ON MARCZEWSKI–BURSTIN REPRESENTABLE ALGEBRAS

BY

MAREK BALCERZAK (Łódź), ARTUR BARTOSZEWICZ (Łódź)
and PIOTR KOSZMIDER (São Paulo)

Abstract. We construct algebras of sets which are not MB-representable. The existence of such algebras was previously known under additional set-theoretic assumptions. On the other hand, we prove that every Boolean algebra is isomorphic to an MB-representable algebra of sets.

Introduction. Marczewski in [Sz] introduced operations $S$ and $S_0$, and applied them to the family of all perfect subsets of a Polish space $X$. The results of these operations yielded an interesting pair of a $\sigma$-algebra and a $\sigma$-ideal of sets, investigated by several authors (see e.g. [Mo], [Mi]). As observed in [P] (see also [BBRW]), if the same operations are applied to an arbitrary family $\mathcal{F}$ of nonempty subsets of a set $X \neq \emptyset$, that is,

$$S(\mathcal{F}) = \{ E \subseteq X : (\forall A \in \mathcal{F})(\exists B \in \mathcal{F})B \subseteq A \cap E \vee B \subseteq A \setminus E \},$$

$$S_0(\mathcal{F}) = \{ E \subseteq X : (\forall A \in \mathcal{F})(\exists B \in \mathcal{F})B \subseteq A \setminus E \},$$

then $S(\mathcal{F})$ and $S_0(\mathcal{F})$ constitute an algebra and an ideal of subsets of $X$, respectively. An old result of Burstin [Bu] states that the pair consisting of the $\sigma$-algebra of Lebesgue measurable sets in $\mathbb{R}$ and the $\sigma$-ideal of Lebesgue null sets in $\mathbb{R}$ is of the form $(S(\mathcal{F}), S_0(\mathcal{F}))$, where $\mathcal{F}$ consists of the perfect sets of positive measure. Note that Burstin worked earlier than and independently of Marczewski, and he did not use operations $S$ and $S_0$ explicitly.

A new direction of study that appeared in [R] and [BR] was devoted to characterization of the families $S(\mathcal{F})$, $S_0(\mathcal{F})$ when $\mathcal{F}$ denotes the collection of perfect sets in various known topologies. Another trend initiated in [BBRW], [BBC], [BET] was concerned with the problem of how to express known algebras and/or ideals of sets in the form $S(\mathcal{F})$, $S_0(\mathcal{F})$ where sometimes $\mathcal{F}$ is required to be a “good” family in an appropriate sense. This is the question of Marczewski–Burstin representability (for short, MB-representability) of a given algebra of sets, or of a pair consisting of an algebra and an ideal of sets.

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Recently, the following algebras have been proved MB-representable:

- the algebra of sets with the Baire property in a Polish space [BET],
- the interval algebra on \([0, 1]\) [BBRW],
- the algebra of Borel sets in an uncountable Polish space (under \(2^\omega = \omega_1\) and \(2^{\omega_1} = \omega_2\)) [BBC],
- the algebra of clopen sets in some topological spaces [BR1].

The latest result of Baldwin [Bd] states that if a pair \((\mathcal{A}, \mathcal{I})\) of an algebra and an ideal has the hull property then \(\mathcal{A} = \mathcal{S}(\mathcal{A} \setminus \mathcal{I})\) and \(\mathcal{I} = \mathcal{S}_0(\mathcal{A} \setminus \mathcal{I})\), which (by the argument based on mutual coinitiality of generating families [BBRW]) easily recovers the results of Burstin [Bu] and Brown–Elalaoui Talibi [BET]. For other recent results concerning operations \(\mathcal{S}, \mathcal{S}_0\) and MB-representability, see [Sch], [ET], [BC]. Interestingly, Wroński [W] proved that the maximal number of different algebras which can be obtained by the successive repetitions of the operation \(\mathcal{S}(\cdot \setminus \{0\})\) is three.

In [BBC] it was shown that if \(2^\kappa = \kappa^+\) (a part of GCH) and \(|X| = \kappa \geq \omega\) then there is a non-MB-representable algebra on \(X\). In the present paper, we propose another method of constructing non-MB-representable algebras. In particular we obtain one such algebra on \(\omega\) in ZFC. Surprisingly, we show that every Boolean algebra is isomorphic to an MB-representable algebra of sets.

