COLLOQUIUM MATHEMATICUM

VOL. 99 2004 NO. 1

STRONGLY SIMPLY CONNECTED COIL ALGEBRAS

BY

FLÁVIO U. COELHO (São Paulo), MA. I. R. MARTINS (São Paulo), and BERTHA TOMÉ (México)

Abstract. We study the simple connectedness and strong simple connectedness of the following classes of algebras: (tame) coil enlargements of tame concealed algebras and n-iterated coil enlargement algebras.

The notion of simple connectedness has arisen in the representation theory of algebras in the study of the so-called representation-finite algebras. Using covering techniques, one can reduce the study of a representation-finite algebra to a simply connected one. Although there are nice results involving simply connected representation-infinite algebras [7, 17], there is no analogue in this case of the above-cited result. In [17], Skowroński then proposed to study a class of simply connected algebras, the so-called strongly simply connected. For such a class, many promising results have appeared lately (e.g. [1–5, 17]).

Recall that a *coil* is a translation quiver which is obtained from a stable tube by a sequence of admissible operations. Such a notion first appeared in [9], where the authors also defined a class of algebras called multicoil algebras. In [12], the coil enlargements of algebras were introduced and studied, in particular, the tame coil enlargements of tame concealed algebras which are called *coil algebras*. A further step was made in [19], where Tomé iterates the process of constructing tame coil enlargements of tame concealed algebras to define the n-iterated coil enlargement algebras which we shall call *n-iterated coil algebras*.

Coil algebras appear naturally in the representation theory of algebras of polynomial growth. In fact, it was shown in [18] that if A is a polynomial growth strongly simply connected algebra, then every non-directing indecomposable finite-dimensional A-module lies in a coil of a multicoil component of the Auslander–Reiten quiver of A, and the support of such a coil is a coil algebra. This fact is crucial for some characterizations of strongly simply connected algebras of polynomial growth given in [15].

2000 *Mathematics Subject Classification*: 16G70, 16G20.

Key words and phrases: strongly simply connected, simply connected, coil enlargements.

In the present work, we are interested in the characterization of coil algebras and n-iterated coil algebras which are simply connected and strongly simply connected. Our main results can be stated as follows (see below for definitions).

THEOREM A. *Let A be a coil enlargement of a tame concealed algebra C. Then A is strongly simply connected if and only if A^- and A^+ are strongly simply connected.*

THEOREM B. *Let A be a tame coil enlargement of a tame concealed algebra C. The following conditions are equivalent*:

(a) *A is strongly simply connected.*
(b) *A^+ and A^- are strongly simply connected.*
(c) *A is strongly $\widetilde{\mathbb{A}}$-free.*
(d) *A^+ and A^- are strongly $\widetilde{\mathbb{A}}$-free.*
(e) *The orbit graph of each directed component of Γ_{A^+} and Γ_{A^-} is a tree.*
(f) *A^+ and A^- satisfy the separation and coseparation conditions.*

THEOREM C. *Let A be an n-iterated coil algebra. The following conditions are equivalent*:

(a) *A is strongly simply connected.*
(b) *A is strongly $\widetilde{\mathbb{A}}$-free.*
(c) *Each B_i is strongly $\widetilde{\mathbb{A}}$-free.*
(d) *No C_i is hereditary of type $\widetilde{\mathbb{A}}_n$.*
(e) *For each i, the orbit graph of each directed component of Γ_{B_i} is a tree.*
(f) *Each B_i is strongly simply connected.*
(g) *Each B_i satisfies the separation and coseparation conditions.*

This paper is organized as follows. After recalling some basic notions in Section 1, we devote Section 2 to some preliminary results involving the coil enlargements of algebras. In Section 3 we prove Theorem A above, while the proofs of the other two main theorems are given in Sections 4 and 5, respectively.

This work was done using an exchange project between Brazil and Mexico (CNPq-CONACyT). The first author would also like to acknowledge partial support from CNPq and FAPESP.

1. Preliminaries

1.1. Throughout this paper, k denotes an algebraically closed field. By *algebra* is meant an associative, finite-dimensional k-algebra with an identity, which we assume to be basic and, unless otherwise specified, connected.

We recall that a *quiver* Q is defined by its set of *vertices* Q_0 and its set of *arrows* Q_1. A *relation* from a point x to a point y is a linear combination $\varrho = \sum_{i=1}^{m} \lambda_i w_i$, where, for each i such that $1 \le i \le m$, λ_i is a non-zero scalar and w_i is a path of length at least two from x to y. Assume that Q has no oriented cycles. Then any set of relations generates an ideal I, called *admissible*, in the path algebra kQ. The pair (Q, I) is called a *bound quiver*. An algebra A is called *triangular* if its ordinary quiver Q_A has no oriented cycle. In this paper, we deal exclusively with triangular algebras. It is well known that, for an algebra A, there exists a surjective morphism $\nu : kQ_A \to A$ of k-algebras (induced by the choice of a set of representatives of basis vectors in the k-vector space $\operatorname{rad} A/\operatorname{rad}^2 A$) whose kernel I_ν is admissible. Thus $A \cong kQ_A/I_\nu$. The bound quiver (Q_A, I_ν) is called a *presentation* of A. An algebra $A = kQ/I$ can equivalently be considered as a locally bounded k-category, whose object class, denoted by A_0, is the set Q_0, and where the set of morphisms $A(x, y)$ from x to y is the k-vector space $kQ(x, y)$ of all linear combinations of paths in Q from x to y modulo the subspace $I(x, y) = I \cap kQ(x, y)$ (see [14]). A full subcategory B of A is called *convex* if any path in A with source and target in B lies entirely in B. An algebra A is called *strongly* $\widetilde{\mathbb{A}}$*-free* if it contains no full convex subcategory which is hereditary of type $\widetilde{\mathbb{A}}_n$.

By an A-*module* is meant a finitely generated right A-module. We denote by $\operatorname{mod} A$ their category. For $x \in A_0$, we denote by S_x the corresponding simple A-module, and by P_x (or I_x) the projective cover (or injective envelope, respectively) of S_x.

1.2. *Simple connectedness.* Let (Q, I) be a connected bound quiver. A relation $\varrho = \sum_{i=1}^{m} \lambda_i w_i \in I(x, y)$ is *minimal* if $m \ge 2$ and, for any non-empty proper subset $J \subset \{1, \dots, m\}$, we have $\sum_{j \in J} \lambda_j w_j \notin I(x, y)$. We denote by α^{-1} the formal inverse of an arrow $\alpha \in Q_1$. A *walk* in Q from x to y is a formal composition $\alpha_1^{\varepsilon_1} \alpha_2^{\varepsilon_2} \cdots \alpha_t^{\varepsilon_t}$ (where $\alpha_i \in Q_1$ and $\varepsilon_i \in \{1, -1\}$ for all i) with source x and target y. We denote by e_x the trivial path at x. Let $\sim$ be the least equivalence relation on the set of all walks in Q such that:

(a) If $\alpha : x \to y$ is an arrow, then $\alpha^{-1}\alpha \sim e_y$ and $\alpha\alpha^{-1} \sim e_x$.
(b) If $\varrho = \sum_{i=1}^{m} \lambda_i w_i$ is a minimal relation, then $w_i \sim w_j$ for all i, j.
(c) If $u \sim v$, then $wuw' \sim wvw'$ whenever these compositions make sense.

