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SOME NOTIONS OF AMENABILITY FOR CERTAIN PRODUCTS OF BANACH ALGEBRAS

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Abstract. For two Banach algebras $\mathcal{A}$ and $\mathcal{B}$, an interesting product $\mathcal{A} \times_\theta \mathcal{B}$, called the θ-Lau product, was recently introduced and studied for some nonzero characters θ on $\mathcal{B}$. Here, we characterize some notions of amenability as approximate amenability, essential amenability, n-weak amenability and cyclic amenability between $\mathcal{A}$ and $\mathcal{B}$ and their θ-Lau product.

1. Introduction. Let $\mathcal{A}$ and $\mathcal{B}$ be two Banach algebras and $\theta \in \sigma(\mathcal{B})$, the spectrum of $\mathcal{B}$ of all nonzero characters on $\mathcal{B}$. Then the *θ-Lau product* of $\mathcal{A}$ and $\mathcal{B}$, denoted by $\mathcal{A} \times_\theta \mathcal{B}$, is defined as the space $\mathcal{A} \times \mathcal{B}$ equipped with the multiplication

$$(a, b)(a', b') = (aa' + \theta(b)a' + \theta(b')a, bb'),$$

and the norm

$$\|(a, b)\| = \|a\| + \|b\|,$$

for all $a, a' \in \mathcal{A}$ and $b, b' \in \mathcal{B}$. The θ-Lau product $\mathcal{A} \times_\theta \mathcal{B}$ is a Banach algebra.

This product was first introduced by Lau [L1] for Lau algebras; recall that a *Lau algebra* is a Banach algebra which is the predual of a von Neumann algebra for which the identity of the dual is a multiplicative linear functional. The study of this large class of Banach algebras originated with a paper published in 1983 by Lau [L1] in which he referred to them as "F-algebras"; see also Lau [L2]. Later on, in his useful monograph Pier [Pi] introduced the name "Lau algebra". Examples of Lau algebras include the group algebra and the measure algebra of a locally compact group or hypergroup (see Lau [L1]), and also the Fourier algebra and

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the Fourier–Stieltjes algebra of a topological group (see Lau and Ludwig [LL]).

The algebraic and topological properties of the Banach algebra $\mathcal{A} \times_\theta \mathcal{B}$ were recently studied by Monfared [M]. If we allow $\theta = 0$, we obtain the usual direct product of Banach algebras. Since direct products often exhibit different properties, we exclude the case $\theta = 0$. In $\mathcal{A} \times_\theta \mathcal{B}$ we identify $\mathcal{A} \times \{0\}$ with $\mathcal{A}$, and $\{0\} \times \mathcal{B}$ with $\mathcal{B}$. Hence, $\mathcal{A}$ is a closed two-sided ideal while $\mathcal{B}$ is a closed subalgebra of $\mathcal{A} \times_\theta \mathcal{B}$; moreover, $(\mathcal{A} \times_\theta \mathcal{B})/\mathcal{A}$ is isometrically isomorphic to $\mathcal{B}$.

We note that if $\mathcal{B}$ is the Banach algebra $\mathbb{C}$ of all complex numbers and θ is the identity map on $\mathbb{C}$, then $\mathcal{A} \times_\theta \mathcal{B}$ is the unitization $\mathcal{A}^\sharp$ of $\mathcal{A}$.

Furthermore, the dual $(\mathcal{A} \times_\theta \mathcal{B})^{(1)}$ of $\mathcal{A} \times_\theta \mathcal{B}$ can be identified with $\mathcal{A}^{(1)} \times \mathcal{B}^{(1)}$ in the natural way

$$\langle (a^{(1)}, b^{(1)}), (a, b) \rangle = \langle a^{(1)}, a \rangle + \langle b^{(1)}, b \rangle,$$

for all $a \in \mathcal{A},\ b \in \mathcal{B},\ a^{(1)} \in \mathcal{A}^{(1)}$ and $b^{(1)} \in \mathcal{B}^{(1)}$. The dual norm on $\mathcal{A}^{(1)} \times \mathcal{B}^{(1)}$ is of course the maximum norm $\|(a^{(1)}, b^{(1)})\| = \max\{\|a^{(1)}\|, \|b^{(1)}\|\}$. Moreover, suppose that the second duals $\mathcal{A}^{(2)}$, $\mathcal{B}^{(2)}$ and $(\mathcal{A} \times_\theta \mathcal{B})^{(2)}$ are equipped with their first Arens products (see [M]). Then $(\mathcal{A} \times_\theta \mathcal{B})^{(2)}$ is isometrically isomorphic to $\mathcal{A}^{(2)} \times_{\theta^{[2]}} \mathcal{B}^{(2)}$, where $\theta^{[2]} \in \sigma(\mathcal{B}^{(2)})$. Now, take $\mathcal{A}^{(n)} \times \mathcal{B}^{(n)}$ as the underlying space of $(\mathcal{A} \times_\theta \mathcal{B})^{(n)}$. By induction, the $(\mathcal{A} \times_\theta \mathcal{B})$-bimodule actions on $(\mathcal{A} \times_\theta \mathcal{B})^{(n)}$ are as follows:

