ON TOPOLOGICAL PROPERTIES OF THE SPACES OF DARBOUX BAIRE 1 FUNCTIONS AND BOUNDED DERIVATIVES

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Abstract. We investigate the topological structure of the space $\mathcal{DB}_1$ of bounded Darboux Baire 1 functions on $[0, 1]$ with the metric of uniform convergence and with the $p^*$-topology. We also investigate some properties of the set $\Delta$ of bounded derivatives.

The class $\mathcal{DB}_1$ of bounded Darboux Baire 1 functions on $[0, 1]$ contains subclasses of functions important for differentiation theory such as derivatives. For that reason many mathematicians have investigated this class. In [2], [8], [3] “typical” properties in this class were considered, where a property $\Phi$ is called typical in $\mathcal{DB}_1$ if the class of all functions satisfying $\Phi$ is residual in $\mathcal{DB}_1$. Therefore, the topological structure of $\mathcal{DB}_1$ is worth investigating, and this is one of the purposes of this article. First we shall consider some properties of the set $\Delta$ of all bounded derivatives on $[0, 1]$. One of these properties (superporosity at each point of $\mathcal{DB}_1$) plays an important role in further considerations connected with $\mathcal{DB}_1$.

We apply the classical terminology and notation. We adopt the following definition of a Darboux function ([9], [5]):

A function $F : X \to Y$ (where $X$, $Y$ are topological spaces) is called a Darboux function if $F(C)$ is a connected set for each connected set $C \subset X$.

By $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{N}$, $\mathbb{I}$ we denote the sets of real numbers, rational numbers, natural numbers, and the segment $[0, 1]$ respectively. The symbol $m_1$ stands for the Lebesgue measure on the real line. By $C_f$ (resp. $D_f$) we denote the set of all points of continuity (resp. discontinuity) of a function $f : X \to Y$. For $x_0 \in Y$, we denote by const$_{x_0} : X \to Y$ the constantly $x_0$ function.

A subset $L \subset X$ is called an arc if there exists a homeomorphism $h$ from $\mathbb{I}$ onto $L$. The elements $h(0)$ and $h(1)$ are called the endpoints of $L$. The arc with endpoints $a$ and $b$ is denoted by $L(a, b)$.

We say that a set $A \subset \mathbb{I}$ is bilaterally $c$-dense in itself if $\text{card } A \cap (x, x+\delta) = \text{card } A \cap (x-\delta, x) = c$ for all $x \in A$ and $\delta > 0$.


Key words and phrases: Darboux Baire 1 function, $p^*$-topology, superporosity, derivative.
By $\mathcal{D}$ (resp. $\mathcal{B}_1$) we denote the set of bounded Darboux (resp. Baire 1) functions $f : \mathbb{I} \to \mathbb{R}$. By $g$ we denote the metric of uniform convergence.

We say that a function $f : \mathbb{I} \to \mathbb{R}$ satisfies the Young condition if

- for every $x \in (0, 1)$ there exist sequences $x_n \searrow x$ and $y_n \nearrow x$ such that both $f(x_n)$ and $f(y_n)$ converge to $f(x)$,
- there exists a sequence $x_n \searrow 0$ such that $f(x_n)$ converges to $f(0)$,
- there exists a sequence $y_n \nearrow 1$ such that $f(y_n)$ converges to $f(1)$.

We say that $A \subset X$ is a stationary set for the class $\mathcal{F}$ of functions from $X$ to $Y$ provided that, for each $f \in \mathcal{F}$, if $f$ is constant on $A$, then $f$ must be constant on the whole domain.

