GOOD-\(\lambda\) INEQUALITIES FOR WAVELETS OF COMPACT SUPPORT

BY

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Abstract. For a wavelet \(\psi\) of compact support, we define a square function \(S_w\) and a maximal function \(N.A\). We then obtain the \(L_p\) equivalence of these functions for \(0 < p < \infty\). We show this equivalence by using good-\(\lambda\) inequalities.

1. Introduction. In 1970, Burkholder and Gundy [4] showed, among other results, that if for a martingale \(f = (f_n)\) the square function \(Sf\) and maximal function \(f^*\) are given by

\[
Sf(x) = \left( \sum_{n=1}^{\infty} (f_n(x) - f_{n-1}(x))^2 \right)^{1/2}, \quad f^*(x) = \sup_n |f_n(x)|,
\]

then \(\|Sf\|_p \approx \|f^*\|_p\) for \(0 < p < \infty\). Previously this result was known only for \(1 < p < \infty\).

Burkholder and Gundy proved their results by first showing that \(Sf\) controls \(f^*\) and conversely \(f^*\) controls \(Sf\) by what is now commonly known as a good-\(\lambda\) inequality.

Definition 1. For positive measurable functions \(f\) and \(g\), we say that \(g\) controls \(f\) by a good-\(\lambda\) inequality if there exist constants \(K > 1, 0 < \varepsilon_0 \leq 1\) and a function \(C(\varepsilon)\), with \(C(\varepsilon) \to 0\) as \(\varepsilon \to 0\), such that for all \(\varepsilon > 0\) and for some fixed constant \(C\) depending only on \(\lambda\) such that for all \(\lambda > 0\) and \(0 < \varepsilon < \varepsilon_0\) we have

\[
\left| \{x \in \mathbb{R} : f(x) > K\lambda, g(x) < \varepsilon\lambda \} \right| \leq C(\varepsilon) \left| \{x \in \mathbb{R} : f(x) > \lambda \} \right|.
\]

Burkholder ([2], [3]) later refined these results and in particular gave in [2] the following lemma, which demonstrates the usefulness of a good-\(\lambda\) inequality.

Lemma 1. Consider a non-decreasing continuous function \(\Phi\) defined on \([0, \infty)\) such that \(\Phi(0) = 0\) and \(\Phi\) is not identically zero. Suppose \(\Phi\) satisfies \(\Phi(2\lambda) \leq C\Phi(\lambda)\) for all \(\lambda > 0\) and for some fixed constant \(C\) depending only on \(\Phi\). Suppose also that \(g\) controls \(f\) by a good-\(\lambda\) inequality. For a fixed \(\varepsilon\), \(0 < \varepsilon < \varepsilon_0\), there exist real numbers \(\varphi\) and \(\nu\) which satisfy

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\( \Phi(K \lambda) \leq \varrho \Phi(\lambda) \) and \( \Phi(\varepsilon^{-1} \lambda) \leq \nu \Phi(\lambda) \) for every \( \lambda > 0 \). (The growth condition on \( \Phi \) ensures the existence of \( \varrho \) and \( \nu \).) Finally, suppose \( \varrho C(\varepsilon) < 1 \) and \( \int_{\mathbb{R}} \Phi(\min\{1, f(x)\}) \, dx < \infty \). Then

\[
\int_{\mathbb{R}} \Phi(f(x)) \, dx \leq \frac{\varrho \nu}{1 - \varrho C(\varepsilon)} \int_{\mathbb{R}} \Phi(g(x)) \, dx.
\]

Burkholder and Gundy’s results along with the fact that a martingale is essentially a Haar wavelet provide us with a reason to believe that good-\( \lambda \) inequalities exist for a more generalized wavelet. As further justification, we have the following theorem from Meyer [8] which shows the \( L_p \) equivalence of two square functions for an \( r \)-regular wavelet. In the statement of the theorem and throughout the paper, for a dyadic interval \( Q = [k/2^n, (k+1)/2^n) \), let \( \psi_Q(x) := \psi_{n,k}(x) = 2^n/2^j \psi(2^n x - k) \) be the standard dilation and translation of a wavelet \( \psi \). Also, for each dyadic \( Q \), let \( a_Q \) denote the corresponding wavelet coefficient.

**Theorem 1.** For \( \psi \) an \( r \)-regular wavelet and \( 1 < p < \infty \), the norms \( \| (\sum_J |a_J|^2 |\psi(x)|^2)^{1/2} \|_p \) and \( \| (\sum_J |a_J|^2 |J^{-1}X_J(x)|^2)^{1/2} \|_p \) are equivalent.

We now make some definitions which will be used throughout the paper. Fix \( \psi \) to be a wavelet with compact support. Then there exists \( M \in \mathbb{Z} \) such that if \( \overline{Q} \) is the unique interval that has the same center as \( Q \) and length \( |\overline{Q}| = 2^M |Q| \), then \( \text{supp} (\psi_Q) \subseteq \overline{Q} \). Fix such an \( M \) and for a dyadic \( Q \), define \( \hat{Q} \) in this manner. Similarly, define \( \hat{\hat{Q}} \) to be the interval of \( \mathbb{R} \) that has the same center as \( Q \) and length \( |\hat{\hat{Q}}| = 2^{M+3} |Q| \). Further, let \( Q_n(x) \) be the unique dyadic interval that contains \( x \) and has length \( |Q_n(x)| = 2^{-n} \).