Let $x \in Q_0$ be arbitrary. The set $\pi_1(Q, I, x)$ of equivalence classes $\widetilde{u}$ of closed walks u starting and ending at x has a group structure defined by the operation $\widetilde{u} \cdot \widetilde{v} = \widetilde{uv}$. Since Q is connected, $\pi_1(Q, I, x)$ does not depend on the choice of x. We denote it by $\pi_1(Q, I)$ and call it the *fundamental group* of (Q, I).

Let (Q_A, I_ν) be a presentation of a triangular algebra A. The fundamental group $\pi_1(Q_A, I_\nu)$ depends essentially on I_ν—thus is not an invariant of A. A triangular algebra A is *simply connected* if, for any presentation (Q_A, I_ν) of A, the fundamental group $\pi_1(Q_A, I_\nu)$ is trivial [7].

1.3. *Strong simple connectedness.* Following [17], we say that an algebra A is *strongly simply connected* if it satisfies one of the following equivalent conditions:

(a) Any full convex subcategory of A is simply connected.
(b) Any full convex subcategory of A satisfies the separation condition.
(c) Any full convex subcategory of A satisfies the coseparation condition.
(d) For any full convex subcategory C of A, its first Hochchild cohomology group $\mathrm{H}^1(C)$ vanishes.

We shall, however, use a characterization by Assem–Liu of strongly simply connected algebras [4] which we include below. Let A be an algebra, and (Q_A, I) be a presentation of A. A *contour* (p, q) in Q_A from x to y consists of a pair of non-trivial paths p, q from x to y. It is *interlaced* if p, q have a common point besides x and y. It is *irreducible* if there exists no sequence of paths $p = p_0, p_1, \dots, p_m = q$ from x to y such that each of the contours (p_i, p_{i+1}) is interlaced. Let C be a simple cycle which is not a contour, and let $\sigma(C)$ denote the number of sources in C. Then C is *reducible* if there exist x, y on C and a path $p : x \to \cdots \to y$ in Q_A such that if w_1 and w_2 denote the subwalks of C from x to y (so that $C = w_1 w_2^{-1}$), then $w_1 p^{-1}$ and $w_2 p^{-1}$ are cycles and $\sigma(w_1 p^{-1}) < \sigma(C)$, $\sigma(w_2 p^{-1}) < \sigma(C)$. A cycle C is *irreducible* if either it is an irreducible contour, or it is not a contour, but it is not reducible in the above sense. Finally, a contour (p, q) from x to y is *naturally contractible* in (Q_A, I) if there exists a sequence of paths $p = p_0, p_1, \dots, p_m = q$ in Q_A such that, for each i, the paths p_i and p_{i+1} have subpaths q_i and q_{i+1}, respectively, which are involved in the same minimal relation in (Q_A, I).

THEOREM ([4]). *An algebra A is strongly simply connected if and only if, for any presentation (Q_A, I) of A, any irreducible cycle in Q_A is an irreducible contour, and any irreducible contour in Q_A is naturally contractible in (Q_A, I).*

1.4. *Auslander–Reiten quivers.* We denote by $\mathrm{D} = \mathrm{Hom}_k(-, k)$ the standard duality between $\operatorname{mod} A$ and $\operatorname{mod} A^{\mathrm{op}}$, and by $\tau = \mathrm{DTr}$ and $\tau^{-1} = \mathrm{TrD}$ the Auslander–Reiten translations in $\operatorname{mod} A$. The Auslander–Reiten quiver of A is denoted by Γ_A (for details, see [13, 16]). A component Γ of Γ_A is called *directed* if, for any indecomposable module M in Γ, there exists no sequence $M = M_0 \xrightarrow{f_1} M_1 \xrightarrow{f_2} \cdots \xrightarrow{f_t} M_t = M$ of non-zero non-isomorphisms between indecomposable A-modules. It is called *standard* if Γ is equivalent to the mesh category $k(\Gamma)$ of Γ (see [14], [16]). We finish this section by recalling the notion of a weakly separating family of components of Γ_A.

DEFINITION. Let A be an algebra. A family $\mathcal{T} = (\mathcal{T}_i)_{i\in I}$ of components of Γ_A is called a *weakly separating family* in $\operatorname{mod} A$ if the indecomposable A-modules not in $\mathcal{T}$ split into two classes $\mathcal{P}$ and $\mathcal{Q}$ such that:

(i) the components of $(\mathcal{T}_i)_{i\in I}$ are standard and pairwise orthogonal;
(ii) $\operatorname{Hom}_A(\mathcal{Q},\mathcal{P}) = \operatorname{Hom}_A(\mathcal{Q},\mathcal{T}) = \operatorname{Hom}_A(\mathcal{T},\mathcal{P}) = 0$;
(iii) any morphism from $\mathcal{P}$ to $\mathcal{Q}$ factors through $\operatorname{add}\mathcal{T}$.

2. Coil enlargements of algebras

2.1. *Admissible operations.* We now recall the notion of admissible operations introduced in [9]. Let A be an algebra and let Γ be a standard component of Γ_A. For an indecomposable module X in Γ, called *pivot*, the following three admissible operations are defined. In each case, we will get a modified algebra A' of A and a modified component Γ' of Γ.

(ad1) Suppose the support of $\operatorname{Hom}_A(X,-)|_\Gamma$ is of the form

$$X = X_0 \to X_1 \to X_2 \to \cdots$$

In this case, X is called an (ad1)-*pivot* and we set $A' = (A\times D)[X\oplus Y_1]$, where D is the full $t\times t$ lower triangular matrix algebra, and Y is the unique indecomposable projective-injective D-module. The component Γ' is obtained in this case from Γ and Γ_D by inserting a rectangle consisting of the modules $Z_{ij} = (k, X_i\oplus Y_j, (1,1)^t)$ for $i\ge 0$, $1\le j\le t$, and $X_i' = (k, X_i, 1)$ for $i\ge 0$, where Y_j, $1\le j\le t$, denote the indecomposable injective D-modules. In case $t=0$, we just set $A' = A[X]$, and the rectangle above reduces to the ray formed by the modules of the form X_i'.

(ad2) Suppose the support of $\operatorname{Hom}_A(X,-)|_\Gamma$ is of the form

$$Y_t \leftarrow \cdots \leftarrow Y_1 \leftarrow X = X_0 \to X_1 \to X_2 \to \cdots$$

with $t\ge 1$ (so that X is injective). Here, we call X an (ad2)-*pivot* and we set $A' = A[X]$. The component Γ' is obtained by inserting in Γ a rectangle consisting of the modules $Z_{ij} = (k, X_i\oplus Y_j, (1,1)^t)$ for $i\ge 1$, $1\le j\le t$, and $X_i' = (k, X_i, 1)$ for $i\ge 0$.

(ad3) Suppose the support of $\operatorname{Hom}_A(X,-)|_\Gamma$ is of the form

$$\begin{array}{ccccccccccccc}
 & Y_1 & \to & Y_2 & \to & \cdots & \to & Y_t & & & & & \\
 & \uparrow & & \uparrow & & & & \uparrow & & & & & \\
X = & X_0 & \to & X_1 & \to & \cdots & \to & X_{t-1} & \to & X_t & \to & X_{t+1} & \to \ \cdots
\end{array}$$

with $t\ge 2$ (so that X_{t-1} is injective). In this case, we call X an (ad3)-*pivot* and we set $A' = A[X]$. Here, Γ' is obtained by inserting in Γ a rectangle consisting of the modules $Z_{ij} = (k, X_i \oplus Y_j, (1,1)^t)$ for $i \geq 1$, $1 \leq j \leq i \leq t$ and $i > t$, $1 \leq j \leq t$, and $X'_i = (k, X_i, 1)$ for $i \geq 0$.