$$(a, b) \cdot (a^{(n)}, b^{(n)}) = \begin{cases} (a \cdot a^{(n)} + \theta^{[n]}(b) a^{(n)} + \theta^{[n]}(b^{(n)}) a, b \cdot b^{(n)}) & \text{if } n \text{ is even,} \\ (a \cdot a^{(n)} + \theta^{[n-1]}(b) a^{(n)}, a^{(n)}(a) \theta^{[n-1]} + b \cdot b^{(n)}) & \text{if } n \text{ is odd,} \end{cases}$$

and

$$(a^{(n)}, b^{(n)}) \cdot (a, b) = \begin{cases} (a^{(n)} \cdot a + \theta^{[n]}(b) a^{(n)} + \theta^{[n]}(b^{(n)}) a, b^{(n)} \cdot b) & \text{if } n \text{ is even,} \\ (a^{(n)} \cdot a + \theta^{[n-1]}(b) a^{(n)}, a^{(n)}(a) \theta^{[n-1]} + b^{(n)} \cdot b) & \text{if } n \text{ is odd,} \end{cases}$$

for all $(a, b) \in \mathcal{A} \times_\theta \mathcal{B}$ and $(a^{(n)}, b^{(n)}) \in \mathcal{A}^{(n)} \times \mathcal{B}^{(n)}$, where $\theta^{[2k]} \in \sigma(\mathcal{B}^{(2k)})$ denotes the kth adjoint of θ. Also, for $(m, n), (p, q) \in (\mathcal{A} \times_\theta \mathcal{B})^{(2)}$ we have

$$(m, n) \,\Box\, (p, q) = (m \,\Box\, p + n(\theta) p + q(\theta) m, n \,\Box\, q)$$

(see [M, Proposition 2.12]).

On the other hand, recently several important notions of amenability have been defined and studied on Banach algebras. In this paper, we are going to investigate these concepts on $\mathcal{A} \times_\theta \mathcal{B}$ and their relations with $\mathcal{A}$ and $\mathcal{B}$.

2. Approximate amenability. Let $\mathcal{A}$ be a Banach algebra and $\mathcal{X}$ be an $\mathcal{A}$-bimodule. A *derivation* is a linear map $D : \mathcal{A} \to \mathcal{X}$ such that

$$D(aa') = D(a) \cdot a' + a \cdot D(a')$$

for all $a, a' \in \mathcal{A}$. For $x \in \mathcal{X}$ set $\mathrm{ad}_x : a \mapsto a \cdot x - x \cdot a$ from $\mathcal{A}$ into $\mathcal{X}$. Hence, ad_x is a derivation; these are the *inner derivations*.

A derivation $D : \mathcal{A} \to \mathcal{X}$ is *approximately inner* if there exists a net $(x_\alpha)_\alpha \subseteq \mathcal{X}$ such that

$$D(a) = \lim_\alpha (a \cdot x_\alpha - x_\alpha \cdot a)$$

for each $a \in \mathcal{A}$, so that $D = \lim_\alpha \mathrm{ad}_{x_\alpha}$ in the strong operator topology.

The dual space $\mathcal{X}^{(1)}$ of a Banach $\mathcal{A}$-bimodule $\mathcal{X}$ becomes a Banach $\mathcal{A}$-bimodule with the module actions

$$\langle a \cdot x^{(1)}, x \rangle = \langle x^{(1)}, x \cdot a \rangle, \quad \langle x^{(1)} \cdot a, x \rangle = \langle x^{(1)}, a \cdot x \rangle,$$

for all $a \in \mathcal{A}$, $x \in \mathcal{X}$ and $x^{(1)} \in \mathcal{X}^{(1)}$. A Banach algebra $\mathcal{A}$ is called *amenable* if for any $\mathcal{A}$-bimodule $\mathcal{X}$, every continuous derivation $D : \mathcal{A} \to \mathcal{X}^{(1)}$ is inner. This notion was first introduced and studied by Johnson [J1] in 1972. Amenability has known hereditary properties (see [D], [J1] and [R]). In particular, $\mathcal{A} \times_\theta \mathcal{B}$ is amenable if and only if both $\mathcal{A}$ and $\mathcal{B}$ are amenable (see [M]).

The Banach algebra $\mathcal{A}$ is called *weakly amenable* if every continuous derivation from $\mathcal{A}$ into $\mathcal{A}^{(1)}$ is inner. The notion of weak amenability for an arbitrary Banach algebra was defined by Johnson [J2]; the study of this notion was pursued by several authors: see for example [G], [J2], [LE], [M], [NS] and [R]. Monfared [M] shows that weak amenability of $\mathcal{A}$ and $\mathcal{B}$ implies weak amenability of $\mathcal{A} \times_\theta \mathcal{B}$, but in the general case the converse is not true. However, he proves that weak amenability of $\mathcal{A} \times_\theta \mathcal{B}$ implies weak amenability of $\mathcal{B}$ and cyclic amenability of $\mathcal{A}$.

