_0, say $|Q_0| = 2^{-m_0}$. We define

$$\mathcal{T} := \{Q : |Q| > 2^{-m_0},\ \widehat{Q} \cap E \neq \emptyset\}, \quad \mathcal{B} := \{Q : |Q| \le 2^{-m_0},\ \widehat{Q} \cap E \neq \emptyset\}.$$

For fixed $x \in \mathbb{R}$, set

$$\widetilde{S}_{\mathrm{w}}^{\mathcal{T}}(x) := \Big(\sum_{Q \in \mathcal{T}} a_Q^2 \frac{1}{|Q|} \mathcal{X}_{\widehat{Q}}(x) \Big)^{1/2}, \quad \widetilde{S}_{\mathrm{w}}^{\mathcal{B}}(x) := \Big(\sum_{Q \in \mathcal{B}} a_Q^2 \frac{1}{|Q|} \mathcal{X}_{\widehat{Q}}(x) \Big)^{1/2}.$$

Note that $(\widetilde{S}_{\mathrm{w}}(x))^2 - (\widetilde{S}_{\mathrm{w}}^{\mathcal{T}}(x))^2 = (\widetilde{S}_{\mathrm{w}}^{\mathcal{B}}(x))^2$ and that $\widetilde{S}_{\mathrm{w}}^{\mathcal{T}}(x)$ is constant on Q_0. Thus,

$$\begin{aligned}
\int_{Q_0} ((\widetilde{S}_{\mathrm{w}}(x))^2 - (\widetilde{S}_{\mathrm{w}}^{\mathcal{T}}(x))^2)\, dx &= \int_{Q_0} (\widetilde{S}_{\mathrm{w}}^{\mathcal{B}}(x))^2\, dx \\
&= \int_{Q_0} \sum_{\substack{\widehat{Q} \cap E \neq \emptyset \\ |Q| \le |Q_0|}} a_Q^2 \frac{1}{|Q|} \mathcal{X}_{\widehat{Q}}(x)\, dx \\
&\le 2^{M+3} \sum_{\substack{\widehat{Q} \cap E \neq \emptyset,\ \widehat{Q} \cap Q_0 \neq \emptyset \\ |Q| \le |Q_0|}} a_Q^2 \le 2^{M+3} \sum_{Q \in \mathcal{G}} a_Q^2,
\end{aligned}$$

where $\mathcal{G} := \{Q : \overline{\overline{Q}} \cap E \neq \emptyset,\ \overline{\overline{Q}} \cap Q_0 \neq \emptyset,\ |Q| \le |Q_0|\}$.

By orthonormality, we have

$$\sum_{Q\in\mathcal{G}} a_Q^2 = \int_{\mathbb{R}} \sum_{Q\in\mathcal{G}} a_Q^2 |\psi_Q(x)|^2\,dx = \int_{\mathbb{R}} \Big|\sum_{Q\in\mathcal{G}} a_Q\psi_Q(x)\Big|^2\,dx. \tag{7}$$

To estimate (7), we first note that $\sum_{Q\in\mathcal{G}} a_Q\psi_Q$ has support of length $C|Q_0|$, where C depends only on M. To complete the estimate of (7), we need to make a pointwise estimate for the partial sum of wavelets associated to dyadics in $\mathcal{G}$. We will make this estimate by taking the summation over a larger collection of dyadic intervals. We remark that this procedure is similar to one used by Gundy and Iribarren [7]. Define

$$\begin{aligned}\mathcal{H} &:= \{Q : Q\cap \overline{\overline{J}} \neq \emptyset \text{ for some } J\in\mathcal{G}, |J| = |Q|\},\\ \mathcal{I} &:= \{Q : Q\cap \overline{\overline{J}} \neq \emptyset \text{ for some } J\in\mathcal{H}, |J| = |Q|\},\\ \mathcal{J} &:= \{Q : Q\cap \overline{\overline{J}} \neq \emptyset \text{ for some } J\in\mathcal{I}, |J| = |Q|\}.\end{aligned}$$

Note that clearly $\mathcal{G}\subseteq\mathcal{H}\subseteq\mathcal{I}\subseteq\mathcal{J}$.