The dual operations (ad1*), (ad2*) and (ad3*) are also called *admissible*.

2.2. *Coils*

Definition. A translation quiver Γ is called a *coil* if there exists a sequence of translation quivers $\Gamma_0, \Gamma_1, \ldots, \Gamma_m = \Gamma$ such that Γ_0 is a stable tube and, for each $0 \leq i < m$, Γ_{i+1} is obtained from Γ_i by an admissible operation.

Definition. Let A be an algebra, and $\mathcal{T}$ be a weakly separating family of stable tubes of Γ_A. An algebra B is called a *coil enlargement* of A through modules in $\mathcal{T}$ provided there is a finite sequence of algebras $A = A_0, A_1, \ldots, A_m = B$ such that, for each $0 \leq j < m$, A_{j+1} is obtained from A_j by an admissible operation with pivot either in a stable tube of $\mathcal{T}$ or in a coil of Γ_{A_j}, obtained from a stable tube of $\mathcal{T}$ by means of admissible operations done so far. The sequence $A = A_0, A_1, \ldots, A_m = B$ is then called an *admissible sequence*.

2.3. The following result is taken (not literally) from [12].

Theorem. *Let A be a coil enlargement of a tame concealed algebra C. Then*:

(a) *There is a unique maximal branch extension A^+ of C inside A which is full and convex in A, and A is obtained from A^+ by a sequence of admissible operations of type* (ad1*), (ad2*) *or* (ad3*).
(b) *There is a unique maximal branch coextension A^- of C inside A which is full and convex in A, and A is obtained from A^- by a sequence of admissible operations of type* (ad1), (ad2) *or* (ad3).
(c) $\operatorname{ind} A = \mathcal{P} \vee \mathcal{T} \vee \mathcal{I}$, *where $\mathcal{T}$ is a weakly separating family of coils separating $\mathcal{P}$ from $\mathcal{I}$, $\mathcal{P}$ consists of A^--modules and $\mathcal{I}$ of A^+-modules.*

2.4. To prove Theorem A in the next section, we shall require the following lemmata. For a given vertex x in a quiver, we indicate by $x^{\rightarrow}$ the set of all arrows of the quiver starting at x.

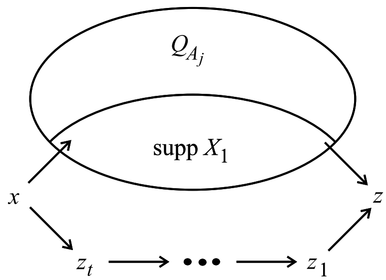
Lemma. *Let A be a coil enlargement of a tame concealed algebra C and $C = A_0, A_1, \ldots, A_m = A^-, A_{m+1}, \ldots, A_n = A$ be an admissible sequence for A. Let $j \geq m$, $X \in \operatorname{ind} A_j$ be an* (ad2) *or* (ad3)*-pivot and $A_{j+1} = A_j[X]$. If x is the corresponding extension point then there is a unique vertex $z \in A^- \setminus A^+$ that satisfies*:

(i) *Each $\alpha \in x^{\rightarrow}$ is the starting point of a non-zero path $\omega_\alpha \in A(x, z)$.*
(ii) *There are at least two different arrows in $x^{\rightarrow}$. Moreover, if $\alpha, \beta \in x^{\rightarrow}$, and $\alpha \neq \beta$, then $\omega_\alpha - \omega_\beta \in I$.*

Proof. Assume first that X is an (ad2)-pivot and that $S(X)$ is

$$Y_t \leftarrow \cdots \leftarrow Y_1 \leftarrow X = X_0 \rightarrow X_1 \rightarrow X_2 \rightarrow \cdots$$

where $t \geq 1$, and $X, Y_1, \ldots, Y_t$ are injectives in a coil of $\mathit{\Gamma}_{A_j}$. Let $z, z_1, \ldots, z_t \in A^-$ be such that $X = I_z$ and $Y_i = I_{z_i}$ for $1 \leq i \leq t$. Since $X = \operatorname{rad} P_x$, there must be a non-zero path from x to each vertex y which is a predecessor of z. Hence, each $\alpha \in x^{\rightarrow}$ is the starting arrow of a non-zero path from x to z, and there are at least two arrows in $x^{\rightarrow}$: one from x to z_t and one from x to a point in $\operatorname{supp} X_1$. Moreover, since $P_x(z) = X(z) = k$, all paths are congruent modulo I_{j+1}. The bound quiver of A_{j+1} has the form



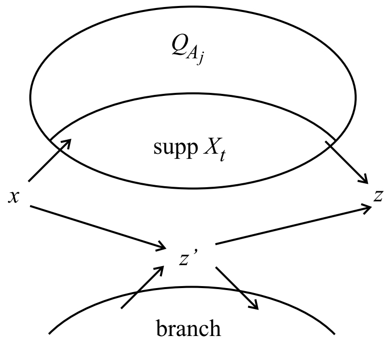
with $A_{j+1}(x, z)$ one-dimensional. By [12, Section 3], $z \in A^- \setminus A^+$, $x \in A^+ \setminus A^-$, and $z_1, \ldots, z_t \in A^- \cap A^+$. Assume now that X is an (ad3)-pivot and that $S(X)$ is

$$\begin{array}{ccccccccccccc}
 & Y_1 & \rightarrow & Y_2 & \rightarrow & \cdots & \rightarrow & Y_t & & & & & \\
 & \uparrow & & \uparrow & & & & \uparrow & & & & & \\
X = & X_0 & \rightarrow & X_1 & \rightarrow & \cdots & \rightarrow & X_{t-1} & \rightarrow & X_t & \rightarrow & X_{t+1} & \rightarrow \ \cdots
\end{array}$$

where $t \geq 2$, and X_{t-1}, Y_t are injectives in a coil of $\mathit{\Gamma}_{A_j}$. Let $z, z' \in A^-$ be such that $X_{t-1} = I_z$ and $Y_t = I_{z'}$. Then X is the indecomposable A_j-module given by

$$X(y) = \begin{cases} 0 & \text{if } y < z', \\ k & \text{if } z' < y, \\ X_{t-1}(y) & \text{in any other case.} \end{cases}$$

Since $X = \operatorname{rad} P_x$, there must be a non-zero path from x to each vertex y which is a predecessor of z, but those which are predecessors of z'. Hence each $\alpha \in x^{\rightarrow}$ is the starting arrow of a non-zero path from x to z, and there are at least two arrows in $x^{\rightarrow}$: one from x to z' and one from x to a point in $\operatorname{supp} X_t$. Moreover, since $P_x(z) = X_{t-1}(z) = k$, all paths from x to z are congruent modulo I_{j+1}. The bound quiver of A_{j+1} has the form



with $A_{j+1}(x,z)$ one-dimensional. By [12, Section 3], $z \in A^- \setminus A^+$, $x \in A^+ \setminus A^-$, z' and the vertices of the branch belong to $A^- \cap A^+$. This finishes the proof. ■

2.5. Let b be a coextension point of C and B be the branch with root vertex b. We denote the arrows in B by γ and δ, where $\gamma\delta = 0$. If $z \in B$ is the end point of an arrow δ (respectively, the starting point of an arrow γ), let $\sigma(z)$ (respectively, $\tau(z)$) be the vertex $y \in B$ for which the length of a path $y \xrightarrow{\delta} \cdots \xrightarrow{\delta} z$ (respectively, $z \xrightarrow{\gamma} \cdots \xrightarrow{\gamma} y$) is the largest possible. Following [16], we call the maximal subbranch of B of the form $b \xrightarrow{\delta} \cdots \xrightarrow{\delta} y$ the *factor space branch* of b. Note that if $|{\to}z| = 2$ and z does not lie on the factor space branch of b, then $\tau(\sigma(z))$ is defined.