A Banach algebra $\mathcal{A}$ is called *approximately amenable* if any continuous derivation $D : \mathcal{A} \to \mathcal{X}^{(1)}$ is approximately inner for all Banach $\mathcal{A}$-bimodules $\mathcal{X}$. Moveover, $\mathcal{A}$ is called *approximately weakly amenable* if any continuous derivation $D : \mathcal{A} \to \mathcal{A}^{(1)}$ is approximately inner. The concepts of approximate amenability and approximate weak amenability were introduced and studied by Ghahramani and Loy [GL] (see also [GLZ]).

PROPOSITION 2.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be two Banach algebras and $\theta \in \sigma(\mathcal{B})$. If $\mathcal{A} \times_\theta \mathcal{B}$ is approximately amenable, then $\mathcal{A}$ and $\mathcal{B}$ are approximately amenable.*

Proof. Suppose that $\mathcal{A} \times_\theta \mathcal{B}$ is approximately amenable. Then it is clear that $(\mathcal{A} \times_\theta \mathcal{B})/\mathcal{A}$ is isometrically isomorphic to $\mathcal{B}$, and so $\mathcal{B}$ is approximately amenable by [GL, Corollary 2.1]. Now, we show that $\mathcal{A}$ is also approximately amenable. Let $\mathcal{X}$ be a Banach $\mathcal{A}$-bimodule and $D : \mathcal{A} \to \mathcal{X}^{(1)}$ be a derivation.

Then it is easy to show that $\mathcal{X}$ is an $(\mathcal{A}\times_\theta\mathcal{B})$-bimodule with the module actions

$$(a,b)\cdot x = a\cdot x+\theta(b)x, \qquad x\cdot(a,b)=x\cdot a+\theta(b)x,$$

for all $a\in\mathcal{A}$, $b\in\mathcal{B}$ and $x\in\mathcal{X}$. We prove that the map

$$\tilde{D}:\mathcal{A}\times_\theta\mathcal{B}\to\mathcal{X}^{(1)}$$

defined by $\tilde{D}((a,b))=D(a)$ is a derivation for all $a\in\mathcal{A}$ and $b\in\mathcal{B}$. In fact, for every (a,b) and (a',b') in $\mathcal{A}\times_\theta\mathcal{B}$ we have

$$\begin{aligned}(1)\qquad \tilde{D}((a,b)(a',b')) &= \tilde{D}((aa'+\theta(b')a+\theta(b)a',bb'))\\ &= D(aa')+\theta(b')D(a)+\theta(b)D(a')\\ &= a\cdot D(a')+D(a)\cdot a'+\theta(b')D(a)+\theta(b)D(a').\end{aligned}$$

On the other hand,

$$(2)\qquad (a,b)\cdot\tilde{D}((a',b'))=(a,b)\cdot D(a')=a\cdot D(a')+\theta(b)D(a'),$$

$$(3)\qquad \tilde{D}((a,b))\cdot(a',b')=D(a)\cdot(a',b')=D(a)\cdot a'+\theta(b')D(a),$$

for each $a,a'\in\mathcal{A}$ and $b,b'\in\mathcal{B}$. Adding (2) to (3) and comparing with (1), we conclude that $\tilde{D}$ is a derivation. From the approximate amenability of $A\times_\theta\mathcal{B}$, it follows that $\tilde{D}=\lim_\alpha \mathrm{ad}_{x_\alpha^{(1)}}$ for some net $(x_\alpha^{(1)})_\alpha\subseteq\mathcal{X}^{(1)}$ in the strong operator topology. We claim that $D=\lim_\alpha\mathrm{ad}_{x_\alpha^{(1)}}$ in the strong operator topology; indeed,

$$D(a)=\tilde{D}((a,0))=\lim_\alpha((a,0)\cdot x_\alpha^{(1)}-x_\alpha^{(1)}\cdot(a,0))=\lim_\alpha(a\cdot x_\alpha^{(1)}-x_\alpha^{(1)}\cdot a)$$

for all $a\in\mathcal{A}$, as required. ∎

We do not know if the converse of Proposition 2.1 is valid; here, we prove the converse under an extra assumption.