If $(X, d)$ is a metric space, then we denote by $B(x, R)$ the open ball with center at $x$ and radius $r > 0$. Let $M \subset X$, $x \in X$, $R > 0$. Then $\gamma(x, R, M)$ denotes the supremum of the set of all $r > 0$ for which there exists $z \in X$ such that $B(z, r) \subset B(x, R) \setminus M$. The set $M$ is called porous at $x$ if $\limsup_{R \to 0^+} \gamma(x, R, M)/R > 0$. We say that $E \subset X$ is superporous at $x \in X$ if $E \cap F$ is porous at $x$ whenever $F$ is porous at $x$. A set $G \subset X$ is said to be $p$-open if $X \setminus G$ is superporous at each point of $G$. The system of all superporous sets at $x$ forms an ideal. Therefore the system of all $p$-open sets forms a topology, called the $p$-topology ([12]). A set $H \subset X$ is said to be $p^*$-open if $H = G \setminus N$, where $G$ is $p$-open and $N$ is $p$-meager. The system of all $p^*$-open sets forms a topology, called the $p^*$-topology. Clearly the $p^*$-topology is stronger than the $p$-topology, and the $p$-topology is stronger than the topology generated by the metric $d$ ([12]).

The notion of an abstract density topology (in the category sense) is understood as in [6].

It is known that $\Delta \subset DB_1$ ([1], [10]). It is easy to see that $\text{card}(\Delta) = \text{card}(DB_1) = c$. But it turns out that $\Delta$ is a “small” subset of $DB_1$ in the topological sense. To prove this we need two lemmas.

First from [7, Theorem 1.1.9(3) and Corollary 1.7.12] we infer

**Lemma 1.** If $f : [a, b] \to \mathbb{R}$ $(a < b)$ is a Darboux (resp. Baire 1) function, then for every $\alpha \in \mathbb{R}$, the functions $f^* = \max(f, \text{const}_\alpha)$ and $f_* = \min(f, \text{const}_\alpha)$ are Darboux (resp. Baire 1) functions. 

Let $\{a_i\}_{i \in K}$ ($K = \{1, \ldots, n\}$) be a finite increasing sequence of real numbers from an interval $(a, b)$. Put $F_1 = [a, a_1]$, $F_i = [a_{i-1}, a_i]$ for $i \in K \setminus \{1, n\}$, $F_n = [a_{n-1}, b]$. Obviously the family $\{F_i\}_{i \in K}$ is a closed covering of $[a, b]$.

It is easy to check

**Lemma 2.** Let $\{F_i\}_{i \in K}$ be the sequence of sets defined above and let $f_i : F_i \to \mathbb{R}$, where $i \in K$, be a family of compatible Darboux (resp. Baire 1)
functions. Then the common extension $f = \nabla_{i\in K} f_i$ is a Darboux (resp. Baire 1) function.

**Theorem 3.** The set $\Delta$ is superporous at each point of the space $(DB_1, g)$.

**Proof.** Let $f \in DB_1$ and let $\Phi \subset DB_1$ be porous at $f$. Let $R > 0$. Put $r'_1 = \gamma(f, R, \Phi)/2 > 0$. Then there exist $r_1 > r'_1$ and $g \in DB_1$ such that
\begin{equation}
B(g, r_1) \subset B(f, R) \setminus \Phi.
\end{equation}
We shall show that there exists $h \in DB_1$ such that
\begin{equation}
B(h, r_1/8) \subset B(g, r_1) \setminus \Delta.
\end{equation}
Since $g \in B_1$, there exists a point $x_0 \in (0, 1)$ of continuity of $g$. Consequently, there exists $\delta > 0$ such that $[x_0 - \delta, x_0 + \delta] \subset (0, 1)$ and
\[
g([x_0 - \delta, x_0 + \delta]) \subset (g(x_0) - r_1/4, g(x_0) + r_1/4).
\]
Let $C_\delta \subset (x_0 - \delta/2, x_0 + \delta/2)$ be a bilaterally $c$-dense in itself $F_\sigma$ set of null Lebesgue measure. Then ([1, Theorem II.2.4]) there exists a Darboux Baire 1 function $s : [x_0 - \delta/2, x_0 + \delta/2] \rightarrow \mathbb{I}$ such that $s(x) = 0$ for $x \notin C_\delta$ and $0 < s(x) \leq 1$ for $x \in C_\delta$.

Fix $\alpha \in (0, 1] \cap s(C_\delta)$. Put $s_1(x) = \min(1, \alpha^{-1}s(x))$ for $x \in [x_0 - \delta/2, x_0 + \delta/2]$. Obviously $s_1 : [x_0 - \delta/2, x_0 + \delta/2] \rightarrow \mathbb{I}$ is a bounded Darboux Baire 1 function ([1, Theorem II.3.2] and Lemma 1). Note that $1 \in s_1(C_\delta)$.