LEMMA. *Let A be a coil enlargement of a tame concealed algebra C, and $C = A_0, A_1, \ldots, A_m = A^-, A_{m+1}, \ldots, A_n = A$ be an admissible sequence for A. Each operation of type* (ad2) *or* (ad3) *in the corresponding sequence of admissible operations yields a pair of vertices (x,z) with $x \in A^+ \setminus A^-$ and $z \in A^- \setminus A^+$ that satisfy*:

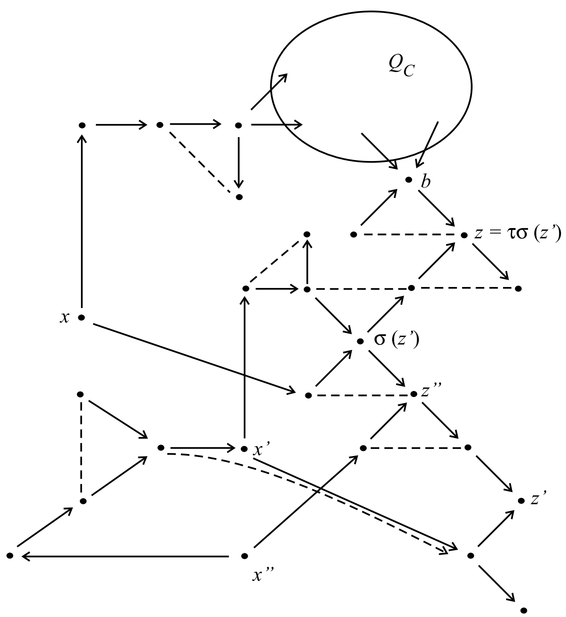
(a) *If (x',z') is such a pair and z' does not lie on the factor space branch of a coextension vertex of C, then there must be a previous operation of type* (ad2) *in the sequence for which the associated pair of vertices is $(x, \tau\sigma(z'))$.*

(b) *If (x',z') and (x'',z'') are two such pairs and there is a path in Q_A from x'' to x', then z' and z'' belong to the same coextension branch B and there is a path $z'' \xrightarrow{\delta} \cdots \xrightarrow{\delta} z'$ in B.*

Proof. (a) Let $\mathcal{C}$ be the coil of Γ_A containing the pivot X of the operation that yields (x',z'). Then $\mathcal{C}$ contains a ray passing through X. On the other hand, on the coinserted tube $\mathcal{T}$ of Γ_{A^-} which is transformed in $\mathcal{C}$ by the sequence of admissible operations, there is a sectional path from $I_{z'}$ to $S_{\sigma(z')}$, but there is no ray passing through both. The only way to create that ray is by the application of an operation (ad2) with pivot $I_{\tau\sigma(z')}$.

(b) If there is a path in Q_A from x'' to x', the operation that yields (x', z') must have been applied before the one that yields (x'', z''). Moreover, $P_{x'}$ and $P_{x''}$ lie on the same coil $\mathcal{C}$ of Γ_A and there is a sectional path from $P_{x'}$ to $P_{x''}$ in $\mathcal{C}$. Now, since either $P_{x'} = I_{z'}$ or $I_{z'}$ lies on the sectional path from $P_{x'}$ to $P_{x''}$, and either $P_{x''} = I_{z''}$ or there is a sectional path from $P_{x''}$ to $I_{z''}$, we infer that there is a sectional path from $I_{z'}$ to $I_{z''}$ in $\mathcal{C}$. Consequently, there is a non-zero path from z'' to z' in Q_A. Since these vertices lie on coextension branches and there is no path between them, z' and z'' belong to the same coextension branch. As $|^{\to}z''| = 2$, the only possible path is of the form $z'' \xrightarrow{\delta} \cdots \xrightarrow{\delta} z'$. ∎

The following diagram illustrates the lemma above.



2.6. Lemma. *Let A be a coil enlargement of a tame concealed algebra C, and $C = A_0, A_1, \ldots, A_m = A^-, A_{m+1}, \ldots, A_n = A$ be an admissible sequence for A. Let $j > m+1$ and $X \in \operatorname{ind} A_j$ be an* (ad1)*-pivot such that $X|_C = 0$. Then X is uniserial.*

Proof. A case by case inspection shows that X is either simple or $\operatorname{supp} X$ is a linearly ordered quiver of type $\mathbb{A}_n$. ∎

2.7. Corollary. *Let A be a coil enlargement of a tame concealed algebra C, and $C = A_0, A_1, \ldots, A_m = A^-, A_{m+1}, \ldots, A_n = A$ be an admissible sequence for A. The only vertices in $A^- \setminus A^+$ which can be sinks of a simple cycle in Q_A are the coextension vertices of C and the vertices z that appear* *in a pair* (x, z) *associated to an operation of type* (ad2) *or* (ad3) *in the corresponding sequence of admissible operations that leads from* A^- *to* A *and which lie on the factor space branch of a coextension vertex of* C.

Proof. Let $z \in A^- \setminus A^+$ be a sink of a simple (reduced) cycle w in Q_A. Then z lies on a coextension branch B and $|{}^{\to}z| = 2$.

Assume that w has t sinks $y_1, \ldots, y_t$, where $y_1 = z$, and t sources $x_1, \ldots, x_t$. For $1 \leq i \leq t$, let w_i be the path from x_i to y_i and w'_i the path from x_i to y_{i+1} (where $y_{t+1} = y_1$). Assume first that $y_1 = z$ is not the only sink in w belonging to B. Since w cannot lie entirely inside B and the paths in Q_A starting at B end also in B, there must be at least two sources in w which do not belong to B. Let i (respectively, j) be the least (respectively, the largest) index such that $x_i \notin B$ (respectively, $x_j \notin B$). Then x_i and x_j belong to C or are extension vertices corresponding to the operations of type (ad1), (ad2) or (ad3) in the sequence of admissible operations that leads from A^- to A. If x_i (or x_j) is an extension vertex corresponding to an operation of type (ad1) then w_i (or w'_j) passes through the root vertex b of B. Indeed, otherwise x_i (or x_j) cannot be a source in w, for the other path starting at x_i (or x_j), if it exists, ends inside the extension branch with root vertex x_i (or x_j). Therefore we may assume that x_j is an extension vertex corresponding to an operation (ad2) or (ad3), and the subwalk $w_i w'^{-1}_{i-1} \cdots w_1 w'^{-1}_t w_t \cdots w_{j+1} w'^{-1}_j$ of w may be any of those shown in the figure below. But then either w_{j+1} intersects w_j, or else it intersects w'_{j-1} in several vertices, including x_{j+1} (see the figure opposite). This contradicts the fact that w is a simple cycle.