PROPOSITION 2.2. *Let $\mathcal{A}$ and $\mathcal{B}$ be two Banach algebras and $\theta\in\sigma(\mathcal{B})$. If $\mathcal{A}$ is amenable and $\mathcal{B}$ is approximately amenable, then $\mathcal{A}\times_\theta\mathcal{B}$ is approximately amenable.*

Proof. Since $\mathcal{A}$ is amenable and $(\mathcal{A}\times_\theta\mathcal{B})/\mathcal{A}$ is approximately amenable, $\mathcal{A}\times_\theta\mathcal{B}$ is approximately amenable. So, the result follows from the fact that $\mathcal{A}$ is a closed two-sided ideal of $\mathcal{A}\times_\theta\mathcal{B}$ and that $(\mathcal{A}\times_\theta\mathcal{B})/\mathcal{A}$ is isometrically isomorphic to $\mathcal{B}$ (see [GL], Corollary 2.1). ∎

3. Essential amenability. An $\mathcal{A}$-bimodule $\mathcal{X}$ is called *neo-unital* if $\mathcal{X}=\mathcal{A}\cdot\mathcal{X}\cdot\mathcal{A}$, where

$$\mathcal{A}\cdot\mathcal{X}\cdot\mathcal{A}=\{a\cdot x\cdot b: a,b\in\mathcal{A},\ x\in\mathcal{X}\}.$$

Recall from [GL] that a Banach algebra $\mathcal{A}$ is called *essentially amenable* if for any neo-unital $\mathcal{A}$-bimodule $\mathcal{X}$, every continuous derivation $D : \mathcal{A} \to \mathcal{X}^{(1)}$ is inner. Moreover, a Banach algebra $\mathcal{A}$ is called *approximately essentially amenable* if every continuous derivation $D : \mathcal{A} \to \mathcal{X}^{(1)}$ is approximately inner for any neo-unital $\mathcal{A}$-bimodule $\mathcal{X}$. The concepts of essential amenability and approximate essential amenability of Banach algebras were introduced and studied by Ghahramani and Loy [GL].

Note that if $\mathcal{X}$ is a Banach $\mathcal{A}$-bimodule such that $\mathcal{X} = \mathcal{A} \cdot \mathcal{X} \cdot \mathcal{A}$, then $\mathcal{X}$ is a $(\mathcal{A} \times_\theta \mathcal{B})$-bimodule with $\mathcal{X} = (\mathcal{A} \times_\theta \mathcal{B}) \cdot \mathcal{X} \cdot (\mathcal{A} \times_\theta \mathcal{B})$ with the module actions

$$(a, b) \cdot x = a \cdot x + \theta(b)x, \qquad x \cdot (a, b) = x \cdot a + \theta(b)x,$$

for all $a \in \mathcal{A}$, $b \in \mathcal{B}$ and $x \in \mathcal{X}$. Now, we investigate these notions on $\mathcal{A} \times_\theta \mathcal{B}$.

PROPOSITION 3.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be two Banach algebras and $\theta \in \sigma(\mathcal{B})$. If $\mathcal{A} \times_\theta \mathcal{B}$ is essentially amenable, then $\mathcal{A}$ and $\mathcal{B}$ are essentially amenable.*

Proof. The result follows by an argument similar to Proposition 2.1. ∎

The next result proves the converse of Proposition 3.1 under the assumption that $\mathcal{A}$ is amenable. We do not know if it is true for all Banach algebras $\mathcal{A}$.

PROPOSITION 3.2. *Let $\mathcal{A}$ and $\mathcal{B}$ be two Banach algebras and $\theta \in \sigma(\mathcal{B})$. Moreover, suppose that $\mathcal{A}$ is an amenable Banach algebra and $\mathcal{B}$ is an essentially amenable Banach algebra. Then $\mathcal{A} \times_\theta \mathcal{B}$ is essentially amenable.*

Proof. We know that $\mathcal{A}$ is a closed ideal of $\mathcal{A} \times_\theta \mathcal{B}$ and $(\mathcal{A} \times_\theta \mathcal{B})/\mathcal{A}$ is isometrically isomorphic to $\mathcal{B}$. Since $\mathcal{A}$ is amenable and $(\mathcal{A} \times_\theta \mathcal{B})/\mathcal{A}$ is essentially amenable, a standard argument as in [Pa, p. 42] shows that $\mathcal{A} \times_\theta \mathcal{B}$ is essentially amenable.

PROPOSITION 3.3. *Let $\mathcal{A}$ be an essentially amenable Banach algebra and I be a closed two-sided ideal of $\mathcal{A}$ with a bounded approximate identity. Then I is amenable.*

Proof. Suppose that $\mathcal{X}$ is a neo-unital Banach I-bimodule and $D : I \to \mathcal{X}^{(1)}$ is a continuous derivation. Then $\mathcal{X}$ is a neo-unital Banach $\mathcal{A}$-bimodule and D has an extension $\tilde{D} : \mathcal{A} \to \mathcal{X}^{(1)}$ by [R, Proposition 2.1.6]. Since $\mathcal{A}$ is essentially amenable, $\tilde{D}$ is inner and so D is inner. Thus I is essentially amenable. Since I is a Banach algebra with a bounded approximate identity, it follows from [R, Proposition 2.1.5] that I is amenable. ∎

THEOREM 3.4. *Let $\mathcal{A}$ and $\mathcal{B}$ be two Banach algebras for which there is a continuous epimorphism from $\mathcal{A}$ onto $\mathcal{B}$. Then approximate essential amenability of $\mathcal{A}$ implies approximate essential amenability of $\mathcal{B}$.*