We define a function $\mu : [x_0 - \delta/2, x_0 + \delta/2] \rightarrow \mathbb{R}$ as follows:
\[
\mu(x) = \frac{r_1}{4} s_1(x) + g(x_0).
\]
Then $\mu : [x_0 - \delta/2, x_0 + \delta/2] \rightarrow [g(x_0), g(x_0) + r_1/4]$ is a bounded Darboux Baire 1 function ([1, Theorem II.3.2]). Note that $r_1/4 + g(x_0) \in \mu(C_\delta)$.

We define a function $h : \mathbb{I} \rightarrow \mathbb{R}$ as follows:
\[
h(x) = \begin{cases}
g(x) & \text{if } x \in \mathbb{I} \setminus (x_0 - \delta, x_0 + \delta), \\
l_1(x) & \text{if } x \in [x_0 - \delta, x_0 - \delta/2], \\
\mu(x) & \text{if } x \in [x_0 - \delta/2, x_0 + \delta/2], \\
l_2(x) & \text{if } x \in [x_0 + \delta/2, x_0 + \delta],
\end{cases}
\]
where $l_1$ and $l_2$ are linear functions such that $l_1(x_0 - \delta) = g(x_0 - \delta)$, $l_1(x_0 - \delta/2) = g(x_0)$, $l_2(x_0 + \delta) = g(x_0 + \delta)$ and $l_2(x_0 + \delta/2) = g(x_0)$. Then $h \in DB_1$ (Lemma 2). Note that $r_1/4 + g(x_0) \in h(C_\delta)$.

Notice that $g(h, g) \leq r_1/2$, so
\begin{equation}
B(h, r_1/8) \subset B(g, r_1).
\end{equation}

Now, assume that there exists a function $\xi \in B(h, r_1/8) \cap \Delta$. Then
\begin{equation}
(x_0 - \delta/2, x_0 + \delta/2) \setminus C_\delta \subset \xi^{-1}((-\infty, g(x_0) + r_1/8)).
\end{equation}
Let $z_0 \in C_\delta$ be such that $h(z_0) = r_1/4 + g(x_0)$. Hence $\xi(z_0) > g(x_0) + r_1/8$. Therefore $z_0 \in \xi^{-1}((g(x_0) + r_1/8, \infty)) \cap C_\delta$. Let $U_0 \subset (x_0 - \delta/2, x_0 + \delta/2)$ be a unilateral neighbourhood of $z_0$. Note that by (4),

$$U_0 \cap \xi^{-1}((g(x_0) + r_1/8, \infty)) \subset C_\delta,$$

so

$$m_1(U_0 \cap \xi^{-1}((g(x_0) + r_1/8, \infty))) \leq m_1(C_\delta) = 0.$$

Thus $\xi \notin \mathcal{M}_2$, which contradicts the fact that $\Delta \subset \mathcal{M}_2$. Hence $B(h, r_1/8) \cap \Delta = \emptyset$. This equality and (3) finish the proof of (2). From (1) and (2) we infer that

$$\gamma(f, R, \Delta \cup \Phi) \geq r_1/8.$$

Therefore

$$\limsup_{R \to 0^+} \frac{\gamma(f, R, \Delta \cup \Phi)}{R} \geq \frac{1}{8} > 0. \blacksquare$$

It is easy to observe that $\Delta$ is a nowhere dense and perfect subset of $\mathcal{DB}_1$. So its topological structure is similar to that of the Cantor set. There are several constructions of Darboux functions from $[0, 1]$ to $\mathbb{R}$ in which the Cantor set plays an important role. This suggests that $\Delta$ can play a similar role in constructions of Darboux functions from $\mathcal{DB}_1$ to $\mathbb{R}$. It turns out that in some cases we can obtain analogous results (Theorem 4), in others it is impossible (Corollary 6).

**Theorem 4.** There exists a Darboux function $F : \mathcal{DB}_1 \to \mathbb{R}$ such that $D_F = \Delta$ and $F(B(g, \varepsilon) \cap \Delta) = \mathbb{R}$ for any $g \in \Delta$ and $\varepsilon > 0$.