**Results.** Let \(X \neq \emptyset\). For an algebra \(\mathcal{A} \subseteq \mathcal{P}(X)\) we put

\[
\mathcal{H}(\mathcal{A}) = \{ A \in \mathcal{A}: (\forall B \subseteq A)B \in \mathcal{A}\};
\]

this is the ideal of sets that are hereditary in \(\mathcal{A}\).

**Theorem 1.** If an algebra \(\mathcal{A} \subseteq \mathcal{P}(X), \mathcal{A} \neq \mathcal{P}(X)\), satisfies the condition

\[
(\forall B \in \mathcal{P}(X) \setminus \mathcal{H}(\mathcal{A}))(\exists A \in \mathcal{A} \setminus \mathcal{H}(\mathcal{A}))A \subseteq B,
\]

then \(\mathcal{A}\) is not MB-representable.

**Proof.** Suppose \(\mathcal{A}\) is MB-representable and \(\mathcal{A} = \mathcal{S}(\mathcal{F})\) for some \(\mathcal{F} \subseteq \mathcal{P}(X) \setminus \{\emptyset\}\). First observe that for each \(B \in \mathcal{P}(X) \setminus \mathcal{H}(\mathcal{A})\) there is an \(F \in \mathcal{F}\) such that \(F \subseteq B\). Indeed, suppose there is a \(B \in \mathcal{P}(X) \setminus \mathcal{H}(\mathcal{A})\) with \(F \setminus B \neq \emptyset\) for each \(F \in \mathcal{F}\). By \((*)\), we pick an \(A \in \mathcal{A} \setminus \mathcal{H}(\mathcal{A})\) contained in \(B\). Since \(A \in \mathcal{S}(\mathcal{F})\) and \(F \setminus A \neq \emptyset\) for each \(F \in \mathcal{F}\), we have \(A \in \mathcal{S}_0(\mathcal{F}) \subseteq \mathcal{H}(\mathcal{S}(\mathcal{F})) = \mathcal{H}(\mathcal{A})\), a contradiction.

Next, we show that \(\mathcal{P}(X) = \mathcal{S}(\mathcal{F})\), which is also a contradiction. Let \(Y \in \mathcal{P}(X)\). Take an \(F \in \mathcal{F}\). If \(F \notin \mathcal{H}(\mathcal{A})\) then at least one of the sets \(F \cap Y\) and \(F \setminus Y\) is not in \(\mathcal{H}(\mathcal{A})\). Thus, by our first observation, there is an \(F_1 \in \mathcal{F}\) such that either \(F_1 \subseteq F \cap Y\) or \(F_1 \subseteq F \setminus Y\). Hence \(Y \in \mathcal{S}(\mathcal{F})\). If \(F \in \mathcal{H}(\mathcal{A})\) then for \(Z = Y \cap F\) we have \(Z \in \mathcal{H}(\mathcal{A}) \subseteq \mathcal{A}\). Thus there is an \(F_2 \in \mathcal{F}\) such
that either $F_2 \subseteq Z \cap F = Y \cap F$ or $F_2 \subseteq F \setminus Z = F \setminus (Y \cap F) = F \setminus Y$. Hence $Y \in S(\mathcal{F})$. ■

For an ideal $\mathcal{I} \subseteq \mathcal{P}(X)$, a family $\mathcal{F} \subseteq \mathcal{P}(X)$ is called $\mathcal{I}$-almost disjoint if $F_1 \cap F_2 \in \mathcal{I}$ for any distinct $F_1, F_2 \in \mathcal{F}$.

**Theorem 2.** Let $\mathcal{I} \subseteq \mathcal{P}(X)$ be an ideal such that $|\mathcal{P}(X) \setminus \mathcal{I}| = \kappa$, and 

$$
(\ast) \quad \text{for each } Y \in \mathcal{P}(X) \setminus \mathcal{I} \text{ there is an } \mathcal{I}\text{-almost disjoint family } \mathcal{G} \subseteq \mathcal{P}(Y) \setminus \mathcal{I} \text{ of cardinality } \kappa.
$$

Then there is an algebra $A \subseteq \mathcal{P}(X)$ such that $\mathcal{H}(A) = \mathcal{I}$ and $A$ satisfies $(\ast)$.