In this paper, we show the \( L_p \) equivalence, \( 0 < p < \infty \), of a maximal function and a square function for our wavelet by using good-\( \lambda \) inequalities. We define our maximal function, \( NA \), by

\[
NA(x) := \sup_n \sup_{y \in Q_n(x)} |A_n(y)|, \quad \text{where} \quad A_n(x) := \sum_{|J| > 2^{-n}} a_J \psi_J(x).
\]

We also define the square function, \( S_w \), by

\[
S_w(x) := \left( \sum_{Q \subset \mathbb{R}} a_Q^2 \frac{1}{|Q|} X_Q(x) \right)^{1/2}.
\]

In Section 2, we show \( S_w \) controls \( NA \) by a good-\( \lambda \) inequality. The proof of this inequality roughly follows the proof of the martingale case. In fact, we shall make use of the following theorem, which is a variation of that found in [4].

**Theorem 2.** There exists a constant \( K > 1 \), and constants \( C \) and \( c \) possibly depending on \( K \), such that for \( 0 < \varepsilon < 1, \lambda > 0 \) we have

\[
|\{x \in Q_0 : f^*(x) > K \lambda, S f < \varepsilon \lambda\}| \leq C \exp\left( \frac{-c}{\varepsilon^2} \right) |\{x \in Q_0 : f^*(x) > \lambda\}|.
\]
In the third section, we will define a new maximal function $N_\alpha A$ and show $N_\alpha A$ controls $S_w$. We shall then estimate $NA$ by $N_\alpha A$. In that section we will use the theory of dyadic bounded mean oscillation to obtain our good-$\lambda$ inequality. In particular, we will use the following corollary to the John–Nirenberg Theorem [9].

**Corollary 1.** Suppose $g \in \text{BMO}_d$, $g \not\equiv 0$, with $\|g\|_d \leq C(\varepsilon \lambda)^2$ for some $0 < \varepsilon < 1$ and $\lambda > 0$. Then there exist constants $K > 1$, $c_1 > 0$, and $c_2 > 0$ independent of $\varepsilon$ and $\lambda$ such that

$$|\{x \in \mathbb{R} : |g(x)| > K\lambda^2\}| \leq c_1 \exp\left(-\frac{c_2}{\varepsilon^2}\right) |\{x \in \mathbb{R} : |g(x)| > \lambda^2\}|.$$

**2. Control of $NA$ by $S_w$.** Our goal in this section is to prove the following good-$\lambda$ inequality:

**Theorem 3.** There exist $k > 1$, $0 < \varepsilon_0 \leq 1$ and constants $C$ and $c$ such that for $0 < \varepsilon < \varepsilon_0$, $\lambda > 0$ we have

$$|\{x \in \mathbb{R} : NA(x) > k\lambda, S_w(x) < \varepsilon\lambda\}| \leq C \exp\left(-\frac{c}{\varepsilon^2}\right) |\{x \in \mathbb{R} : NA(x) > \lambda\}|.$$

To prove Theorem 3, we shall divide the dyadic intervals of $\mathbb{R}$ into a finite number of sets and examine the square function indexed over these sets. To this end, we make use of the following which is a slight variation of a lemma found in [6].

**Lemma 2.** Let $F$ denote the set of all dyadic intervals of $\mathbb{R}$ and for $m \in \mathbb{Z}$ let $F_m = \{Q \in F : |Q| = 2^{-m}\}$. For $x \in \mathbb{R}$, set $F^x = \{x + Q : Q \in F\}$ and $F^x_m = \{x + Q : Q \in F_m\}$. For a dyadic interval $Q_0$, there exist $N \in \mathbb{N}$, \{x_j\}_{j=1}^N \subseteq \mathbb{R}$, and disjoint subsets $(B^j)_{j=1}^N$ of $F$ such that

$$\{Q \in F : Q \subseteq Q_0\} = \bigcup_{j=1}^N B^j.$$

Furthermore, if $Q \in B^j$, then we have $\overline{Q} \subseteq Q'$ for a unique $Q' \in F^{x_j}$ with $|Q'| = 2^{M+2}|Q|$. Also, if $Q_1, Q_2 \in B^j$ and $Q_1 \neq Q_2$, then $Q_1' \neq Q_2'$. Of importance here is the fact that for any dyadic $Q_0$, we have subsets $B^j$, $j \in \{1, \ldots, N\}$, where $N$ depends only on $M$. For simplification, we shall assume in what follows that $Q_0$ is the unit dyadic interval $[0, 1)$. Similar results for an arbitrary dyadic $Q_0$ also hold and we shall be free to use this later on.