Proof. Suppose that $\mathcal{A}$ is approximately essentially amenable and that $\mathcal{X}$ is a neo-unital Banach $\mathcal{B}$-bimodule. Then $\mathcal{X}$ is a neo-unital Banach $\mathcal{A}$-bimodule via the module actions defined by

$$a \cdot x = \Phi(a) \cdot x, \qquad x \cdot a = x \cdot \Phi(a),$$

for all $a \in \mathcal{A}$ and $x \in \mathcal{X}$. If $D : \mathcal{B} \to \mathcal{X}^{(1)}$ is a derivation, then it is clear that the map $D \circ \Phi : \mathcal{A} \to \mathcal{X}$ is a derivation, where $\Phi : \mathcal{A} \to \mathcal{B}$ is a continuous epimorphism. Therefore, there exists a net $(x_\alpha^{(1)})_\alpha \subseteq \mathcal{X}^{(1)}$ such that for each $a \in \mathcal{A}$ we have

$$(D \circ \Phi)(a) = \lim_\alpha (a \cdot x_\alpha^{(1)} - x_\alpha^{(1)} \cdot a) = \lim_\alpha (\Phi(a) \cdot x_\alpha^{(1)} - x_\alpha^{(1)} \cdot \Phi(a)).$$

Since Φ is epimorphism, we have $D(b) = \lim_\alpha (b \cdot x_\alpha^{(1)} - x_\alpha^{(1)} \cdot b)$ for every $b \in \mathcal{B}$. So, $\mathcal{B}$ is approximately essentially amenable. ∎

THEOREM 3.5. *Let $\mathcal{A}$ and $\mathcal{B}$ be two Banach algebras and $\theta \in \sigma(\mathcal{B})$. If $\mathcal{A} \times_\theta \mathcal{B}$ is approximately essentially amenable, then $\mathcal{A}$ and $\mathcal{B}$ are approximately essentially amenable.*

Proof. Approximate essential amenability of $\mathcal{B}$ follows from Theorem 3.4. Now, suppose that the $\mathcal{A}$-bimodule $\mathcal{X}$ is neo-unital. Then via the module actions defined by

$$x \cdot (a, b) = x \cdot a + \theta(b)x, \qquad (a, b) \cdot x = a \cdot x + \theta(b)x,$$

it is clear that $(\mathcal{A} \times_\theta \mathcal{B})$-bimodule $\mathcal{X}$ is neo-unital for all $a \in \mathcal{A}$, $b \in \mathcal{B}$ and $x \in \mathcal{X}$. If $D : \mathcal{A} \to \mathcal{X}^{(1)}$ is a continuous derivation, then we can extend it to $\tilde{D} : \mathcal{A} \times_\theta \mathcal{B} \to \mathcal{X}^{(1)}$ via

$$\tilde{D}((a, b)) = D(a)$$

for all $a \in \mathcal{A}$ and $b \in \mathcal{B}$. Clearly, $\tilde{D}$ is a derivation. Thus, there exists $(x_\alpha^{(1)})_\alpha \subseteq \mathcal{X}^{(1)}$ such that

$$\tilde{D}((a, b)) = \lim_\alpha \operatorname{ad}_{x_\alpha^{(1)}}(a, b) = \lim_\alpha ((a, b) \cdot x_\alpha^{(1)} - x_\alpha^{(1)} \cdot (a, b))$$

for all $a \in \mathcal{A}$ and $b \in \mathcal{B}$. Therefore,

$$\begin{aligned} D(a) = \tilde{D}((a, 0)) &= \lim_\alpha ((a, 0) \cdot x_\alpha^{(1)} - x_\alpha^{(1)} \cdot (a, 0)) \\ &= \lim_\alpha (a \cdot x_\alpha^{(1)} - x_\alpha^{(1)} \cdot a) \end{aligned}$$

for all $a \in \mathcal{A}$. So, $\mathcal{A}$ is approximately essentially amenable. ∎

4. n-Weak amenability. For $n \in \mathbb{N}$, the concept of n-weak amenability was initiated and intensively developed by Dales, Ghahramani and Grønbæk [DGG].

A Banach algebra $\mathcal{A}$ is said to be *n-weakly amenable* if every continuous derivation from $\mathcal{A}$ into $\mathcal{A}^{(n)}$ is inner. Trivially, 1-weak amenability is nothing other than weak amenability, which was first introduced and intensively studied by Bade, Curtis and Dales [BCD] for commutative Banach algebras, and then by Johnson [J3] for a general Banach algebra.