**Proof.** Our argument is motivated by [Ha]. Let $X_\alpha$, $\alpha < \kappa$, be an enumeration of sets from $\mathcal{P}(X) \setminus \mathcal{I}$. Denote by $A_0$ the algebra generated by $\mathcal{I}$. We will construct sequences $\langle A_\alpha \colon \alpha < \kappa \rangle$ and $\langle B_\alpha \colon \alpha < \kappa \rangle$ of sets from $\mathcal{P}(X) \setminus \mathcal{I}$ such that for all $\alpha < \kappa$ we have:

1. $A_\alpha, B_\alpha \subseteq X_\alpha$,
2. $\forall \beta < \alpha \exists B_\beta \not\in A_\alpha$, where $A_\alpha$ stands for the algebra generated by $A_0 \cup \{A_\gamma \colon \gamma < \alpha\}$.

Suppose the construction is finished; then we define $A = \bigcup_{\alpha < \kappa} A_\alpha$. Let us check that $A$ is as desired. Obviously $\mathcal{I} \subseteq \mathcal{H}(A)$. To check the converse, take any $X_\alpha \in \mathcal{P}(X) \setminus \mathcal{I}$. Thus $X_\alpha \not\in \mathcal{H}(A)$ since $B_\alpha \subseteq X_\alpha$ and $B_\alpha \not\in A$ by (1) and (2). To show $(\ast)$ consider $X_\alpha \in \mathcal{P}(X) \setminus \mathcal{I}$ and observe that $A_\alpha \subseteq X_\alpha$ and $A_\alpha \in A$.

Now, let us describe the construction. Let $\alpha < \kappa$ and assume that $A_\beta, B_\beta$ for $\beta < \alpha$ have been defined. We will find a set $A_\alpha \subseteq X_\alpha$ with $B_\beta \not\in A_{\alpha+1}$ for all $\beta < \alpha$, and a set $B_\alpha \subseteq X_\alpha$ with $B_\alpha \not\in A_{\alpha+1}$. By $(\ast)$ there is an $\mathcal{I}$-almost disjoint family $\mathcal{G} = \{C_\xi \colon \xi < \kappa\} \subseteq \mathcal{P}(X_\alpha) \setminus \mathcal{I}$. We claim that we may take for $A_\alpha$ one of the sets $C_\xi$. If not, for each $\xi < \kappa$ there is a $\beta < \alpha$, $\beta = \beta(\xi)$, such that $B_\beta$ is in the algebra generated by $A_\alpha \cup \{C_\xi\}$, that is,

$$
B_\beta = (E_\xi \cap C_\xi) \cup (F_\xi \setminus C_\xi) \cup G_\xi,
$$

where $E_\xi, F_\xi, G_\xi \in A_\alpha$ are pairwise disjoint. Recall that $A_\alpha$ is generated by $A_0 \cup \{A_\gamma \colon \gamma < \alpha\}$, which means that the quotient algebra $A_\alpha/\mathcal{I}$ is generated by $\{[A_\gamma] \colon \gamma < \alpha\}$, where $[A_\gamma]$ stands for the corresponding equivalence class. Since $|\mathcal{G}| = \kappa > \alpha$, there are two distinct ordinals $\xi, \xi' < \kappa$ such that $\beta(\xi) = \beta(\xi') := \beta$ and $[E_\xi] = [E_{\xi'}], [F_\xi] = [F_{\xi'}], [G_\xi] = [G_{\xi'}]$ for the corresponding representations of $B_\beta$. Write $C \sim D$ whenever $(C \setminus D) \cup (D \setminus C) \in \mathcal{I}$. We have

$$
B_\beta \cap E_\xi \sim B_\beta \cap E_{\xi'},
$$

$$
(B_\beta \cap E_\xi) \cap (B_\beta \cap E_{\xi'}) = (E_\xi \cap C_\xi) \cap (E_{\xi'} \cap C_{\xi'}) \subseteq (C_\xi \cap C_{\xi'}) \in \mathcal{I},
$$

which is a contradiction. Thus $A_\alpha \subseteq X_\alpha$ and $A_\alpha \in A$. Therefore $\mathcal{H}(A) = \mathcal{I}$ and $A$ satisfies $(\ast)$.
and
\[ B_\beta \cap F_\xi \sim B_\beta \cap F_{\xi'} \]
\[ \sim (B_\beta \cap F_\xi) \cup (B_\beta \cap F_{\xi'}) = (F_\xi \setminus C_\xi) \cup (F_{\xi'} \setminus C_{\xi'}) \]
\[ \sim (F_\xi \setminus C_\xi) \cup (F_\xi \setminus C_{\xi'}) = F_\xi \setminus (C_\xi \cap C_{\xi'}) \sim F_\xi. \]

Hence
\[ B_\beta = (B_\beta \cap E_\xi) \cup (B_\beta \cap F_\xi) \cup G_\xi \in A_\alpha, \]
which contradicts the assumption.