Fix $j \in \{1, \ldots, N\}$, where $N$ is as in Lemma 2. Continuing with the notation from the lemma, we re-index the wavelet coefficients and functions by

$$c_{Q'}^{(j)} := a_Q, \quad \omega_{Q'}^{(j)}(x) := \psi_Q(x).$$
Similarly, we also subdivide the function $A_n$ by defining

$$A_{Q_0, m}^{(j)}(x) := \sum_{Q \in B^j} a_Q \psi_Q(x) = \sum_{Q' \in G^x_{i-j}} c_Q^{(j)} \omega_Q^{(j)}(x),$$

for $m \geq 0$ and where $G^x_{i-j} := \{Q' \in F^x_{i-j} : Q \subseteq Q_0, Q \in B^j\}$.

Note that we now have $\text{supp} \omega_Q^{(j)} \subseteq Q'$ and $\int_{Q'} \omega_Q^{(j)}(x) \, dx = 0$. Thus, if we set $f_{Q_0, m}^{(j)} := E(A_{Q_0, m}^{(j)} | G^x_{i-j})$ where $G^x_{i-j}$ is the $\sigma$-field generated by intervals in $F^x_{m-M+2}$, then $f_{Q_0}^{(j)} = (f_{Q_0, 0}^{(j)})$ is a martingale with $f_{Q_0, 0}^{(j)} = 0$. For each martingale in this indexed collection, we denote the martingale maximal function and square function as $(f_{Q_0}^{(j)})^*(x)$ and $Sf_{Q_0}^{(j)}(x)$ respectively.

We wish to estimate the wavelet square function by the martingale square functions and apply Theorem 2. Since we have a collection of indexed martingales, it is necessary to define an indexed set of functions similar to the wavelet square function by

$$S^{(j)}_{Q_0, w}(x) := \left( \sum_{Q' \in G^x_{i-j}} (c_Q^{(j)})^2 \frac{1}{|Q'|} \chi_{Q'}(x) \right)^{1/2}.$$

The following lemmas relate the wavelet square function to the martingale. Lemma 3 is due to Bañuelos and Moore [1].

**Lemma 3.** There exist $c_1$ and $c_2$ depending only on $\psi$ such that

1. $$(Sf_{Q_0}^{(j)}(x))^2 \leq c_1(S^{(j)}_{Q_0, w}(x))^2$$

2. $$|f_{Q_0, k}^{(j)}(x) - A_{Q_0, k}^{(j)}(x)|^2 \leq c_2(S^{(j)}_{Q_0, w}(x))^2.$$  

**Lemma 4.** Fix $Q_n$ to be a dyadic interval with length $2^{-n}$. There exists a constant $c_3$, depending only on $\psi$, such that if $x, y \in Q_n$ then

$$\left| \sum_{j=1}^{N} (A_{Q_0, n}^{(j)}(y) - A_{Q_0, n}^{(j)}(x)) \right| \leq c_3 S_w(x).$$

**Proof.** We have

$$\left| \sum_{j=1}^{N} (A_{Q_0, n}^{(j)}(y) - A_{Q_0, n}^{(j)}(x)) \right| = \left| \sum_{J \subseteq Q_0 \atop |J| > 2^{-n}} a_J (\psi_J(y) - \psi_J(x)) \right|.$$

Since $x, y \in Q_n$ it follows that if either $x$ or $y$ is in $\text{supp} \psi_J$ for some dyadic interval $J$ with $|J| > 2^{-n}$, then both $x, y \in J$. Hence we may approximate the above by

$$\left| \sum_{J \subseteq Q_0 \atop |J| > 2^{-n}} a_J (\psi_J(y) - \psi_J(x)) \right| \leq \sum_{J \subseteq Q_0 \atop |J| > 2^{-n}} |a_J| 2^{-n} \|\psi_J\|_\infty \chi_J(x).$$
\[ \leq \sum_{|J|>2^{-n}} |a_J|2^{-n} \|\psi'\|_{\infty}|J|^{-3/2}\mathcal{X}(x) \]
\[ \leq \|\psi'\|_{\infty} \left( \sum_{|J|>2^{-n}} |a_J|^2 |J|^{-1}\mathcal{X}(x) \right)^{1/2} \left( \sum_{|J|>2^{-n}} 2^{-2n} |J|^{-2}\mathcal{X}(x) \right)^{1/2} \]
\[ \leq 2\|\psi'\|_{\infty} S_w(x). \]

To prove Theorem 3, we must obtain constants $k$ and $\varepsilon_0$. We shall explain how these constants are obtained later. In what follows, we simply assume $k > 1$, $\varepsilon_0 \leq 1$, and $\varepsilon < \varepsilon_0$.

Choose a maximal dyadic $Q \subseteq \mathbb{R}$ with $|\{x \in Q : N\Lambda(x) > \lambda\}| > \frac{1}{4}|Q|$. By maximality of $Q$, there exists $x_0 \in Q$ such that $N\Lambda(x_0) \leq \lambda$. In particular, for $y \in Q$ we have
\[ \sum_{|J|>|Q|} a_J \psi_J(y) \leq \lambda. \]