We may now select α_0 large enough so that if $\alpha > \alpha_0$ and $Q\in\mathcal{J}$, $|Q| = 2^{-j}$, then $Q\subseteq \Gamma_{j,\alpha}(x_0)$ for some $x_0\in E$. We remark that α_0 depends only on M.

There are three properties of this collection of sets which we state now as propositions. The proof of Proposition 2 is clear from the definition of the relevant sets and is omitted.

Proposition 2. *For dyadic Q, let $\widetilde{Q}$ be the parent dyadic interval of Q. If $|Q| < |Q_0|$ and $Q\in\mathcal{G}$ (respectively $\mathcal{H},\mathcal{I},\mathcal{J}$) then $\widetilde{Q}\in\mathcal{G}$ (respectively $\mathcal{H},\mathcal{I},\mathcal{J}$).*

Proposition 3. *Let $Q\in\mathcal{J}$ with $|Q| = 2^{-j}$. For for all $t\in Q$, and all $l_1, l_2\in\mathbb{N}$ with $m_0\le l_1 < l_2\le j$, we have*

$$\Big|\sum_{2^{-l_1}\ge|J|>2^{-l_2}} a_J\psi_J(t)\Big| \le 2\varepsilon\lambda.$$

Proof. By choice of α_0, $Q\subseteq\Gamma_{j,\alpha}(x_0)$ for some $x_0\in E$. Proposition 2 implies that if J is the unique dyadic interval with $|J| = 2^{-l_2}$ and $Q\subseteq J$, then $J\in\mathcal{J}$. Thus, again by choice of α_0, we get $J\subseteq\Gamma_{l_2,\alpha}(x_0)$ and $|\Lambda_{l_2}(t)|\le\varepsilon\lambda$ for all $t\in J$.

Similar reasoning shows that if I is the unique dyadic interval of length 2^{-l_1+1} containing Q then $|\Lambda_{l_1-1}(t)|\le\varepsilon\lambda$ for all $t\in I$. The proposition then follows from the triangle inequality.

Our last proposition estimates the wavelet coefficient a_Q for $Q\subseteq\mathcal{I}$.

Proposition 4. *If $Q\in\mathcal{I}$ is such that $|Q| = 2^{-i}$, then there exists a constant C, depending only on ψ, such that $|a_Q|\le 2^{-i/2}C\varepsilon\lambda$.*

Proof. Fix such a Q and say $Q = [r/2^i, (r+1)/2^i)$. Let $t \in \overline{\overline{Q}}$ be arbitrary. There must exist some dyadic $I \subseteq \overline{\overline{Q}}$ with $t \in I$ and $|I| = |Q|$. Thus, since $Q \in \mathcal{I}$, we have $I \in \mathcal{J}$. By Proposition 3,

$$\Big|\sum_{J:|J|=|I|} a_J\psi_J(x)\Big| \le 2\varepsilon\lambda$$

for all $x \in I$. In particular, the above inequality holds for t.

Using the orthonormality of $\{\psi_J\}$ and applying the above estimate we get

$$|a_Q| = \Big|\int_{\mathbb{R}} \sum_{J:|J|=|Q|} a_J\psi_J(s)\psi_Q(s)\,ds\Big| \le \int_{\mathbb{R}} |\psi_Q(s)|\Big|\sum_{J:|J|=|Q|} a_J\psi_J(s)\Big|\,ds$$
$$\le \int_{\mathbb{R}} |\psi_Q(s)|\,2\varepsilon\lambda\,ds.$$

Since $\psi_Q(s) = 2^{i/2}\psi(2^i s - k)$, we may substitute $y = 2^i s - k$ in the above to obtain

$$|a_Q| \le \int_{\mathbb{R}} |\psi_Q(s)|\,2\varepsilon\lambda\,ds = 2\varepsilon\lambda\int_{\mathbb{R}} 2^{-i/2}|\psi(y)|\,dy \le 2\varepsilon\lambda 2^{-i/2}\|\psi\|_1.$$

We are now prepared to make the desired pointwise estimate for the partial sum of wavelets associated to dyadic intervals in $\mathcal{G}$.