Now, if \( A_{\alpha+1} \) is defined, we choose a set \( B_\alpha \subseteq X_\alpha \) with \( B_\alpha \not\in A_{\alpha+1} \) by a similar argument (we consider an \( \mathcal{I} \)-almost disjoint subfamily of \( \mathcal{P}(X_\alpha) \setminus \mathcal{I}; \) at least one set in this family is not in \( A_{\alpha+1} \)). This ends the construction. \( \blacksquare \)

Recall the following theorem from [Ku]:

**Theorem 3.** If \( |X| = \kappa \geq \omega \) and \( 2^{<\kappa} = \kappa \), then there is a family \( \mathcal{A} \subseteq [X]^\kappa \) with \( |\mathcal{A}| = 2^\kappa \) and \( |A \cap B| < \kappa \) for all \( A, B \in \mathcal{A}, A \neq B \).

Let us apply Theorem 2 to \( X = \omega \) and \( \mathcal{I} = [\omega]^{<\omega} \). Then by Theorem 3, condition (***) is satisfied. Thus we obtain the following corollary:

**Corollary 1.** There is an algebra \( \mathcal{A} \subseteq \mathcal{P}(\omega) \) such that \( \mathcal{A} \) is not MB-representable and \( \mathcal{H}(\mathcal{A}) = [\omega]^{<\omega} \).

Next, we apply Theorem 2 to the case when \( |X| = c \), so we may assume that \( X = \mathbb{R} \). The equality \( 2^{<c} = c \) is not provable in ZFC but it is ensured by MA (Martin’s Axiom). Assume MA, and let either \( \mathcal{I} = \mathcal{M} \), the ideal of meager sets in \( \mathbb{R} \), or \( \mathcal{I} = \mathcal{N} \), the ideal of Lebesgue null sets in \( \mathbb{R} \). Thus for any \( Y \in \mathcal{P}(\mathbb{R}) \setminus \mathcal{I} \) we have \( |Y| = c \) since, under MA, if \( Y \subseteq \mathbb{R} \) and \( |Y| < c \) then \( Y \in \mathcal{I} \). Consequently, from Theorems 1–3 we derive

**Corollary 2.** Assume MA. Let \( \mathcal{I} = \mathcal{M} \) or \( \mathcal{I} = \mathcal{N} \). Then there is an algebra \( \mathcal{A} \subseteq \mathcal{P}(\mathbb{R}) \) such that \( \mathcal{A} \) is not MB-representable and \( \mathcal{H}(\mathcal{A}) = \mathcal{I} \).

Although, as we have seen, not every algebra of sets is MB-representable, we have the following positive result:

**Theorem 4.** For every Boolean algebra \( \mathcal{A} \) there is a set \( X \neq \emptyset \) and a family \( \mathcal{F} \subseteq \mathcal{P}(X) \) such that \( S(\mathcal{F}) \) is isomorphic to \( \mathcal{A} \) and \( S_0(\mathcal{F}) = \{\emptyset\} \).

**Proof.** Let \( Y \) be the Stone space of \( \mathcal{A} \), that is, the unique zero-dimensional compact Hausdorff space whose algebra of clopen sets is isomorphic to \( \mathcal{A} \). Let \( X = Y \times Y \). Define \( \mathcal{A}^+ \) to be the algebra of all subsets of \( X \) of the form \( A \times Y \), where \( A \in \mathcal{A} \). It is clear that \( \mathcal{A}^+ \) and \( \mathcal{A} \) are isomorphic. For each \( y \in Y \) and clopen \( A \) such that \( y \in A \), define
\[ F(y, A) = (\{y\} \times Y) \cup ((A \setminus \{y\}) \times (Y \setminus \{y\})). \]
Put $\mathcal{F} = \{ F(y, A) : y \in A, \ A \text{ is clopen} \}$. Note that for any $F(x, A), F(y, B) \in \mathcal{F}$ we have

$$F(x, A) \subseteq F(y, B) \Rightarrow x = y. \quad (***$$

This is because $\{x\} \times Y \subseteq F(x', A)$ if and only if $x = x'$.