Fix $x_Q \in Q$ such that $N\Lambda(x_Q) > k\lambda$ and $S_w(x_Q) < \varepsilon \lambda$. Say $Q$ has length $|Q| = 2^{-m}$. Note that since $x_0, x_Q \in Q$, it follows that for every $n$ with $n < m$, both $x_0$ and $x_Q$ must be in the same dyadic interval of length $2^{-n}$. Thus, if $\sup_{y \in Q_n(x_Q)} |A_n(y)| > k\lambda$, then $\sup_{y \in Q_n(x_0)} |A_n(y)| > k\lambda$, which implies $N\Lambda(x_0) > k\lambda$. But by choice of $x_0$, $N\Lambda(x_0) \leq \lambda < k\lambda$. Hence, for $N\Lambda(x_Q)$, it suffices to take the supremum over only those $n$ where $n \geq m$. Thus
\[ N\Lambda(x_Q) = \sup_{m \leq n} \sup_{y \in Q_n(x_Q)} |A_n(y)| \]
\[ \leq \sup_{m \leq n} \sup_{y \in Q_n(x_Q)} \left| \sum_{|Q| \geq |J| > 2^{-n}} a_J \psi_J(y) \right| + \sum_{|J|=|Q|} a_J \psi_J(y) \]
\[ \leq \lambda + \sup_{m \leq n} \sup_{y \in Q_n(x_Q)} \left| \sum_{|Q| \geq |J| > 2^{-n}} a_J \psi_J(y) \right| \]
where the last inequality comes from (4).

Since $N\Lambda(x_Q) > k\lambda$, we then have
\[ (k-1)\lambda \leq \sup_{m \leq n} \sup_{y \in Q_n(x_Q)} \left| \sum_{|Q| \geq |J| > 2^{-n}} a_J \psi_J(y) \right|. \]

By applying Lemma 2 to $Q$, there must exist $n_0 > m$ and $y_{n_0} \in Q_{n_0}(x_Q)$ where
\[ (k-1)\lambda \leq \left| \sum_{j=1}^{N} A_{Q,n_0}^{(j)}(y_{n_0}) \right| \]
\[ \leq \left| \sum_{j=1}^{N} A_{Q,n_0}^{(j)}(x_Q) \right| + \left| \sum_{j=1}^{N} (A_{Q,n_0}^{(j)}(y_{n_0}) - A_{Q,n_0}^{(j)}(x_Q)) \right|. \]
It is clear that $\sum_{j=1}^{N}(S_{Q,w}^{(j)}(x_Q))^2 \leq 2^{-M}(S_w(x_Q))^2$. Using this with (2) and $S_w(x_Q) < \varepsilon \lambda$, we may estimate the first summation in (5) by

$$\sum_{j=1}^{N} |A_{Q,k_0}^{(j)}(x_Q)| \leq c_\psi \varepsilon \lambda + \sum_{j=1}^{N} f_{Q,k_0}^{(j)}(x_Q).$$

Using (3) and (6) in inequality (5) we obtain

$$(k - 1)\lambda \leq c_\psi \varepsilon \lambda + \sum_{j=1}^{N} f_{Q,k_0}^{(j)}(x_Q) + c_3 \varepsilon \lambda,$$

which implies $k^* \lambda \leq \sum_{j=1}^{N} f_{Q,k_0}^{(j)}(x_Q) \leq \sum_{j=1}^{N} (f_{Q}^{(j)})^*(x_Q)$. We then have

$$\left| \left\{ x \in Q : N\Lambda(x) > k\Lambda, S_w(x) < \varepsilon \lambda \right\} \right| \leq \sum_{j=1}^{N} \left| \left\{ x \in Q : (f_{Q}^{(j)})^*(x) > \frac{k^* \lambda}{N}, S_{Q,w}^{(j)}(x) < \frac{\varepsilon \lambda}{\sqrt{2M+2}} \right\} \right|$$

$$\leq \sum_{j=1}^{N} \left| \left\{ x \in Q : (f_{Q}^{(j)})^*(x) > \frac{k^* \lambda}{N}, S f_{Q}^{(j)}(x) < c_1 \frac{\varepsilon \lambda}{\sqrt{2M+2}} \right\} \right|$$

where the last estimate is from (1). Now set $k$ large enough and $\varepsilon_0$ small enough so that we may apply Theorem 2 to the above to obtain

$$\left| \left\{ x \in Q : N\Lambda(x) > k\Lambda, S_w(x) < \varepsilon \lambda \right\} \right| \leq C \exp \left( -\frac{c}{\varepsilon^2} \right) |Q|.$$

Note that our constants will depend on $N$. However, recall that for all dyadic $Q$, $N$ depends only on $M$. Summing over all such maximal dyadic $Q$ yields Theorem 3.

### 3. Control of $S_w$ by $N\Lambda$.

Ideally, in this third section we would prove a version of Theorem 3 with the roles of $N\Lambda$ and $S_w$ reversed. Unfortunately, we have not been able to obtain this inequality. However, we will obtain our desired $L_p$ equivalence of $S_w$ and $N\Lambda$ by showing $S_w$ is controlled by a new maximal function, $N_\alpha \Lambda$, where we shall take the supremum over non-dyadic intervals.

Before we specifically define the maximal function $N_\alpha \Lambda$, we require a few definitions. Since $\psi$ is of compact support, there is a smallest integer $L$ such that for a dyadic interval $J = [k/2^j, (k+1)/2^j)$, we have $\widehat{J} \subseteq [(k - L)/2^j, (k + L)/2^j)$. Fix this $L$ and for a dyadic interval $J$, define $\overline{J} = [(k - L - 1)/2^j, (k + L + 2)/2^j)$. Thus supp $\psi_J \subseteq \widehat{J} \subseteq \overline{J}$.