**Proof.** In $\mathbb{R}$ we define an equivalence relation $\ast$ in the following way: $x \ast y \iff x - y \in \mathbb{Q}$. Denote by $\mathcal{E}$ the set of equivalence classes of this relation and let $\xi : \mathcal{E} \to \mathbb{R}$ be a bijection. Define a function $\chi : \mathbb{R} \to \mathbb{R}$ by $\chi(x) = \xi([x]_\ast)$. Then $\chi$ is a Darboux function such that $\chi((a, b)) = \mathbb{R}$ for all $a < b$. Let $\varphi(x) = (1/x) \sin(1/x)$ for $x \in (0, \infty)$. We define $F : \mathcal{DB}_1 \to \mathbb{R}$ by

$$F(f) = \begin{cases} \chi(f(0)) & \text{if } f \in \Delta, \\ \varphi(\theta_{\Delta}(f)) & \text{if } f \in \mathcal{DB}_1 \setminus \Delta. \end{cases}$$

First we shall show that

(5)

$F$ is a Darboux function.

Let $C \subset \mathcal{DB}_1$ be a connected set. Consider the following three cases.

**Case 1:** $C \subset \Delta$. Suppose that $F(C)$ is disconnected. Then there exist $r_1 < r_0 < r_2$ and $f_1, f_2 \in C$ such that $F(f_1) = r_1$, $F(f_2) = r_2$ and $F(f) \neq r_0$.

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(1) A function $f : \mathbb{I} \to \mathbb{R}$ is said to be of class $\mathcal{M}_2$ if for each $a \in \mathbb{R}$ the set $E = \{x \in \mathbb{I} : f(x) > a\}$ is either empty or an $F_\sigma$ and $m_1(E \cap (x - \delta, x)) > 0$ and $m_1(E \cap (x + \delta)) > 0$ for each $x \in E$ and each $\delta > 0$. Zahorski proved that every bounded derivative is of class $\mathcal{M}_2$ ([10]).
for each \( f \in C \). Consequently, there exists \( E^* \in \mathcal{E} \) such that \( \xi(E^*) = r_0 \). Then
\[
(6) \quad f(0) \notin E^* \quad \text{for each } f \in C.
\]
As \( r_1 \neq r_2 \), we have \( F(f_1) \neq F(f_2) \). Then \( [f_1(0)]_\ast \neq [f_2(0)]_\ast \), so \( f_1(0) \neq f_2(0) \).
Let, for instance, \( f_1(0) < f_2(0) \). Then there exists \( y^* \in E^* \cap (f_1(0), f_2(0)) \).
By (6),
\[
C = \{ f \in C : f(0) < y^* \} \cup \{ f \in C : f(0) > y^* \},
\]
where the sets \( \{ f \in C : f(0) < y^* \} \) and \( \{ f \in C : f(0) > y^* \} \) are nonempty (they contain \( f_1, f_2 \) respectively) and separated, which contradicts the connectedness of \( C \).

**Case 2:** \( C \subset DB_1 \setminus \Delta \). If \( g_\Delta(C) \) is a singleton, so is \( F(C) \). In the opposite case, let \( r_1 = \inf\{ r > 0 : \exists f \in C \quad g_\Delta(f) = r \} \) and \( r_2 = \sup\{ r > 0 : \exists f \in C \quad g_\Delta(f) = r \} \). It is evident that \( r_1 \neq r_2 \) and \( r_1 \geq 0, r_2 > 0 \). Note that (by the connectedness of \( C \))
\[
\forall r \in (r_1, r_2) \quad C \cap \{ f \in DB_1 : g_\Delta(f) = r \} \neq \emptyset.
\]
Consider the following subcases:

(a) \( \forall f \in C \quad (g_\Delta(f) \neq r_1 \land g_\Delta(f) \neq r_2) \). Then \( F(C) = \varphi((r_1, r_2)) \) is connected because \( \varphi \) is continuous on \( (0, \infty) \).

(b) \( \forall f \in C \quad g_\Delta(f) \neq r_1 \land (\exists f_0 \in C \quad g_\Delta(f_0) = r_2) \). Then \( F(C) = \varphi((r_1, r_2)) \) is connected.