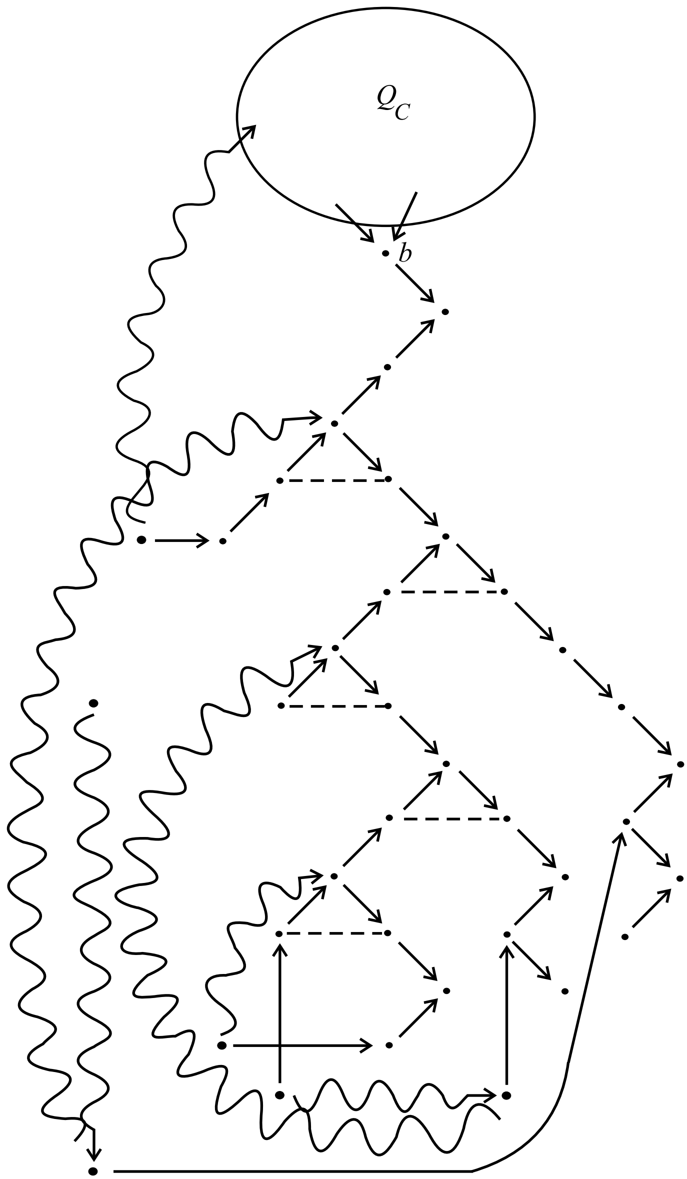
Hence $y_1 = z$ is the only sink in w belonging to B, $x_i = x_1$, and $x_j = x_t$. If y_1 does not lie on the factor space branch of b, then both x_1 and x_t must be extension vertices corresponding to operations (ad2) or (ad3). But then again either w_1 intersects w_t in x_1, or else it intersects w'_{t-1} in several vertices including x_1, thus contradicting the simplicity of w. This shows that y_1 lies on the factor space branch of b, and there remain only two possible cases: either w_1 and w'_t pass through b, or w_1 passes through b and w'_t does not. In the first case, $y_1 = b$, and in the second case, x_t is an extension point corresponding to an operation (ad2) or (ad3) which has (x_t, y_1) as its associated pair of vertices. ■

3. Strong simple connectedness of coil enlargements of tame concealed algebras

3.1. We shall prove one of our main results.

THEOREM A. *Let* A *be a coil enlargement of a tame concealed algebra* C. *Then* A *is strongly simply connected if and only if* A^- *and* A^+ *are strongly simply connected.*

Proof. The necessity is clear. To prove the sufficiency, observe first that by [12, (3.5)], we know that A can be obtained from A^- by a finite sequence of admissible operations of type (ad1), (ad2) and (ad3). We proceed by induction on the number of these operations. Let $C = A_0, A_1, \ldots, A_m = A^-, A_{m+1}, \ldots, A_n = A$ be an admissible sequence for A, and assume that $A = A_{m+1}$. Let x_0 be the corresponding extension point (x_0 is the root of a branch of A^+ if the operation performed is (ad1)).



Assume that A is not strongly simply connected. Then, by [4, Theorem 1.3], the bound quiver (Q, I) of A contains an irreducible cycle which is not a contour, or an irreducible contour which is not naturally contractible. As in [2, (4.3)], we consider two cases.

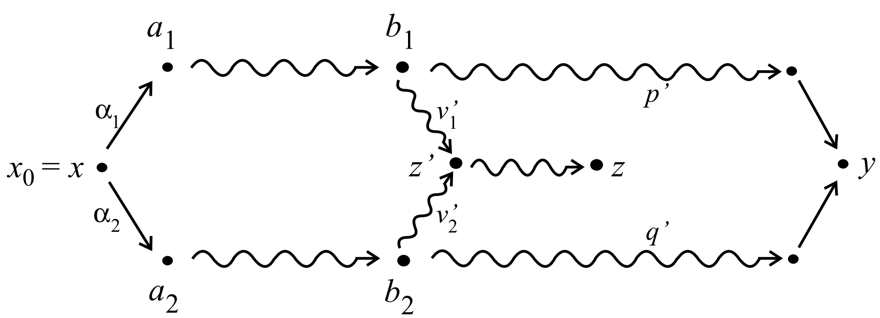
CASE 1. Let $w = (p, q)$ be an irreducible contour from x to y which is not naturally contractible. Since w lies neither inside A^- nor inside A^+, and since A^- is closed under successors and A^+ under predecessors, it follows that $x = x_0$ and $y \in A^- \setminus A^+$.

Let $\alpha_1 \colon x \to a_1$ and $\alpha_2 \colon x \to a_2$ be the starting arrows of p and q, respectively. Since $x = x_0$ is separating, by [6, (2.2)], there exists a minimal relation

$$\lambda_1 \alpha_1 v_1 + \lambda_2 \alpha_2 v_2 + \sum_{j \geq 3} \lambda_j u_j$$

from x to a vertex $z \in A^-$, with $0 \neq \lambda_i \in k$. If the admissible operation performed is (ad1), then $z \in C$. Since $y \notin A^+$, it follows that $z \neq x$ and there is no path from y to z. Moreover, there is no path from z to y because w is irreducible. If the operation performed is (ad2) or (ad3), we can take z as the vertex in $A^- \setminus A^+$ described in (2.4). Since w is not naturally contractible, it follows that $z \neq y$ and there is no path from y to z. Also, there is no path from z to y because w is irreducible.

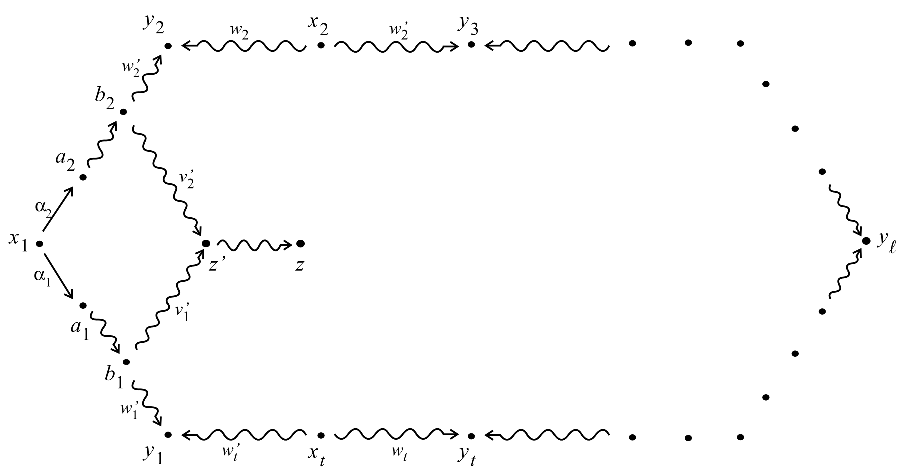
Let b_1 (or b_2) be the last common vertex between v_1 and p (or v_2 and q, respectively) and z' be the first common vertex between v_1 and v_2. Denote by v_1' (or v_2') the subpath of v_1 (or v_2) from b_1 (or b_2, respectively) to z', and by p' (or q') the subpath of p (or q) from b_1 (or b_2) to y.