For $\alpha > 0$, $n \in \mathbb{Z}$, and $x \in \mathbb{R}$, define $\Gamma_{n,\alpha}(x) := \{ t \in \mathbb{R} : |t - x| < \alpha 2^{-n} \}$, the open interval of length $2^{-n+1}\alpha$ that has its center at $x$. We now define the maximal function $N_\alpha \Lambda$ by
As mentioned before, our intermediate goal is to prove $N_\alpha A$ controls $S_w$ by a good-$\lambda$ inequality. We record this as Theorem 4.

**Theorem 4.** There exist $\alpha_0 > 0$, $K > 1$ and constants $C, c$ such that

$$|\{x : S_w(x) > K\lambda, N_\alpha A(x) < \varepsilon\lambda\}| \leq C \exp\left(\frac{-c}{\varepsilon^2}\right)|\{x : S_w(x) > \lambda\}|$$

for $0 < \varepsilon < 1$, $\lambda > 0$, $\alpha > \alpha_0$.

To prove this theorem, we shall define a function similar to $S_w$ that is of dyadic bounded mean oscillation. To obtain our desired BMO function, set $E := \{x \in \mathbb{R} : N_\alpha A(x) \leq \varepsilon\lambda\}$ for fixed $\lambda > 0$, $0 < \varepsilon < 1$, and define

$$\tilde{S}_w(x) := \left( \sum_{J \subseteq \mathbb{R}, \tilde{J} \cap E \neq \emptyset} a_J^2 \frac{1}{|J|} \mathcal{X}_J(x) \right)^{1/2}.$$

Note that $S_w(x) \geq \tilde{S}_w(x)$ for all $x \in \mathbb{R}$, and $S_w(x) = \tilde{S}_w(x)$ for all $x \in E$. We shall show

**Proposition 1.** $(\tilde{S}_w)^2 \in \text{BMO}_d$ and $\|\tilde{(S)_w}^2\|_d \leq c_4(\varepsilon\lambda)^2$, where $c_4$ depends only on $\text{supp}\, \psi$.

Fix an arbitrary dyadic interval $Q_0$, say $|Q_0| = 2^{-m_0}$. We define

$T := \{Q : |Q| > 2^{-m_0}, \tilde{Q} \cap E \neq \emptyset\}$, $\mathcal{B} := \{Q : |Q| \leq 2^{-m_0}, \tilde{Q} \cap E \neq \emptyset\}$.

For fixed $x \in \mathbb{R}$, set

$$\tilde{S}_w^T(x) := \left( \sum_{Q \in T} a_Q^2 \frac{1}{|Q|} \mathcal{X}_{\tilde{Q}}(x) \right)^{1/2}$$

and

$$\tilde{S}_w^B(x) := \left( \sum_{Q \in \mathcal{B}} a_Q^2 \frac{1}{|Q|} \mathcal{X}_{\tilde{Q}}(x) \right)^{1/2}.$$

Note that $(\tilde{S}_w(x))^2 - (\tilde{S}_w^T(x))^2 = (\tilde{S}_w^B(x))^2$ and that $\tilde{S}_w^T(x)$ is constant on $Q_0$.

Thus,

$$\int_{Q_0} ((\tilde{S}_w(x))^2 - (\tilde{S}_w^T(x))^2) \, dx = \int_{Q_0} (\tilde{S}_w^B(x))^2 \, dx$$

$$= \int_{Q_0} \sum_{\tilde{Q} \cap E \neq \emptyset} a_Q^2 \frac{1}{|Q|} \mathcal{X}_{\tilde{Q}}(x) \, dx$$

$$\leq 2^{M+3} \sum_{\tilde{Q} \cap E \neq \emptyset, |\tilde{Q} \cap Q_0| \neq \emptyset, \, |Q| \leq |Q_0|} a_Q^2 \sum_{Q \in \mathcal{G}} a_Q^2,$$

where $\mathcal{G} := \{Q : \bar{\tilde{Q}} \cap E \neq \emptyset, \, \bar{\tilde{Q}} \cap Q_0 \neq \emptyset, \, |Q| \leq |Q_0|\}$.
By orthonormality, we have

\[ \sum_{Q \in \mathcal{G}} a_Q^2 = \int \sum_{Q \in \mathcal{G}} a_Q^2 |\psi_Q(x)|^2 \, dx = \int \sum_{Q \in \mathcal{G}} a_Q \psi_Q(x) \, dx. \]

To estimate (7), we first note that \( \sum_{Q \in \mathcal{G}} a_Q \psi_Q \) has support of length \( C |Q_0| \), where \( C \) depends only on \( M \). To complete the estimate of (7), we need to make a pointwise estimate for the partial sum of wavelets associated to dyadics in \( \mathcal{G} \). We will make this estimate by taking the summation over a larger collection of dyadic intervals. We remark that this procedure is similar to one used by Gundy and Iribarren [7]. Define

\[ \mathcal{H} := \{ Q : Q \cap \tilde{J} \neq \emptyset \text{ for some } J \in \mathcal{G}, |J| = |Q| \}, \]

\[ \mathcal{I} := \{ Q : Q \cap \tilde{J} \neq \emptyset \text{ for some } J \in \mathcal{H}, |J| = |Q| \}, \]

\[ \mathcal{J} := \{ Q : Q \cap \tilde{J} \neq \emptyset \text{ for some } J \in \mathcal{I}, |J| = |Q| \}. \]

Note that clearly \( \mathcal{G} \subseteq \mathcal{H} \subseteq \mathcal{I} \subseteq \mathcal{J} \).