THEOREM 4.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be two Banach algebras, $\theta \in \sigma(\mathcal{B})$ and $n \in \mathbb{N}$.*

(i) *If $\mathcal{A} \times_\theta \mathcal{B}$ is $(2n)$-weakly amenable, then $\mathcal{A}$ is $(2n)$-weakly amenable.*
(ii) *If $\mathcal{A}$ and $\mathcal{B}$ are $(2n+1)$-weakly amenable, then $\mathcal{A} \times_\theta \mathcal{B}$ is $(2n+1)$-weakly amenable.*

Proof. (i) Let $\mathcal{A} \times_\theta \mathcal{B}$ be $(2n)$-weakly amenable. We show that $\mathcal{A}$ is $(2n)$-weakly amenable. If $D : \mathcal{A} \to \mathcal{A}^{(2n)}$ is a continuous derivation, then we can extend this derivation to $\tilde{D} : \mathcal{A} \times_\theta \mathcal{B} \to \mathcal{A}^{(2n)} \times_{\theta[2n]} \mathcal{B}^{(2n)}$ via

$$\tilde{D}((a,b)) = (d(a), 0),$$

for all $a \in \mathcal{A}$ and $b \in \mathcal{B}$. Clearly, $\tilde{D}$ is a derivation on $\mathcal{A} \times_\theta \mathcal{B}$. Thus, there exists $(a^{(2n)}, b^{(2n)}) \in \mathcal{A}^{(2n)} \times_{\theta[2n]} \mathcal{B}^{(2n)}$ such that

$$\tilde{D}((a,b)) = (a,b) \cdot (a^{(2n)}, b^{(2n)}) - (a^{(2n)}, b^{(2n)}) \cdot (a,b)$$

for all $a \in \mathcal{A}$ and $b \in \mathcal{B}$. Therefore, $D(a) = a \cdot a^{(2n)} - a^{(2n)} \cdot a$ and $b \cdot b^{(2n)} = b^{(2n)} \cdot b$ for all $a \in \mathcal{A}$ and $b \in \mathcal{B}$. So, $\mathcal{A}$ is $(2n)$-weakly amenable.

(ii) Suppose that $D : \mathcal{A} \times_\theta \mathcal{B} \to \mathcal{A}^{(2n+1)} \times \mathcal{B}^{(2n+1)}$ is a continuous derivation. Moreover, suppose that $\imath : \mathcal{A} \to \mathcal{A} \times_\theta \mathcal{B}$ is the natural embedding,

$$\imath^{(2n+1)} : \mathcal{A}^{(2n+1)} \times \mathcal{B}^{(2n+1)} \to \mathcal{A}^{(2n+1)}$$

is the $(2n+1)$-th adjoint of $\imath$, and $\pi : \mathcal{A} \times_\theta \mathcal{B} \to (\mathcal{A} \times_\theta \mathcal{B})/\mathcal{A}$ is the quotient map. Then

$$\imath^{(2n+1)} \circ D \circ \imath : \mathcal{A} \to \mathcal{A}^{(2n+1)}$$

is a continuous derivation. So, there exists $a^{(2n+1)} \in \mathcal{A}^{(2n+1)}$ such that

$$(\imath^{(2n+1)} \circ D)(a) = \mathrm{ad}_{a^{(2n+1)}}(a)$$

for all $a \in \mathcal{A}$. We can extend $a^{(2n+1)}$ to an element of $\mathcal{A}^{(2n+1)} \times \mathcal{B}^{(2n+1)}$. Thus, if we put

$$D_{a^{(2n+1)}} := D - \mathrm{ad}_{a^{(2n+1)}},$$

then $(\imath^{(2n+1)} \circ D) = 0$ on $\mathcal{A}$.

Now, for any $a, a' \in \mathcal{A}$ and $(a^{(2n)}, b^{(2n)}) \in \mathcal{A}^{(2n)} \times_{\theta^{[2n]}} \mathcal{B}^{(2n)}$,

$$\begin{aligned}\langle D(aa'), (a^{(2n)}, b^{(2n)})\rangle &= \langle D(a), (a', 0) \cdot (a^{(2n)}, b^{(2n)})\rangle \\ &\quad + \langle D(a'), (a^{(2n)}, b^{(2n)}) \cdot (a, 0)\rangle \\ &= \langle D(a), \imath^{(2n)}((a', 0) \cdot (a^{(2n)}, b^{(2n)}))\rangle \\ &\quad + \langle D(a'), \imath^{(2n)}((a^{(2n)}, b^{(2n)}) \cdot (a, 0))\rangle \\ &= \langle (\imath^{(2n+1)} \circ D)(a), (a', 0) \cdot (a^{(2n)}, b^{(2n)})\rangle \\ &\quad + \langle (\imath^{(2n+1)} \circ D)(a'), (a^{(2n)}, b^{(2n)}) \cdot (a, 0)\rangle \\ &= 0,\end{aligned}$$

where $\theta^{[2n]} \in \sigma(\mathcal{B}^{(2n)})$. Thus, $D = 0$ on $\mathcal{A}^2 := \mathcal{A}\mathcal{A}$. By the $(2n+1)$-weak amenability of $\mathcal{A}$ and Proposition 2.8.63(i) of [D], we have $D = 0$ on $\mathcal{A}$ since $\overline{\mathcal{A}^2} = \mathcal{A}$.