The walk $w' = v_2'^{-1} q' p'^{-1} v_1'$ is a cycle. Indeed, there is no intersection between p' and q', nor is there one between v_1' and v_2', and the existence of an intersection between $v_2' v_1'^{-1}$ and $p' q'^{-1}$ would contradict the irreducibility of w. Moreover, w' is irreducible because w is. Finally, w' is not a contour because it has two different sinks z' and y. Since w' lies inside A^-, we obtain a contradiction.

CASE 2. Let w be an irreducible cycle which is not a contour. Denote by $x_1,\ldots,x_t$ its sources, by $y_1,\ldots,y_t$ its sinks and for $1\le i\le t$, by w_i the path from x_i to y_i, and by w_i' the path from x_i to y_{i+1} (where $y_{t+1}=y_1$). Since w lies neither inside A^- nor inside A^+, we may assume that $x_1=x_0$ and $y_l\in A^-\setminus A^+$ (where $1\le l\le t$). As before, if $\alpha_1\colon x_1\to a_1$ and $\alpha_2\colon x_1\to a_2$ are the starting arrows of w_1 and w_1', respectively, there exists a minimal relation $\lambda_1\alpha_1v_1+\lambda_2\alpha_2v_2+\sum_{j\ge 3}\lambda_ju_j$ from x_1 to a vertex $z\in A^-$, with $0\ne\lambda_i\in k$. If the admissible operation performed is (ad1), then $z\in C$ and hence $z\ne y_l$ and there is no path from y_l to z. Also, by the irreducibility of w, there is no path from z to y_l. If the operation performed is (ad2) or (ad3), then z can be taken as the vertex in $A^-\setminus A^+$ described in (2.4). Since w is irreducible, it follows that $z\ne y_l$, there is no path from z to y_l and none from y_l to z (otherwise, y_l is the root of the branch in which z lies and there is a path from x_1 to y_l that reduces the cycle).

Let b_1 (or b_2) be the last common vertex of v_1 (or v_2) and w_1 (or w_1', respectively), and z' be the first common vertex of v_1 and v_2. Denote by v_1' (or v_2') the subpath of v_1 (or v_2) from b_1 (or b_2, respectively) to z', and by w_1'' (or w_2'') the subpath of w_1 (or w_1') from b_1 (or b_2) to y_1 (or y_2, respectively).



Note that either $t\ge 3$ or, if $t=2$, then $b_1\ne y_1$ or $b_2\ne y_2$. Indeed, if $t=2$, then one of y_1 or y_2 is y_l and we have just shown that there is no path from y_l to z. Consider now the walk $w'=w_1''w_t'^{-1}w_t\cdots w_2w_2''^{-1}v_2'v_1'^{-1}$. This walk is a cycle: the walk $w_1''w_t'^{-1}w_t\cdots w_2w_2''^{-1}$ has no self-intersection because it is a subwalk of w, the walk $v_2'v_1'^{-1}$ has no self-intersection by definition, and these two do not intersect because w is irreducible. Moreover, w' is irreducible because w is. Finally, it is not a contour because it has at least two sinks (z' and y_l). Since w' lies inside A^-, we get a contradiction.

The contradictions obtained in both cases show that A_{m+1} is strongly simply connected.

Let $k > m$, $A = A_{k+1}$ and assume that A_k is strongly simply connected, but A is not. Let x_0 be the extension point of A_k and $X \in \operatorname{ind} A_k$ be the pivot of the admissible operation. If X is an (ad1)-pivot, then $X|_C \neq 0$. Indeed, otherwise, by (2.6), X would be uniserial, and since A is obtained from A_k by the one-point extension by X followed by several one-point coextensions by simple modules, by [4, (3.4)], A would be strongly simply connected, contrary to our assumption.

As before, the bound quiver (Q, I) of A contains an irreducible cycle which is not a contour or an irreducible contour which is not naturally contractible.

Case 1. Let $w = (p, q)$ be an irreducible contour from x to y which is not naturally contractible. Since w lies neither inside A_k nor inside A^+, we deduce that $x = x_0$ and $y \in A^- \setminus A^+$. We can then proceed as in the first step of induction to obtain an irreducible cycle which is not a contour lying inside A_k, thus contradicting our assumption.

Case 2. Let w be an irreducible cycle which is not a contour. Denote by $x_1, \ldots, x_t$ its sources, by $y_1, \ldots, y_t$ its sinks, and for $1 \leq i \leq t$, by w_i the path from x_i to y_i, and by w_i' the path from x_i to y_{i+1} (where $y_{t+1} = y_1$). As above, we may assume that $x_1 = x_0$, and $y_l \in A^- \setminus A^+$ (where $1 \leq l \leq t$). Further, if $\alpha_1 \colon x_1 \to a_1$ and $\alpha_2 \colon x_1 \to a_2$ are the starting arrows of w_1 and w_1', respectively, there is a minimal relation $\lambda_1\alpha_1 v_1 + \lambda_2\alpha_2 v_2 + \sum_{j\geq 3} \lambda_j u_j$, from x_1 to a vertex $z \in A_k$, with $0 \neq \lambda_i \in k$.

If the admissible operation applied is (ad1), then $z \in C$ and we can proceed as in the first step of induction. If the operation applied is (ad2) or (ad3), then z is the vertex in $A^- \setminus A^+$ described in (2.4). Again the irreducibility of w shows that $z \neq y_l$, there is no path from z to y_l and none from y_l to z in case y_l is a coextension point of C. If y_l is not a coextension point of C, then by (2.7) there is no path from y_l to z. Indeed, otherwise x_0 is not constructible since $S(X)$ does not have the proper shape. We can now proceed as in the first step of induction to obtain an irreducible cycle which is not a contour lying inside A_k, contrary to our assumption.

Again, both contradictions show that A_{k+1} is strongly simply connected, thus completing our proof. ■

4. Strong simple connectedness of coil algebras

4.1. In our next result, we characterize the tame coil enlargements of tame concealed algebras (called coil algebras) which are strongly simply connected.

THEOREM B. *Let A be a tame coil enlargement of a tame concealed algebra C. The following conditions are equivalent*:

(a) *A is strongly simply connected.*
(b) *A^+ and A^- are strongly simply connected.*
(c) *A is strongly $\widetilde{\mathbb{A}}$-free.*
(d) *A^+ and A^- are strongly $\widetilde{\mathbb{A}}$-free.*
(e) *The orbit graph of each directed component of Γ_{A^+} and Γ_{A^-} is a tree.*
(f) *A^+ and A^- satisfy the separation and coseparation conditions.*

Proof. By Theorem A, (a) and (b) are equivalent. Clearly, (a) implies (c), and (c) implies (d). Since A^+ and A^- are either tilted or cotilted of Euclidean type or tubular, the equivalence of (b), (d) and (e) follows from [3, (2.3)] and [1, (1.7)]. By [17, (4.1)], (a) implies (f), therefore it remains to show that (f) implies (b). If A^+ (or A^-) is tubular, the separation condition for A^+ and $(A^+)^{\mathrm{op}}$ (or A^- and $(A^-)^{\mathrm{op}}$) implies that A^+ (or A^-) is strongly simply connected according to [1, (1.7)]. If A^+ (or A^-) is domestic, the separation condition for A^+ (or $(A^-)^{\mathrm{op}}$) implies that A^+ (or A^-) is strongly $\widetilde{\mathbb{A}}$-free according to [1, (1.6)]. Then, by [3, (2.3)], A^+ (or A^-) is strongly simply connected. ∎