Lemma 5. *There is a constant c_5 depending only on ψ such that for all $t \in \mathbb{R}$,*

$$\Big|\sum_{Q\in\mathcal{G}} a_Q\psi_Q(t)\Big| \le c_5\varepsilon\lambda.$$

Proof. Fix $t \in \operatorname{supp}(\sum_{Q\in\mathcal{G}} a_Q\psi_Q)$. Then $t \in J$ for some $J \in \mathcal{H}$. By Proposition 2 applied to $\mathcal{H}$, we have the following two cases.

Case 1: There exists a sequence $\{J_k\}_{k=m_0}^{\infty}$ of dyadic intervals with $t \in J_k \in \mathcal{H}$ and $|J_k| = 2^{-k}$. For each $k \ge m_0$, there exist $x_k \in E$ and $y_k \in Q_0$ with $|x_k - t| \le C2^{-k}$ and $|y_k - t| \le C2^{-k}$ where C depends only on M. Since E is closed we have $t \in E$, which implies $|\sum_{|I|>2^{-n}} a_I\psi_I(t)| \le \varepsilon\lambda$ for all n and consequently, for every $n > m_0$,

$$\Big|\sum_{2^{-m_0}\ge|I|>2^{-n}} a_I\psi_I(t)\Big| \le 2\varepsilon\lambda. \tag{8}$$

Fix $n > m_0$. We wish to show

$$\{I \in \mathcal{G} : t \in \operatorname{supp}\psi_I,\ |I| > 2^{-n}\} = \{I : t \in \operatorname{supp}\psi_I,\ 2^{-m_0} \ge |I| > 2^{-n}\}.$$

To see this, let I be dyadic with $|I| = 2^{-l}$, $m_0 \le l < n$, and let $t \in \operatorname{supp}\psi_I$. Note that $t \in \operatorname{supp}\psi_I \subseteq \overline{\overline{I}}$ and, from the above, $t \in E$. Thus, $\overline{\overline{I}} \cap E \ne \emptyset$. By definition of $\widehat{I}$, it must be that $\widehat{I} \cap Q_0 \ne \emptyset$ and hence $\overline{\overline{I}} \cap Q_0 \ne \emptyset$. Thus, $I \in \mathcal{G}$.

This gives half of the desired set equality. The remaining set containment is trivial.

We substitute into (8) to obtain $|\sum_{I\in\mathcal{G},\,|I|>2^{-n}} a_I\psi_I(t)| \le 2\varepsilon\lambda$ for every $n > m_0$. The conclusion of the lemma is then immediate in this case.

Case 2: There exists a maximum integer n_0, with $n_0 \ge m_0$, and dyadic intervals $\{J_k\}_{k=m_0}^{n_0} \subseteq \mathcal{H}$ with $|J_k| = 2^{-k}$, such that $t \in J_k$ for all $n_0 \ge k \ge m_0$. Consider the dyadic interval $J_{n_0} \in \mathcal{H}$. By Proposition 3,

$$\Big|\sum_{2^{-m_0}\ge|I|\ge 2^{-n_0}} a_I\psi_I(t)\Big| \le 2\varepsilon\lambda. \tag{9}$$

Similarly to Case 1, we can show

$$\{I\in\mathcal{I} : t\in\operatorname{supp}\psi_I,\ |I|\ge 2^{-n_0}\} = \{I : t\in\operatorname{supp}\psi_I,\ 2^{-m_0}\ge|I|\ge 2^{-n_0}\}.$$