We may now select \( \alpha_0 \) large enough so that if \( \alpha > \alpha_0 \) and \( Q \in \mathcal{J} \), \( |Q| = 2^{-j} \), then \( Q \subseteq \Gamma_{j,\alpha}(x_0) \) for some \( x_0 \in E \). We remark that \( \alpha_0 \) depends only on \( M \).

There are three properties of this collection of sets which we state now as propositions. The proof of Proposition 2 is clear from the definition of the relevant sets and is omitted.

PROPOSITION 2. For dyadic \( Q \), let \( \tilde{Q} \) be the parent dyadic interval of \( Q \). If \( |Q| < |Q_0| \) and \( Q \in \mathcal{G} \) (respectively \( \mathcal{H}, \mathcal{I}, \mathcal{J} \)) then \( \tilde{Q} \in \mathcal{G} \) (respectively \( \mathcal{H}, \mathcal{I}, \mathcal{J} \)).

PROPOSITION 3. Let \( Q \in \mathcal{J} \) with \( |Q| = 2^{-j} \). For all \( t \in Q \), and all \( l_1, l_2 \in \mathbb{N} \) with \( m_0 \leq l_1 < l_2 \leq j \), we have

\[ \left| \sum_{2^{-l_1} \geq |J| > 2^{-l_2}} a_J \psi_J(t) \right| \leq 2 \varepsilon \lambda. \]

Proof. By choice of \( \alpha_0 \), \( Q \subseteq \Gamma_{j,\alpha}(x_0) \) for some \( x_0 \in E \). Proposition 2 implies that if \( J \) is the unique dyadic interval with \( |J| = 2^{-l_2} \) and \( Q \subseteq J \), then \( J \in \mathcal{J} \). Thus, again by choice of \( \alpha_0 \), we get \( J \subseteq \Gamma_{l_2,\alpha}(x_0) \) and \( |A_{l_2}(t)| \leq \varepsilon \lambda \) for all \( t \in J \).

Similar reasoning shows that if \( I \) is the unique dyadic interval of length \( 2^{-l_1+1} \) containing \( Q \) then \( |A_{l_1-1}(t)| \leq \varepsilon \lambda \) for all \( t \in I \). The proposition then follows from the triangle inequality.

Our last proposition estimates the wavelet coefficient \( a_Q \) for \( Q \subseteq \mathcal{I} \).

PROPOSITION 4. If \( Q \in \mathcal{I} \) is such that \( |Q| = 2^{-i} \), then there exists a constant \( C \), depending only on \( \psi \), such that \( |a_Q| \leq 2^{-i/2} C \varepsilon \lambda \).
**Proof.** Fix such a $Q$ and say $Q = [r/2^i,(r + 1)/2^i)$. Let $t \in \overline{Q}$ be arbitrary. There must exist some dyadic $I \subseteq Q$ with $t \in I$ and $|I| = |Q|$. Thus, since $Q \in \mathcal{I}$, we have $I \in \mathcal{J}$. By Proposition 3,
\[
\left| \sum_{J:|J|=|I|} a_J \psi_J(x) \right| \leq 2\varepsilon \lambda
\]
for all $x \in I$. In particular, the above inequality holds for $t$.

Using the orthonormality of $\{\psi_J\}$ and applying the above estimate we get
\[
|a_Q| = \left| \int \sum_{J:|J|=|Q|} a_J \psi_J(s) \psi_Q(s) \, ds \right| \leq \int \left| \psi_Q(s) \right| \sum_{J:|J|=|Q|} a_J \psi_J(s) \, ds
\]
\[
\leq \int \left| \psi_Q(s) \right| 2\varepsilon \lambda \, ds.
\]
Since $\psi_Q(s) = 2^{i/2}\psi(2^i s - k)$, we may substitute $y = 2^i s - k$ in the above to obtain
\[
|a_Q| \leq \int \left| \psi_Q(s) \right| 2\varepsilon \lambda \, ds = 2\varepsilon \lambda \int 2^{-i/2} \left| \psi(y) \right| \, dy \leq 2\varepsilon \lambda 2^{-i/2} \|\psi\|_1.
\]

We are now prepared to make the desired pointwise estimate for the partial sum of wavelets associated to dyadic intervals in $\mathcal{G}$.

**Lemma 5.** There is a constant $c_5$ depending only on $\psi$ such that for all $t \in \mathbb{R}$,
\[
\left| \sum_{Q \in \mathcal{G}} a_Q \psi_Q(t) \right| \leq c_5 \varepsilon \lambda.
\]

**Proof.** Fix $t \in \text{supp}(\sum_{Q \in \mathcal{G}} a_Q \psi_Q)$. Then $t \in J$ for some $J \in \mathcal{H}$. By Proposition 2 applied to $\mathcal{H}$, we have the following two cases.