4.2. COROLLARY. *For a coil algebra A, the following conditions are equivalent*:

(a) *A is strongly simply connected.*
(b) *A is simply connected and strongly $\widetilde{\mathbb{A}}$-free.*
(c) $\mathrm{H}^1(A) = 0$ *and A is strongly $\widetilde{\mathbb{A}}$-free.*

Proof. This follows directly from Theorem B and [17]. ∎

5. Strong simple connectedness of iterated coil enlargements

5.1. Following [19], any tame branch extension or coextension of a tame concealed algebra is called a 0-*iterated coil enlargement.* Let B_0 be a tame branch coextension of a tame concealed algebra C_0. Then we can write

$$\operatorname{ind} B_0 = \mathcal{P}_0 \vee \mathcal{T}_0 \vee \mathcal{I}_0,$$

where $\mathcal{I}_0$ is the preinjective component of Γ_{B_0}, and $\mathcal{T}_0$ is a separating tubular family containing injectives. Applying admissible operations (ad1), (ad2) and (ad3) we insert projectives in the coinserted and stable tubes of $\mathcal{T}_0$. We obtain a coil enlargement A_1 of C_0 with $(A_1)^- = B_0$. If the branch extension $B_1 = (A_1)^+$ of C_0 is tame, we call $\Lambda_1 = A_1$ a 1-*iterated coil enlargement.*

If B_1 is domestic, the iteration process stops. On the contrary, if B_1 is a tubular algebra, then it is a branch coextension of a tame concealed algebra C_1, and we can write

$$\operatorname{ind} A_1 = \mathcal{P}_1 \vee \mathcal{T}_1 \vee \mathcal{I}_1,$$

where $\mathcal{I}_1$ is the preinjective component of Γ_{A_1}, and $\mathcal{T}_1$ is a separating tubular family containing injectives. (Indeed, $\mathcal{T}_1$ and $\mathcal{I}_1$ are, respectively, the last separating tubular family and the preinjective component of Γ_{B_1}; see [19, (3.1)]). Applying admissible operations (ad1), (ad2) and (ad3) we insert projectives in the coinserted and stable tubes of $\mathcal{T}_1$. We obtain a coil enlargement A_2 of C_1 with $(A_2)^- = (A_1)^+ = B_1$. If the branch extension $B_2 = (A_2)^+$ of C_1 is tame, we call the algebra Λ_2 obtained from $\Lambda_1 = A_1$ by inserting projectives in the tubes of $\mathcal{T}_1$ a 2*-iterated coil enlargement*. Again, if B_2 is domestic, the iteration process stops, and if B_2 is a tubular algebra, we are able to iterate the process once more.

Inductively, assume that Λ_{n-1} is an $(n-1)$-iterated coil enlargement in which we are able to iterate the process once more. This means that there is a coil enlargement A_{n-1} of a tame concealed algebra C_{n-2} such that the branch extension $B_{n-1} = (A_{n-1})^+$ of C_{n-2} is a tubular algebra. Hence B_{n-1} is a branch coextension of a tame concealed algebra C_{n-1}, and we can write

$$\operatorname{ind} \Lambda_{n-1} = \mathcal{P}_{n-1} \vee \mathcal{T}_{n-1} \vee \mathcal{I}_{n-1},$$

where $\mathcal{I}_{n-1}$ is the preinjective component of $\Gamma_{\Lambda_{n-1}}$ and $\mathcal{T}_{n-1}$ is a separating tubular family containing injectives. Applying admissible operations (ad1), (ad2) and (ad3) we insert projectives in the coinserted and stable tubes of $\mathcal{T}_{n-1}$. We obtain a coil enlargement A_n of C_{n-1} with $(A_n)^- = (A_{n-1})^+ = B_{n-1}$. If the branch extension $B_n = (A_n)^+$ of C_{n-1} is tame, we call the algebra Λ_n obtained from Λ_{n-1} by inserting projectives in the tubes of $\mathcal{T}_{n-1}$ an n*-iterated coil enlargement*.

5.2. By [19, (3.3)], the n-iterated coil enlargements are tame of polynomial growth. Hence we shall call them simply n-iterated coil algebras. We borrow the following example from [19].

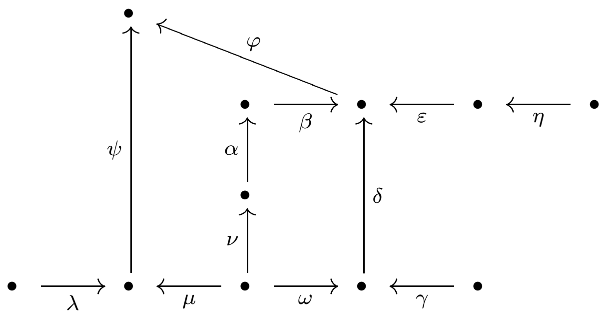
Example. Let Λ_i, for $1 \le i \le 4$, be the k-algebras given respectively by the following quivers with relations:

Λ_0

$$\begin{tikzcd}
\bullet \arrow[r, "\lambda"] & \bullet \arrow[r, "\psi"] & \bullet & & & \\
& \bullet \arrow[r, "\alpha"'] & \bullet \arrow[r, "\beta"'] & \bullet \arrow[ul, "\varphi"'] & \bullet \arrow[l, "\varepsilon"'] & \bullet \arrow[l, "\eta"'] \\
& & & \bullet \arrow[u, "\delta"'] & & \\
& & & \bullet \arrow[u, "\gamma"'] & &
\end{tikzcd}$$

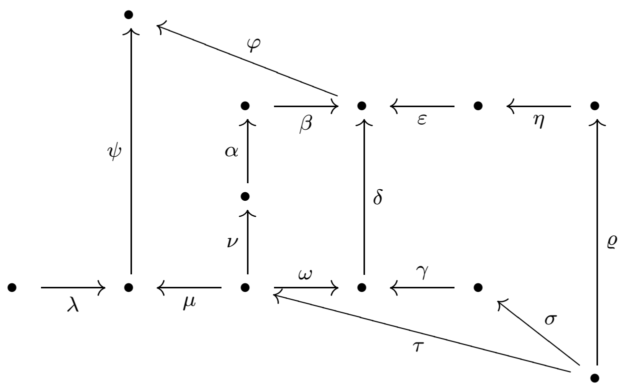
$\gamma\delta\varphi = 0 \qquad \varepsilon\varphi = 0$

Λ_1



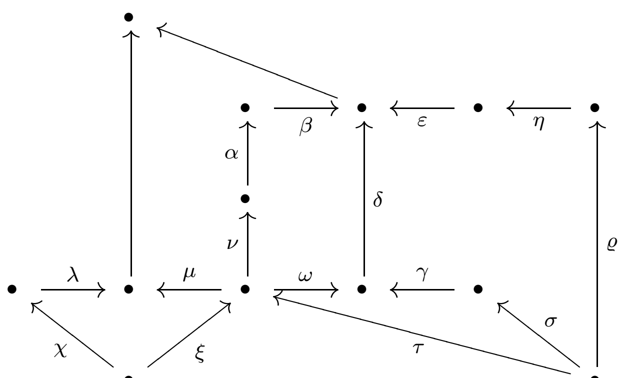
$\gamma\delta\varphi = 0 \qquad \varepsilon\varphi = 0 \qquad \nu\alpha\beta = \omega\delta \qquad \mu\psi = \nu\alpha\beta\varphi$