Hence, (9) becomes $|\sum_{I\in\mathcal{I},\,|I|\ge 2^{-n_0}} a_I\psi_I(t)| \le 2\varepsilon\lambda$ and we obtain our desired inequality by estimating the quantities $|\sum_{I\in\mathcal{I}\setminus\mathcal{H},\,|I|\ge 2^{-n_0}} a_I\psi_I(t)|$ and $|\sum_{I\in\mathcal{H}\setminus\mathcal{G},\,|I|\ge 2^{-n_0}} a_I\psi_I(t)|$ by means of the following claim.

Claim. *There exists an N, depending only on ψ, such that for any dyadic $Q\in\mathcal{I}\setminus\mathcal{H}$ (or $Q\in\mathcal{H}\setminus\mathcal{G}$) with $|Q|\ge 2^N/2^{n_0}$, we have $t\notin\operatorname{supp}\psi_Q$.*

Proof. The condition $t\in J_{n_0}\in\mathcal{H}$ implies that there exists some $x_0\in E$ and some $y_0\in Q_0$ such that $|t-x_0|\le C2^{-n_0}$ and $|y_0-t|\le C2^{-n_0}$. Choose N large enough so that $2^N > 2C$. Then $|y_0-x_0|\le 2^N/2^{n_0}$. Let $Q\in\mathcal{I}\setminus\mathcal{H}$ (respectively $Q\in\mathcal{H}\setminus\mathcal{G}$), with $|Q| = 2^{-i}\ge 2^N/2^{n_0}$. Thus, x_0 and y_0 must be either in the same dyadic interval with sidelength 2^{-i} or in adjacent dyadic intervals with sidelengths 2^{-i}, and t must be in the same dyadic with either x_0 or y_0. If $t\in\operatorname{supp}\psi_Q$, the choice of $\widehat{Q}$ implies that $x_0, y_0\in\widehat{Q}$. This gives us $Q\in\mathcal{G}$, which is impossible since $Q\in\mathcal{I}\setminus\mathcal{H}$ (respectively $Q\in\mathcal{H}\setminus\mathcal{G}$). Hence it must be that $t\notin\operatorname{supp}\psi_Q$, which proves the Claim.

To complete the proof of Lemma 5, fix i. There exist at most $2L$ dyadics Q with $|Q| = 2^{-i}$ and $t\in\operatorname{supp}\psi_Q$. By the above Claim, $t\in\operatorname{supp}\psi_Q$ for at most $2LN$ dyadics Q with $Q\in\mathcal{I}\setminus\mathcal{H}$ (respectively $Q\in\mathcal{H}\setminus\mathcal{G}$).

Finally, if $t\in\operatorname{supp}\psi_Q$ for $Q\in\mathcal{I}\setminus\mathcal{H}$ (respectively $Q\in\mathcal{H}\setminus\mathcal{G}$), then Proposition 4 yields $|a_Q\psi_Q(t)| = |a_Q|\,|\psi_Q(t)| \le 2^{-i/2}C\varepsilon\lambda|\psi_Q(t)|$, where $|Q| = 2^{-i}$. Note that $|\psi_Q(t)|\le 2^{i/2}\sup_{y\in\mathbb{R}}|\psi(y)|$. Since ψ is continuous with compact support, we have $|a_Q\psi_Q(t)|\le C\varepsilon\lambda$.

Thus,

$$\Big|\sum_{\substack{J\in\mathcal{I}\setminus\mathcal{H}\\ |J|\ge 2^{-n_0}}} a_J\psi_J(t)\Big| \le 2NLC\varepsilon\lambda, \qquad \Big|\sum_{\substack{J\in\mathcal{H}\setminus\mathcal{G}\\ |J|\ge 2^{-n_0}}} a_J\psi_J(t)\Big| \le 2NLC\varepsilon\lambda.$$

This combined with (9) completes the proof of the lemma.