(c) \( (\exists f_0 \in C \quad g_\Delta(f_0) = r_1) \land (\forall f \in C \quad g_\Delta(f) \neq r_2) \). Since \( f_0 \in C \subset DB_1 \setminus \Delta \) and \( \Delta \) is a closed set, we have \( r_1 = g_\Delta(f_0) > 0 \). Hence \( [r_1, r_2] \subset (0, \infty) \) and \( F(C) = \varphi([r_1, r_2]) \) is connected.

(d) \( (\exists f_0 \in C \quad g_\Delta(f_0) = r_1) \land (\exists f_0 \in C \quad g_\Delta(f_0) = r_2) \). As in (c) we can show that \( [r_1, r_2] \subset (0, \infty) \). Hence \( F(C) = \varphi([r_1, r_2]) \) is connected.

**Case 3:** \( C \cap \Delta \neq \emptyset \) and \( C \setminus \Delta \neq \emptyset \). Then there exists a function \( \hat{f} \in C \setminus \Delta \).
Let \( \hat{r} = g_\Delta(\hat{f}) > 0 \). Since \( C \) is connected, we have
\[
\forall r \in (0, \hat{r}) \quad C \cap \{ f \in DB_1 : g_\Delta(f) = r \} \neq \emptyset.
\]
Hence \( F(C) \supset \varphi((0, \hat{r})) = \mathbb{R} \) and \( F(C) = \mathbb{R} \) is connected. This ends the proof of (5).

Now we shall show that
\[
(7) \quad \forall g \in \Delta \quad \forall \varepsilon > 0 \quad F(K(g, \varepsilon) \cap \Delta) = \mathbb{R}.
\]
Indeed, if \( g \in \Delta \) and \( \varepsilon > 0 \), then
\[
F(K(g, \varepsilon) \cap \Delta) \supset F(\{ g + \alpha : \alpha \in (-\varepsilon, \varepsilon) \}) = \chi((g(0) - \varepsilon, g(0) + \varepsilon)) = \mathbb{R}.
\]
It is easy to see that \( DB_1 \setminus \Delta \subset C_F \). From (7) we infer that \( \Delta \subset D_F \), so \( D_F = \Delta \), which ends the proof. \( \blacksquare \)
It is known that for each perfect set $P \subset \mathbb{I}$ there exists a bounded Darboux Baire 1 function $h : \mathbb{I} \to \mathbb{R}$ such that $h$ vanishes off $P$ but does not vanish identically ([1, Theorem II.2.4]). This fact leads to the question: Does there exist a Darboux function $F : \mathcal{DB}_1 \to \mathbb{R}$ which vanishes off $\Delta$ but does not vanish identically? The answer is negative (Corollary 6).

The above question is connected with the theory of stationary sets. It is known that $E$ is a stationary set for the family of Darboux functions $f : \mathbb{I} \to \mathbb{R}$ if and only if $\text{card}(\mathbb{I} \setminus E) < c$ ([1, Theorem XII.1.1]). But it turns out that for the family of real Darboux functions defined on $\mathcal{DB}_1$ (with the metric of uniform convergence) this characterization of stationary sets fails.

**Theorem 5.** In the space $(\mathcal{DB}_1, \varrho)$ the set $\Delta' = \mathcal{DB}_1 \setminus \Delta$ is stationary for the class of real Darboux functions.

**Proof.** Let $F : \mathcal{DB}_1 \to \mathbb{R}$ be a Darboux function such that $F(\Delta') = \{\alpha_0\}$ (where $\alpha_0 \in \mathbb{R}$). Let $g \in \Delta$. To prove the theorem it is sufficient to construct an arc $L = L(g, h)$ such that $L \setminus \{g\} \subset \Delta'$.