Λ_2



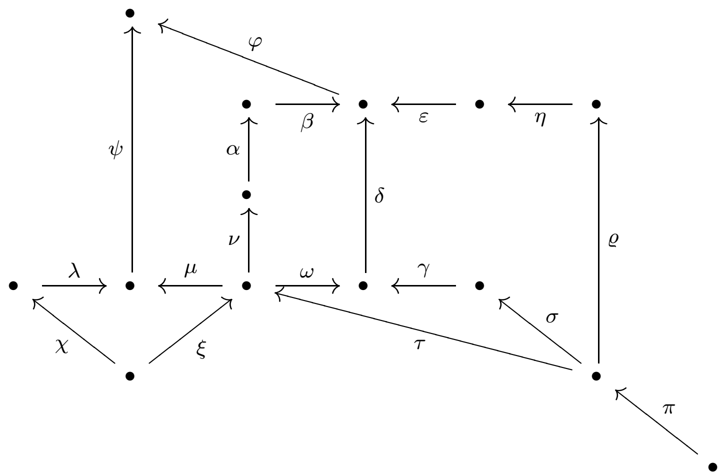
$\gamma\delta\varphi = 0 \qquad \varepsilon\varphi = 0 \qquad \nu\alpha\beta = \omega\delta \qquad \mu\psi = \nu\alpha\beta\varphi$
$\tau\mu = 0 \qquad \tau\omega = \sigma\gamma \qquad \sigma\gamma\delta = \varrho\eta\varepsilon$

Λ_3



$\gamma\delta\varphi = 0 \qquad \varepsilon\varphi = 0 \qquad \nu\alpha\beta = \omega\delta \qquad \mu\psi = \nu\alpha\beta\varphi \qquad \tau\mu = 0$
$\tau\omega = \sigma\gamma \qquad \sigma\gamma\delta = \varrho\eta\varepsilon \qquad \chi\lambda = \xi\mu \qquad \xi\nu = 0 \qquad \xi\omega = 0$

Λ_4



$$\gamma\delta\varphi = 0 \qquad \varepsilon\varphi = 0 \qquad \nu\alpha\beta = \omega\delta \qquad \mu\psi = \nu\alpha\beta\varphi \qquad \tau\mu = 0$$
$$\tau\omega = \sigma\gamma \qquad \sigma\gamma\delta = \varrho\eta\varepsilon \qquad \chi\lambda = \xi\mu \qquad \xi\nu = 0 \qquad \xi\omega = 0$$
$$\pi\tau\nu = 0 \qquad \pi\varrho = 0$$

Each Λ_i is an i-iterated coil algebra.

5.3. We can now state and prove our last result.

THEOREM C. *Let A be an n-iterated coil algebra. The following conditions are equivalent*:

(a) *A is strongly simply connected.*
(b) *A is strongly $\widetilde{\mathbb{A}}$-free.*
(c) *Each B_i is strongly $\widetilde{\mathbb{A}}$-free.*
(d) *No C_i is hereditary of type $\widetilde{\mathbb{A}}_n$.*
(e) *For each i, the orbit graph of each directed component of Γ_{B_i} is a tree.*
(f) *Each B_i is strongly simply connected.*
(g) *Each B_i satisfies the separation and coseparation conditions.*

Proof. Clearly, (a) implies (b), and (b) implies (c). Since B_0 and B_n are, respectively, either cotilted and tilted of Euclidean type or tubular algebras, and the remaining B_i are tubular algebras, the equivalence of (c), (d), (e) and (f) follows from [3, (2.3)] and [1, (1.7)]. By [17, (4.1)], (f) implies (g). If B_i is tubular, then by [1, (1.7)], (g) implies (c), and if B_i is domestic, then by [1, (1.6)] and [3, (2.3)], (g) implies (c).

It remains to show that (f) implies (a). We proceed by induction on n. If $n = 1$, the statement follows from Theorem A. Assume that $n > 1$, and that the statement holds for any $k \leq n - 1$. If $A = \Lambda_n$ is not strongly simply connected, then by [4, Theorem 1.3], it contains an irreducible cycle w which is not a contour, or an irreducible contour w which is not naturally contractible. By induction, w must contain a source lying in B_n but not in C_{n-1}, and a sink lying in B_0 but not in C_0. As in the proof of Theorem A, we may replace w by an irreducible cycle w' which is not a contour, but lies in Λ_{n-1}, a contradiction to the induction hypothesis. ∎

REFERENCES

[1] I. Assem, *Strongly simply connected derived tubular algebras*, in: Representations of Algebras (São Paulo, 1999), Lecture Notes in Pure and Appl. Math. 224, Dekker, 2001, 21–29.

[2] I. Assem, F. U. Coelho and S. Trepode, *Simply connected tame quasi-tilted algebras*, J. Pure Appl. Algebra 172 (2002), 139–160.

[3] I. Assem and S. Liu, *Strongly simply connected tilted algebras*, Ann. Sci. Math. Québec 21 (1997), 13–22.

[4] —, —, *Strongly simply connected algebras*, J. Algebra 207 (1998), 449–477.

[5] I. Assem, S. Liu and J. A. de la Peña, *The strong simple connectedness of a tame tilted algebra*, Comm. Algebra 28 (2000), 1553–1565.

[6] I. Assem and J. A. de la Peña, *The fundamental groups of a triangular algebra*, ibid. 24 (1996), 187–208.

[7] I. Assem and A. Skowroński, *On some classes of simply connected algebras*, Proc. London Math. Soc. 56 (1988), 417–450.

[8] —, —, *Minimal representation-infinite coil algebras*, Manuscripta Math. 67 (1990), 305–331.

[9] —, —, *Indecomposable modules over a multicoil algebra*, Math. Scand. 71 (1992), 31–61.

[10] —, —, *Multicoil algebras*, in: Representations of Algebras, CMS Conf. Proc. 14, Amer. Math. Soc., 1993, 29–68.

[11] —, —, *Coils and multicoil algebras*, in: Representation Theory and Related Topics (Mexico, 1994), CMS Conf. Proc. 19, Amer. Math. Soc., 1996, 1–24.

[12] I. Assem, A. Skowroński and B. Tomé, *Coil enlargements of algebras*, Tsukuba J. Math. 19 (1995), 453–479.

[13] M. Auslander, I. Reiten and S. Smalø, *Representation Theory of Artin Algebras*, Cambridge Stud. Adv. Math. 36, Cambridge Univ. Press, 1995.

[14] K. Bongartz and P. Gabriel, *Covering spaces in representation theory*, Invent. Math. 65 (1981), 331–378.

[15] J. A. de la Peña and A. Skowroński, *Geometric and homological characterizations of polynomial growth strongly simply connected algebras*, Invent. Math. 126 (1996), 287–296.

[16] C. Ringel, *Tame Algebras and Integral Quadratic Forms*, Lecture Notes in Math. 1099, Springer, 1984.

[17] A. Skowroński, *Simply connected algebras and Hochschild cohomologies*, in: Representations of Algebras, CMS Conf. Proc. 14, Amer. Math. Soc., 1993, 431–447.

[18] —, *Simply connected algebras of polynomial growth*, Compositio Math. 109 (1997), 99–133.

[19] B. Tomé, *Iterated coil enlargements of algebras*, Fund. Math. 146 (1995), 251–266.

Departamento de Matemática–IME
Universidade de São Paulo
CP 66281
São Paulo, SP, 05315-970, Brazil
E-mail: fucoelho@ime.usp.br
bel@ime.usp.br

Departamento de Matemáticas
Facultad de Ciencias
UNAM
Ciudad Universitaria
México 04510, D.F., México
E-mail: bta@hp.fciencias.unam.mx

Received 8 September 2003;
revised 14 January 2004 (4377)