On the other hand, if $\mathcal{X}_{\mathcal{A}}$ is the closed linear subspace of $\mathcal{A}^{(2n)} \times_{\theta^{[2n]}} \mathcal{B}^{(2n)}$ spanned by

$$\mathcal{A}(\mathcal{A}^{(2n)} \times_{\theta^{[2n]}} \mathcal{B}^{(2n)}) \cup (\mathcal{A}^{(2n)} \times_{\theta^{[2n]}} \mathcal{B}^{(2n)})\mathcal{A},$$

then for all $(a, b) \in \mathcal{A} \times_\theta \mathcal{B}$ and $a' \in \mathcal{A}$ we have

$$0 = D((a, b)(a', 0)) = D((a, b)) \cdot (a', 0),$$
$$0 = D((a', 0)(a, b)) = (a', 0) \cdot D((a, b)).$$

Moreover, for all $a, a' \in \mathcal{A}$, $b \in \mathcal{B}$ and $(a^{(2n)}, b^{(2n)}) \in \mathcal{A}^{(2n)} \times_{\theta^{[2n]}} \mathcal{B}^{(2n)}$,

$$\langle D((a, b)), (a^{(2n)}, b^{(2n)})(a', 0)\rangle = \langle (a', 0) \cdot D((a, b)), (a^{(2n)}, b^{(2n)})\rangle = 0,$$
$$\langle D((a, b)), (a', 0)(a^{(2n)}, b^{(2n)})\rangle = \langle D((a, b)) \cdot (a', 0), (a^{(2n)}, b^{(2n)})\rangle = 0.$$

So, $D(\mathcal{A} \times_\theta \mathcal{B}) \subseteq \mathcal{X}_{\mathcal{A}}^{\perp}$. Hence, $D(\mathcal{A} \times_\theta \mathcal{B}) \subseteq ((\mathcal{A} \times_\theta \mathcal{B})/\mathcal{A})^{(n)}$. Clearly, $\mathcal{X}_{\mathcal{A}}$ is a closed $(\mathcal{A} \times_\theta \mathcal{B})$-submodule of $\mathcal{A}^{(2n)} \times_{\theta^{[2n]}} \mathcal{B}^{(2n)}$ and $(\mathcal{A}^{(2n)} \times_{\theta^{[2n]}} \mathcal{B}^{(2n)})/\mathcal{X}_{\mathcal{A}}$ is an $((\mathcal{A} \times_\theta \mathcal{B})/\mathcal{A})$-bimodule. Now, we define a map

$$D_{\mathcal{A}} : (\mathcal{A} \times_\theta \mathcal{B})/\mathcal{A} \to ((\mathcal{A} \times_\theta \mathcal{B})/\mathcal{A})^{(n)}$$

via $D_{\mathcal{A}}((a, b) + \mathcal{A}) = D((a, b))$ for all $a \in \mathcal{A}$ and $b \in \mathcal{B}$. Then $D_{\mathcal{A}}$ is continuous derivation. But $(\mathcal{A} \times_\theta \mathcal{B})/\mathcal{A}$ is isometrically isomorphic to $\mathcal{B}$, and $\mathcal{B}$ is $(2n+1)$-weakly amenable. Thus, there exists $f^{(n)} \in ((\mathcal{A} \times_\theta \mathcal{B})/\mathcal{A})^{(n)}$ such that $D_{\mathcal{A}} = \mathrm{ad}_{f^{(n)}}$. It follows that $\mathcal{A} \times_\theta \mathcal{B}$ is $(2n+1)$-weakly amenable. ■

REMARK. Let $\mathcal{A}$ and $\mathcal{B}$ be two Banach algebras, $\theta \in \sigma(\mathcal{B})$ and $n \in \mathbb{N}$. An argument similar to the proof of Theorem 4.1 shows that:

(i) If $\mathcal{A} \times_\theta \mathcal{B}$ is approximately $(2n)$-weakly amenable, then $\mathcal{A}$ is approximately $(2n)$-weakly amenable.

(ii) If $\mathcal{A}$ and $\mathcal{B}$ are approximately $(2n+1)$-weakly amenable, then $\mathcal{A} \times_\theta \mathcal{B}$ is approximately $(2n+1)$-weakly amenable.

5. Cyclic amenability. Recall that a derivation $D : \mathcal{A} \to \mathcal{A}^{(1)}$ is called *cyclic* if $\langle D(a), b\rangle + \langle D(b), a\rangle = 0$ for all $a, b \in \mathcal{A}$; the Banach algebra $\mathcal{A}$ is called *cyclic amenable* (resp. *approximately cyclic amenable*) if every cyclic continuous derivation $D : \mathcal{A} \to \mathcal{A}^{(1)}$ is inner (resp. approximately inner).