**Case 1:** There exists a sequence $\{J_k\}_{k=m_0}^{\infty}$ of dyadic intervals with $t \in J_k \in \mathcal{H}$ and $|J_k| = 2^{-k}$. For each $k \geq m_0$, there exist $x_k \in E$ and $y_k \in Q_0$ with $|x_k - t| \leq C 2^{-k}$ and $|y_k - t| \leq C 2^{-k}$ where $C$ depends only on $M$. Since $E$ is closed we have $t \in E$, which implies $|\sum_{|I| \geq 2^{-n}} a_I \psi_I(t)| \leq \varepsilon \lambda$ for all $n$ and consequently, for every $n > m_0$,
\[
(8) \quad \left| \sum_{2^{-m_0} \geq |I| > 2^{-n}} a_I \psi_I(t) \right| \leq 2\varepsilon \lambda.
\]

Fix $n > m_0$. We wish to show
\[
\{I \in \mathcal{G} : t \in \text{supp} \psi_I, |I| > 2^{-n}\} = \{I : t \in \text{supp} \psi_I, 2^{-m_0} \geq |I| > 2^{-n}\}.
\]
To see this, let $I$ be dyadic with $|I| = 2^{-l}$, $m_0 \leq l < n$, and let $t \in \text{supp} \psi_I$. Note that $t \in \text{supp} \psi_I \subseteq \overline{I}$ and, from the above, $t \in E$. Thus, $\overline{I} \cap E \neq \emptyset$. By definition of $\overline{I}$, it must be that $\overline{I} \cap Q_0 \neq \emptyset$ and hence $\overline{I} \cap Q_0 \neq \emptyset$. Thus, $I \in \mathcal{G}$. 
This gives half of the desired set equality. The remaining set containment is trivial.

We substitute into (8) to obtain \(|\sum_{I \in \mathcal{G}, |I| \geq 2^{-n}} a_I \psi_I(t)| \leq 2\varepsilon \lambda| for every \( n > m_0 \). The conclusion of the lemma is then immediate in this case.

**Case 2:** There exists a maximum integer \( n_0 \), with \( n_0 \geq m_0 \), and dyadic intervals \( \{J_k\}_{k \geq m_0}^{n_0} \subseteq \mathcal{H} \) with \(|J_k| = 2^{-k}\), such that \( t \in J_k \) for all \( n_0 \geq k \geq m_0 \). Consider the dyadic interval \( J_{n_0} \in \mathcal{H} \). By Proposition 3,

\[
|a_I \psi_I(t)| \leq 2\varepsilon \lambda.
\]

Similarly to Case 1, we can show

\[
\{I \in \mathcal{I} : t \in \text{supp } \psi_I, |I| \geq 2^{-n_0}\} = \{I : t \in \text{supp } \psi_I, 2^{-n_0} \geq |I| \geq 2^{-n_0}\}.
\]

Hence, (9) becomes \(|\sum_{I \in \mathcal{I}, |I| \geq 2^{-n_0}} a_I \psi_I(t)| \leq 2\varepsilon \lambda \) and we obtain our desired inequality by estimating the quantities \(|\sum_{I \in \mathcal{I} \setminus \mathcal{H}, |I| \geq 2^{-n_0}} a_I \psi_I(t)| \) and \(|\sum_{I \in \mathcal{H} \setminus \mathcal{G}, |I| \geq 2^{-n_0}} a_I \psi_I(t)| \) by means of the following claim.

**Claim.** There exists an \( N \), depending only on \( \psi \), such that for any dyadic \( Q \in \mathcal{I} \setminus \mathcal{H} \) (or \( Q \in \mathcal{H} \setminus \mathcal{G} \)) with \(|Q| \geq 2^N/2^{n_0} \), we have \( t \notin \text{supp } \psi_Q \).

**Proof.** The condition \( t \in J_{n_0} \in \mathcal{H} \) implies that there exists some \( x_0 \in E \) and some \( y_0 \in Q \) such that \(|t - x_0| \leq C2^{-n_0}\) and \(|y_0 - t| \leq C2^{-n_0}\). Choose \( N \) large enough so that \( 2^N > 2C \). Then \(|y_0 - x_0| \leq 2^N/2^{n_0}\). Let \( Q \in \mathcal{I} \setminus \mathcal{H} \) (respectively \( Q \in \mathcal{H} \setminus \mathcal{G} \), with \(|Q| = 2^{-i} \geq 2^N/2^{n_0}\). Thus, \( x_0 \) and \( y_0 \) must be either in the same dyadic interval with sidelength \( 2^{-i}\) or in adjacent dyadic intervals with sidelengths \( 2^{-i}\), and \( t \) must be in the same dyadic with either \( x_0 \) or \( y_0 \). If \( t \in \text{supp } \psi_Q \), the choice of \( Q \) implies that \( x_0, y_0 \in Q \). This gives us \( Q \in \mathcal{G} \), which is impossible since \( Q \in \mathcal{I} \setminus \mathcal{H} \) (respectively \( Q \in \mathcal{H} \setminus \mathcal{G} \)). Hence it must be that \( t \not\in \text{supp } \psi_Q \), which proves the Claim.