Since $g \in \mathcal{B}_1$, there exists a point $x_0 \in (0, 1)$ of continuity of $g$. For $r \in \mathbb{I}$ we define $t_r : \mathbb{I} \to \mathbb{R}$ in the following way:

$$t_r(x) = \begin{cases} g(x_0) + r & \text{if } x = x_0, \\ g(x) + r \sin \frac{1}{x-x_0} & \text{if } x \in \mathbb{I} \setminus \{x_0\}. \end{cases}$$

Obviously, $t_r (r \in \mathbb{I})$ is a bounded Baire 1 function. It is not difficult to see that it satisfies the Young condition, so it is a Darboux function ([1]). Hence $t_r \in \mathcal{DB}_1$ for $r \in \mathbb{I}$.

Note that $t_r = g + d_r$ ($r \in \mathbb{I}$), where

$$d_r(x) = \begin{cases} r & \text{if } x = x_0, \\ r \sin \frac{1}{x-x_0} & \text{if } x \in \mathbb{I} \setminus \{x_0\}. \end{cases}$$

We shall prove that

$$d_r \notin \Delta \quad \text{for } r \in (0, 1). \quad (8)$$

Indeed, for a fixed $r^* \in (0, 1]$, define $k : \mathbb{I} \to \mathbb{R}$ by

$$k(x) = \begin{cases} 0 & \text{if } x = x_0, \\ r^*(x-x_0)^2 \cos \frac{1}{x-x_0} & \text{if } x \in \mathbb{I} \setminus \{x_0\}. \end{cases}$$

Then

$$k'(x) = \begin{cases} 0 & \text{if } x = x_0, \\ 2r^*(x-x_0) \cos \frac{1}{x-x_0} + r^* \sin \frac{1}{x-x_0} & \text{if } x \in \mathbb{I} \setminus \{x_0\}. \end{cases}$$

Consider a function $h : \mathbb{I} \to \mathbb{R}$ defined by

$$h(x) = \begin{cases} 0 & \text{if } x = x_0, \\ 2r^*(x-x_0) \cos \frac{1}{x-x_0} & \text{if } x \in \mathbb{I} \setminus \{x_0\}. \end{cases}$$
Then $h$ is continuous and bounded on $I$, so $h \in \Delta$. Therefore also $f = k' - h \in \Delta$. Hence $d_r \notin \Delta$, because the difference
\[
d_r(x) - f(x) = \begin{cases} 
  r^* & \text{if } x = x_0, \\
  0 & \text{if } x \in I \setminus \{x_0\},
\end{cases}
\]
is not a derivative (it does not have the Darboux property). In this way we have proved condition (8).

Since $d_r = t_r - g$ for $r \in (0, 1]$ and $g \in \Delta$, it follows that $t_r$ is not a derivative for each $r \in (0, 1]$.

Note that for any $r_1, r_2 \in I$,
\[
\varrho(t_{r_1}, t_{r_2}) = |r_1 - r_2|.
\]
Therefore the function $\zeta : I \to \{t_r : r \in I\}$ given by $\zeta(r) = t_r$ is a homeomorphism. Hence $L = \{t_r : r \in I\}$ is an arc in $DB_1$ such that $L = L(g, t_1)$ and $L \setminus \{g\} = \{t_r : r \in (0, 1]\} \subset \Delta'$.

**Corollary 6.** There does not exist a Darboux function $F : DB_1 \to \mathbb{R}$ which is zero for $t \in \Delta'$, but not identically zero.

Now we shall investigate the topological structure of the space $DB_1$ with the metric of uniform convergence and the $p^*$-topology.

It is easy to see that $(DB_1, \varrho)$ is a Baire space. So from [12, Theorem 2], we infer

**Corollary 7.** The $p^*$-topology is an abstract density topology (in the category sense) on $(DB_1, \varrho)$.

Obviously $(DB_1, \varrho)$ is a perfectly normal space. For the $p^*$-topology we have

**Theorem 8.** $(DB_1, p^*)$ is a Hausdorff space but it is not regular.

*Proof.* Since the $p^*$-topology is stronger than the $\varrho$-topology ([12]), we deduce that $(DB_1, p^*)$ is a Hausdorff space.