Using the result of Lemma 5 in (7) we have

$$\int_{\mathbb{R}} \Big| \sum_{Q\in\mathcal{G}} a_Q \psi_Q(x) \Big|^2 \, dx \le C(\varepsilon\lambda)^2 |Q_0|.$$

Thus, $\int_{Q_0} ((\widetilde{S}_{\mathrm{w}}(x))^2 - (\widetilde{S}^{\mathcal{T}}_{\mathrm{w}}(x))^2) \, dx \le C(\varepsilon\lambda)^2|Q_0|$, which finishes the proof of Proposition 1.

From this proposition and Corollary 1, we now have

$$|\{x \in \mathbb{R} : \widetilde{S}_{\mathrm{w}}(x) > K\lambda\}| \le C_1 \exp\left(\frac{-C_2}{\varepsilon^2}\right) |\{x \in \mathbb{R} : \widetilde{S}_{\mathrm{w}}(x) > \lambda\}|.$$

This gives the estimate

$$\begin{aligned} &|\{x \in \mathbb{R} : S_{\mathrm{w}}(x) > K\lambda,\, N_\alpha \Lambda(x) < \varepsilon\lambda\}| \\ &\qquad \le |\{x \in E : \widetilde{S}_{\mathrm{w}}(x) > K\lambda,\, N_\alpha \Lambda(x) \le \varepsilon\lambda\}| \\ &\qquad \le |\{x \in \mathbb{R} : \widetilde{S}_{\mathrm{w}}(x) > K\lambda\}| \\ &\qquad \le C_1 \exp\left(\frac{-C_2}{\varepsilon^2}\right) |\{x \in \mathbb{R} : \widetilde{S}_{\mathrm{w}}(x) > \lambda\}| \\ &\qquad \le C_1 \exp\left(\frac{-C_2}{\varepsilon^2}\right) |\{x \in \mathbb{R} : S_{\mathrm{w}}(x) > \lambda\}|, \end{aligned}$$

where the last inequality comes from the fact that $S_{\mathrm{w}}(x) \ge \widetilde{S}_{\mathrm{w}}(x)$ for all $x \in \mathbb{R}$. We have thus proved Theorem 4.

We shall now show the following corollary to Theorem 4, which will give the desired relationship between S_{w} and $N\Lambda$. Namely, we will show

Corollary 2. *For Φ as in Lemma 1, we have*

$$\int_{\mathbb{R}} \Phi(S_{\mathrm{w}}(x)) \, dx \le C \int_{\mathbb{R}} \Phi(N\Lambda(x)) \, dx,$$

where C depends only on Φ and ψ.

To prove the corollary, we make use of the following lemma. The proof of (a) is given in [5]; (b) follows using similar arguments.

Lemma 6. (a) *For $\gamma > \beta > 0$, there exists a constant $C_{\gamma,\beta}$, depending only on γ, β, such that*

$$|\{x \in \mathbb{R} : N_\gamma \Lambda(x) > \lambda\}| \le C_{\gamma,\beta} |\{x \in \mathbb{R} : N_\beta \Lambda(x) > \lambda\}|.$$

(b) *For $\alpha > 0$, there exists a constant C_α, depending only on α, such that*

$$|\{x \in \mathbb{R} : N_\alpha \Lambda(x) > \lambda\}| \le C_\alpha |\{x \in \mathbb{R} : N\Lambda(x) > \lambda\}|.$$

Applying Lemma 6 with the Lebesgue–Stieltjes measure associated with Φ, we obtain Corollary 2. Combining Corollary 2 with Theorem 3 we have our desired result of $\|S_{\mathrm{w}}\|_p \approx \|N\Lambda\|_p$, $0 < p < \infty$.

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Department of Mathematics and Statistics
Washburn University
Topeka, KS 66621, U.S.A.
E-mail: sarah.cook@washburn.edu

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