To complete the proof of Lemma 5, fix \( i \). There exist at most \( 2L \) dyadics \( Q \) with \(|Q| = 2^{-i}\) and \( t \in \text{supp } \psi_Q \). By the above Claim, \( t \in \text{supp } \psi_Q \) for at most \( 2LN \) dyadics \( Q \) with \( Q \in \mathcal{I} \setminus \mathcal{H} \) (respectively \( Q \in \mathcal{H} \setminus \mathcal{G} \)).

Finally, if \( t \in \text{supp } \psi_Q \) for \( Q \in \mathcal{I} \setminus \mathcal{H} \) (respectively \( Q \in \mathcal{H} \setminus \mathcal{G} \)), then Proposition 4 yields \(|a_Q \psi_Q(t)| = |a_Q||\psi_Q(t)| \leq 2^{-i/2}C\varepsilon \lambda|\psi_Q(t)|\), where \(|Q| = 2^{-i}\). Note that \(|\psi_Q(t)| \leq 2^{i/2} \sup_{y \in R} |\psi(y)| \). Since \( \psi \) is continuous with compact support, we have \(|a_Q \psi_Q(t)| \leq C\varepsilon \lambda \).

Thus,

\[
\sum_{J \in \mathcal{I} \setminus \mathcal{H}, |J| \geq 2^{-n_0}} a_J \psi_J(t) \leq 2NLC \varepsilon \lambda, \quad \sum_{J \in \mathcal{H} \setminus \mathcal{G}, |J| \geq 2^{-n_0}} a_J \psi_J(t) \leq 2NLC \varepsilon \lambda.
\]

This combined with (9) completes the proof of the lemma.
Using the result of Lemma 5 in (7) we have
\[
\int \left| \sum_{Q \in \mathcal{Q}} a_Q \psi_Q(x) \right|^2 dx \leq C(\varepsilon \lambda)^2 |Q_0|.
\]
Thus, \( \int_{Q_0} (\tilde{S}_w(x))^2 - (\tilde{S}_w^T(x))^2 \) dx \( \leq C(\varepsilon \lambda)^2 |Q_0| \), which finishes the proof of Proposition 1.

From this proposition and Corollary 1, we now have
\[
|\{ x \in \mathbb{R} : \tilde{S}_w(x) > K\lambda \} | \leq C_1 \exp \left( \frac{-C_2}{\varepsilon^2} \right) |\{ x \in \mathbb{R} : \tilde{S}_w(x) > \lambda \} |.
\]
This gives the estimate
\[
|\{ x \in \mathbb{R} : S_w(x) > K\lambda, N_\alpha A(x) < \varepsilon\lambda \} | \\
\leq |\{ x \in \mathbb{R} : \tilde{S}_w(x) > K\lambda, N_\alpha A(x) \leq \varepsilon\lambda \} | \\
\leq |\{ x \in \mathbb{R} : \tilde{S}_w(x) > K\lambda \} | \\
\leq C_1 \exp \left( \frac{-C_2}{\varepsilon^2} \right) |\{ x \in \mathbb{R} : \tilde{S}_w(x) > \lambda \} | \\
\leq C_1 \exp \left( \frac{-C_2}{\varepsilon^2} \right) |\{ x \in \mathbb{R} : S_w(x) > \lambda \} |,
\]
where the last inequality comes from the fact that \( S_w(x) \geq \tilde{S}_w(x) \) for all \( x \in \mathbb{R} \). We have thus proved Theorem 4.

We shall now show the following corollary to Theorem 4, which will give the desired relationship between \( S_w \) and \( N \alpha \). Namely, we will show

**Corollary 2.** For \( \Phi \) as in Lemma 1, we have
\[
\int_{\mathbb{R}} \Phi(S_w(x)) dx \leq C \int_{\mathbb{R}} \Phi(N\alpha A(x)) dx,
\]
where \( C \) depends only on \( \Phi \) and \( \psi \).

To prove the corollary, we make use of the following lemma. The proof of (a) is given in [5]; (b) follows using similar arguments.

**Lemma 6.** (a) For \( \gamma > \beta > 0 \), there exists a constant \( C_{\gamma,\beta} \), depending only on \( \gamma, \beta \), such that
\[
|\{ x \in \mathbb{R} : N_\gamma A(x) > \lambda \} | \leq C_{\gamma,\beta} |\{ x \in \mathbb{R} : N_\beta A(x) > \lambda \} |.
\]

(b) For \( \alpha > 0 \), there exists a constant \( C_\alpha \), depending only on \( \alpha \), such that
\[
|\{ x \in \mathbb{R} : N_\alpha A(x) > \lambda \} | \leq C_\alpha |\{ x \in \mathbb{R} : N\alpha A(x) > \lambda \} |.
\]
Applying Lemma 6 with the Lebesgue–Stieltjes measure associated with \( \Phi \), we obtain Corollary 2. Combining Corollary 2 with Theorem 3 we have our desired result of 
\[ \|S_w\|_p \approx \|N\|_p, \quad 0 < p < \infty. \]

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