Suppose that $(DB_1, p^*)$ is a regular space. For $q \in \mathbb{Q}$ put
\[
A_q = \{f \in DB_1 : f(0) = q\}, \quad F = \bigcup_{q \in \mathbb{Q}} A_q.
\]
Obviously $F \subset DB_1$. Note that
\[
F \text{ is } \varrho \text{-meager.}
\]

To see this, it suffices to prove that $A_q$ is $\varrho$-nowhere dense for each $q \in \mathbb{Q}$. So fix $q_0 \in \mathbb{Q}$ and let $B(g, \varepsilon)$ be an arbitrary open ball in $(DB_1, \varrho)$. We shall show that there exists a $\varrho$-open set $U \subset B(g, \varepsilon) \setminus A_{q_0}$. If $B(g, \varepsilon) \cap A_{q_0} = \emptyset$, we obviously put $U = B(g, \varepsilon)$. Hence, we may assume that there exists $f_0 \in B(g, \varepsilon) \cap A_{q_0}$. Put $\delta = \varepsilon - \varrho(g, f_0) > 0$ and $U = B(f_0 + \delta/2, \delta/4)$. 

(10) 

\[
F \text{ is } \varrho \text{-meager.}
\]
Clearly $U \subset B(g, \varepsilon)$. If $h \in U$, then $h(0) > q_0 + \delta/4$. Hence $h \not\in A_{q_0}$, so $U \cap A_{q_0} = \emptyset$. The proof of (10) is thus finished. Hence ([12, Theorem 2])

$$F \text{ is } p^*\text{-closed.}$$

Now we shall show that

$$\tag{11} F \text{ is } g\text{-dense.}$$

Indeed, let $B(g, \varepsilon)$ be an arbitrary open ball in $\mathcal{DB}_1$. Let $q^* \in (g(0) - \varepsilon, g(0) + \varepsilon) \cap \mathbb{Q}$. Put $h^*(x) = g(x) - g(0) + q^*$ for $x \in I$. Clearly $h^* \in \mathcal{DB}_1$ ([1, Theorem II.3.2]) and $h^*(0) = q^* \in \mathbb{Q}$, so $h^* \in A_{q^*} \subset F$. Of course $g(g, h^*) < \varepsilon$, so $h^* \in B(g, \varepsilon)$. Hence $F \cap B(g, \varepsilon) \neq \emptyset$, which proves (11).

Now, let $f^* \in \mathcal{DB}_1 \setminus F$. Since $(\mathcal{DB}_1, p^*)$ is (by assumption) a regular space, there exist $p^*$-open and disjoint sets $U_1, U_2$ such that $F \subset U_1$ and $f^* \in U_2$. Then ([12, Theorem 2])

$$U_1 = H_1 \setminus N_1, \quad U_2 = H_2 \setminus N_2,$$

where $H_1, H_2$ are $p$-open and $N_1, N_2$ are $p$-meager.

From $U_1 \cap U_2 = \emptyset$ we conclude that $(H_1 \cap H_2) \setminus (N_1 \cup N_2) = \emptyset$. Since $(\mathcal{DB}_1, g)$ (and hence $(\mathcal{DB}_1, p)$, see e.g. [12]) is a Baire space, we deduce that $H_1 \cap H_2 = \emptyset$. Therefore

$$F \subset U_1 \subset H_1 \subset \mathcal{DB}_1 \setminus H_2.$$

Since $H_2$ is a $p$-open set, we conclude that $F$ is porous at $f^*$ (in the space $(\mathcal{DB}_1, g)$). This contradicts (11).

In the proof of Theorem 5 we have constructed an arc in $\mathcal{DB}_1$. This leads to the question: Are the spaces considered arcwise connected? Theorems 9 and 12 give an answer to this question.

**Theorem 9.** The space $(\mathcal{DB}_1, g)$ is arcwise connected.

*Proof.* Let $f_1, f_2 \in \mathcal{DB}_1$ and $f_1 \neq f_2$. Consider the following cases:

**Case 1:** $f_1 = \text{const}_{t_0}$ or $f_2 = \text{const}_{t_0}$. Assume, for instance, that $f_1 = \text{const}_{t_0}$. Put $L = \{af_2 : a \in I\}$. Then $L$ is an arc in $(\mathcal{DB}_1, g)$ such that $L = L(f_1, f_2)$.