THEOREM 5.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be two Banach algebras with $\overline{\mathcal{A}^2} = \mathcal{A}$ and let $\theta \in \sigma(\mathcal{B})$. Then $\mathcal{A} \times_\theta \mathcal{B}$ is cyclic amenable* (*resp. approximately cyclic amenable*) *if and only if $\mathcal{A}$ and $\mathcal{B}$ are cyclic amenable* (*resp. approximately cyclic amenable*).

Proof. We give a proof for cyclic amenability; the proof for approximate cyclic amenability is similar.

To this end, suppose that $D : \mathcal{A} \to \mathcal{A}^{(1)}$ is a cyclic derivation. Then we can extend it to a derivation $\tilde{D} : \mathcal{A} \times_\theta \mathcal{B} \to \mathcal{A}^{(1)} \times \mathcal{B}^{(1)}$ defined via

$$\tilde{D}((a, b)) = (D(a), 0)$$

for all $a \in \mathcal{A}$ and $\mathcal{B}$. On the other hand, it is clear that $\tilde{D}$ is a cyclic derivation on $\mathcal{A} \times_\theta \mathcal{B}$. Therefore, there exists $(a^{(1)}, b^{(1)}) \in \mathcal{A}^{(1)} \times \mathcal{B}^{(1)}$ such that for each $a \in \mathcal{A}$ and $b \in \mathcal{B}$,

$$\begin{aligned}\tilde{D}((a, b)) &= (a, b) \cdot (a^{(1)}, b^{(1)}) - (a^{(1)}, b^{(1)}) \cdot (a, b)\\ &= (a \cdot a^{(1)} - a^{(1)} \cdot a, b \cdot b^{(1)} - b^{(1)} \cdot b).\end{aligned}$$

But, on the other hand, $\tilde{D}((a, b)) = (D(a), 0)$ for each $a \in \mathcal{A}$ and $b \in \mathcal{B}$. Therefore, $D(a) = a \cdot a^{(1)} - a^{(1)} \cdot a$ and $b \cdot b^{(1)} - b^{(1)} \cdot b = 0$ for all $a \in \mathcal{A}$ and $b \in \mathcal{B}$; i.e. $\mathcal{A}$ is cyclic amenable. Cyclic amenability of $\mathcal{B}$ is proved similarly.

Conversely, if $\mathcal{A}$ and $\mathcal{B}$ are cyclic amenable and $D : \mathcal{A} \times_\theta \mathcal{B} \to \mathcal{A}^{(1)} \times \mathcal{B}^{(1)}$ is a cyclic derivation, then there are two functions $\alpha : \mathcal{A} \times_\theta \mathcal{B} \to \mathcal{A}^{(1)}$ and $\beta : \mathcal{A} \times_\theta \mathcal{B} \to \mathcal{B}^{(1)}$ are such that

$$D((a, b)) = (\alpha((a, b)), \beta((a, b)))$$

for all $a \in \mathcal{A}$ and $b \in \mathcal{B}$. Now, we define $D_1 : \mathcal{A} \to \mathcal{A}^{(1)}$ via $D_1(a) = \alpha((a, 0))$ for all $a \in \mathcal{A}$ and $D_2 : \mathcal{B} \to \mathcal{B}^{(1)}$ via $D_2(b) = \beta((0, b))$ for all $b \in \mathcal{B}$. Thus, for every $a, a' \in \mathcal{A}$ we have

$$\begin{aligned}\langle D_1(a), a'\rangle + \langle D_1(a'), a\rangle &= \langle (D_1(a), 0), (a', 0)\rangle + \langle (D_1(a'), 0), (a, 0)\rangle\\ &= \langle (\alpha((a, 0)), \beta((a, 0))), (a', 0)\rangle\\ &\quad + \langle (\alpha((a', 0)), \beta((a', 0)), (a, 0)\rangle\\ &= \langle D((a, 0)), (a', 0)\rangle + \langle D((a', 0)), (a, 0)\rangle = 0.\end{aligned}$$

So, D_1 is a cyclic derivation. Thus, there exists $a^{(1)} \in \mathcal{A}^{(1)}$ such that

$$D_1(a) = a \cdot a^{(1)} - a^{(1)} \cdot a$$

for all $a \in \mathcal{A}$. Similarly, D_2 is a cyclic derivation. Therefore, there exists $b^{(1)} \in \mathcal{B}^{(1)}$ such that

$$D_2(b) = b \cdot b^{(1)} - b^{(1)} \cdot b$$

for all $b \in \mathcal{B}$. It follows from the assumption that for each $(a^{(1)}, b^{(1)}) \in \mathcal{A}^{(1)} \times \mathcal{B}^{(1)}$ we have

$$D((a,b)) = (a,b) \cdot (a^{(1)}, b^{(1)}) - (a^{(1)}, b^{(1)}) \cdot (a,b)$$

for all $a \in \mathcal{A}$ and $b \in \mathcal{B}$. So, D is an inner derivation; i.e. $\mathcal{A} \times_\theta \mathcal{B}$ is cyclic amenable. ∎

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