First, let us show that $\mathcal{A}^+ \subseteq S(\mathcal{F})$. Given $A \times Y \in \mathcal{A}^+$ and $F(y, B) \in \mathcal{F}$, if $y \in A$ then $F(y, A \cap B) = F(y, B) \cap (A \times Y)$. If $y \notin A$ then $F(y, B \setminus A) \subseteq F(y, B)$ and $F(y, B \setminus A) \cap (A \times Y) = \emptyset$.

Now, let us prove that $S(\mathcal{F}) \subseteq \mathcal{A}^+$. So, let $B \in \mathcal{P}(X) \setminus \mathcal{A}^+$. Consider two cases.

Case 1: There is $y \in Y$ such that neither $\{y\} \times Y \subseteq B$ nor $(\{y\} \times Y) \cap B = \emptyset$. Take $F(y, Y)$. By $(***)$, any subset of $F(y, Y)$ in $\mathcal{F}$ is of the form $F(y, A)$ for $A \in \mathcal{A}$, but such a set includes $\{y\} \times Y$, so it cannot be disjoint from nor include $B$. Thus $B \notin S(\mathcal{F})$.

Case 2: Case 1 is false. Put $A = \{ y \in Y : \{y\} \times Y \subseteq B \}$. We have $B = A \times Y$. Since $B \notin \mathcal{A}^+$, the set $A$ is not clopen, that is, either $A$ or $Y \setminus A$ is not closed. Since $S(\mathcal{F})$ is an algebra of sets, we may assume without loss of generality that $A$ is not closed. Let $y \in (\text{cl}\, A) \setminus A$ and consider $F(y, Y)$. Again by $(***)$ any element of $\mathcal{F}$ included in $F(y, Y)$ is of the form $F(y, C)$ for a clopen $C \subseteq Y$ with $y \in C$. Then from $y \in (\text{cl}\, A) \cap C$ it follows that $A \cap C \neq \emptyset$. Hence $F(y, C) \cap B \neq \emptyset$. Also since $y \notin A$, we never have $F(y, C) \subseteq B$. So $B \notin S(\mathcal{F})$.

Observe that $S_0(\mathcal{F}) \subseteq \mathcal{H}(S(\mathcal{F})) = \mathcal{H}(\mathcal{A}^+) = \{\emptyset\}$. Thus $S_0(\mathcal{F}) = \{\emptyset\}. \quad \blacksquare$

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REFERENCES


123 (1914), 1525–1551.

[ET] H. Elalaoui-Talibi, On Marczewski–Burstin like characterizations of certain σ-

[Ha] R. Haydon, A nonreflexive Grothendieck space that does not contain \( l_\infty \), Israel


[Mi] A. W. Miller, Special subsets of the real line, in: Handbook of Set-Theoretic
Topology, K. Kunen and J. E. Vaughan (eds.), Elsevier, Amsterdam, 1984,
201–233.


65–73.

[R] P. Reardon, Ramsey, Lebesgue and Marczewski sets and the Baire property,

481.

[Sz] E. Szpilrajn (Marczewski), Sur une classe de fonctions de M. Sierpiński et la

[W] S. Wroński, Marczewski operation can be iterated few times, Bull. Polish Acad.

M. Balcerzak
Institute of Mathematics
Łódź Technical University
Al. Politechniki 11, I-2
90-924 Łódź, Poland
E-mail: mbalce@p.lodz.pl
and
Faculty of Mathematics
University of Łódź
Banacha 22
90-238 Łódź, Poland

A. Bartoszewicz
Instititte of Mathematics
Łódź Technical University
Al. Politechniki 11, I-2
90-924 Łódź, Poland
E-mail: arturbar@p.lodz.pl

P. Koszmider
Departamento de Matemática
Universidade de São Paulo
Caixa Postal 66281
São Paulo, SP, CEP: 05315-970, Brasil
E-mail: piotr@ime.usp.br

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