**Case 2:** $f_1 \neq \text{const}_{t_0}$ and $f_2 \neq \text{const}_{t_0}$. Then there are two possibilities:

- There exists $r^* \in \mathbb{R}$ such that $f_1 = r^*f_2$. Since $f_1 \neq f_2$, we have $r^* \neq 1$. Assume, for instance, that $r^* > 1$ (the other case is similar). Put $L = \{af_2 : a \in [1, r^*]\}$. Then $L$ is an arc in $(\mathcal{DB}_1, g)$ such that $L = L(f_2, f_1)$.

- There is no $r \in \mathbb{R}$ such that $f_1 = rf_2$. Put $L_1 = \{af_1 : a \in I\}$ and $L_2 = \{af_2 : a \in I\}$. Then $L_1$ and $L_2$ are arcs in $(\mathcal{DB}_1, g)$ such that $L_1 = L(\text{const}_{t_0}, f_1), L_2 = L(\text{const}_{t_0}, f_2)$ and $L_1 \cap L_2 = \{\text{const}_{t_0}\}$. Thus $L_1 \cup L_2$ is an arc in $(\mathcal{DB}_1, g)$ such that $L_1 \cup L_2 = L(f_1, f_2)$. ■
Lemma 10. For each \( f \in \mathcal{DB}_1 \) and each \( r > 0 \), there exist arcs \( L_1, L_2, L_3 \) in the space \( (\mathcal{DB}_1, \mathcal{g}) \) contained in \( B(f, r) \) such that \( L_i \cap L_j = \{ f \} \) for \( i, j \in \{ 1, 2, 3 \}, i \neq j \).

Proof. Let \( f \in \mathcal{DB}_1 \) and let \( r > 0 \). Consider the following cases:

- \( f \) is not a constant function. Put \( M = \sup \{|f(x)| : x \in \mathbb{I}\} > 0 \). Then \( L_1 = \{ f + a : a \in [0, r/2] \} \), \( L_2 = \{ f + a : a \in [-r/2, 0] \} \) and \( L_3 = \{ af : a \in [1, 1 + r/(2M)] \} \) satisfy the required conditions.

- \( f \) is a constant function. Put \( L_1 = \{ f + a : a \in [0, r/2] \} \), \( L_2 = \{ f + a : a \in [-r/2, 0] \} \) and \( L_3 = \{ l_a : a \in [0, r/2] \} \), where \( l_a(x) = ax + f(x) \), \( x \in \mathbb{I} \). Then \( L_1, L_2, L_3 \) satisfy the required conditions.

Lemma 11. Let \( X \) be an arbitrary set and let \( T \) and \( T' \) be two topologies on \( X \) such that \( T \subset T' \) and \( (X, T) \) is a Hausdorff space. Then, if \( L \) is an arc in \( (X, T') \), it is also an arc in \( (X, T) \).

Theorem 12. The space \( (\mathcal{DB}_1, p^*) \) is not arcwise connected. Moreover, there exists no arc in \( (\mathcal{DB}_1, p^*) \).

Proof. Suppose that there exists an arc \( L \) in \( (\mathcal{DB}_1, p^*) \). From Lemma 11, we infer that \( L \) is an arc in \( (\mathcal{DB}_1, \mathcal{g}) \). Now we show that

\[
L \text{ has empty interior in } (\mathcal{DB}_1, \mathcal{g}).
\]

Indeed, suppose that there exists an open ball \( B(f, r) \subset L \). By Lemma 10 there exist arcs \( L_1, L_2, L_3 \) in \( (\mathcal{DB}_1, \mathcal{g}) \), contained in \( B(f, r) \), such that \( L_i \cap L_j = \{ f \} \) for \( i \neq j \). Then \( L_1, L_2, L_3 \) are arcs in \( (L, \mathcal{g}) \). It is not difficult to check that this is impossible.

Clearly \( L \) is \( \mathcal{g} \)-closed. Hence (by (12)) \( L \) is \( \mathcal{g} \)-nowhere dense, so \( L \) is \( \mathcal{g} \)-meager. Then each subset of \( L \) is \( p^* \)-closed. It follows that \( L \) is disconnected in \( (\mathcal{DB}_1, p^*) \), which contradicts the fact that \( L \) is an arc.

References


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