OPERATOR ENTROPY INEQUALITIES

BY

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Abstract. We investigate a notion of relative operator entropy, which develops the theory started by J. I. Fujii and E. Kamei [Math. Japonica 34 (1989), 341–348]. For two finite sequences $A = (A_1, \ldots, A_n)$ and $B = (B_1, \ldots, B_n)$ of positive operators acting on a Hilbert space, a real number $q$ and an operator monotone function $f$ we extend the concept of entropy by setting

$$S_q^f(A | B) := \sum_{j=1}^n A_j^{1/2}(A_j^{-1/2}B_jA_j^{-1/2})^q f(A_j^{-1/2}B_jA_j^{-1/2})A_j^{1/2},$$

and then give upper and lower bounds for $S_q^f(A | B)$ as an extension of an inequality due to T. Furuta [Linear Algebra Appl. 381 (2004), 219–235] under certain conditions. As an application, some inequalities concerning the classical Shannon entropy are deduced.

1. Introduction and preliminaries. Throughout the paper, let $\mathcal{B}(\mathcal{H})$ denote the algebra of all bounded linear operators acting on a complex Hilbert space $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ and $I$ is the identity operator. When $\dim \mathcal{H} = n$, we identify $\mathcal{B}(\mathcal{H})$ with the full matrix algebra $\mathcal{M}_n(\mathbb{C})$ of $n \times n$ matrices with complex entries and denote its identity by $I_n$. A self-adjoint operator $A \in \mathcal{B}(\mathcal{H})$ is called positive, written $A \geq 0$, if $\langle Ax, x \rangle \geq 0$ for all $x \in \mathcal{H}$. An operator $A$ is said to be strictly positive (denoted by $A > 0$) if it is positive and invertible. For self-adjoint operators $A, B \in \mathcal{B}(\mathcal{H})$, we write $A \leq B$ if $B - A \geq 0$.

Let $f$ be a continuous real valued function defined on an interval $J$. The function $f$ is called operator decreasing if $B \leq A$ implies $f(A) \leq f(B)$ for all $A, B \in \mathcal{B}(\mathcal{H})$ with spectra in $J$. The function $f$ is said to be operator concave on $J$ if

$$\lambda f(A) + (1 - \lambda)f(B) \leq f(\lambda A + (1 - \lambda)B)$$

for all self-adjoint $A, B \in \mathcal{B}(\mathcal{H})$ with spectra in $J$ and all $\lambda \in [0, 1]$. 

2010 Mathematics Subject Classification: Primary 47A63; Secondary 15A42, 46L05, 47A30.

Key words and phrases: $f$-divergence functional, Jensen inequality, operator entropy, entropy inequality, operator concavity, perspective function, positive linear map.

DOI: 10.4064/cm130-2-2 [159] © Instytut Matematyczny PAN, 2013
In 1850 Clausius [Ann. Physik (2) 79 (1850), 368–397, 500–524] introduced the notion of entropy in thermodynamics. Since then several extensions and reformulations have been developed in various disciplines (cf. [ME, LR, L, NU]). The so-called entropy inequalities have been investigated by several authors (see [BLP, BS, F2] and references therein).

A relative operator entropy of strictly positive operators $A, B$ was introduced in noncommutative information theory by Fujii and Kamei [FK] by

$$S(A \mid B) = A^{1/2} \log(A^{-1/2}BA^{-1/2})A^{1/2}. $$

When $A$ is positive, one may set $S(A \mid B) := \lim_{\epsilon \to 0^+} S(A+\epsilon I \mid B)$ if the limit exists in the strong operator topology. In the same paper, it is shown that $S(A \mid B) \leq 0$ if $A \geq B$. There is an analogous notion called the perspective function (see [E, CK]) If $f : [0, \infty) \to \mathbb{R}$ is an operator convex function, then the perspective function $g$ associated to $f$ is defined by

$$g(B, A) = A^{1/2} f(A^{-1/2}BA^{-1/2})A^{1/2}$$

for any self-adjoint operator $B$ and any strictly positive operator $A$.

One can consider a more general case. Let $\tilde{B} = (B_1, \ldots, B_n)$ and $\tilde{A} = (A_1, \ldots, A_n)$ be $n$-tuples of self-adjoint and strictly positive operators, respectively. Then the noncommutative $f$-divergence functional $\Theta$ is defined by

$$\Theta(\tilde{B}, \tilde{A}) = \sum_{i=1}^{n} A_i^{1/2} f(A_i^{-1/2}B_iA_i^{-1/2})A_i^{1/2}. $$

Next, recall that $X \sharp_q Y$ is defined by $X^{1/2}(X^{-1/2}YX^{-1/2})^q X^{1/2}$ for any real $q$ and any strictly positive operators $X$ and $Y$. For $p \in [0, 1]$, the operator $X \sharp_p Y$ coincides with the well-known $p$-power mean of $X,Y$.

Furuta [F1] defined a parametric extension of the operator entropy by

$$S_p(A \mid B) = A^{1/2} (A^{-1/2}BA^{-1/2})^p \log(A^{-1/2}BA^{-1/2})A^{1/2}, $$

where $p \in [0, 1]$ and $A, B$ are strictly positive operators on a Hilbert space $\mathcal{H}$, and proved some operator entropy inequalities: if $\{A_1, \ldots, A_n\}$ and $\{B_1, \ldots, B_n\}$ are two sequences of strictly positive operators on a Hilbert space $\mathcal{H}$ such that $\sum_{j=1}^{n} A_j \sharp_p B_j \leq I$, then

$$\log \left[ \sum_{j=1}^{n} (A_j \sharp_{p+1} B_j) + t_0 \left( I - \sum_{j=1}^{n} A_j \sharp_p B_j \right) \right]$$

$$- (\log t_0) \left( I - \sum_{j=1}^{n} A_j \sharp_p B_j \right)$$

for any $t_0 > 0$. This is an important result in the theory of operator entropies and inequalities.
for any fixed real number \( t_0 > 0 \).

The object of this paper is to state an operator entropy inequality parallel to the main result of [F1] and refine some known operator entropy inequalities.

2. Operator entropy inequality. The following notion is basic in our work.

**Definition 2.1.** Assume that \( A = (A_1, \ldots, A_n) \) and \( B = (B_1, \ldots, B_n) \) are finite sequences of strictly positive operators on a Hilbert space \( \mathcal{H} \). For \( q \in \mathbb{R} \) and an operator monotone function \( f : (0, \infty) \to [0, \infty) \) the **generalized operator Shannon entropy** is defined by

\[
S^f_q(A \mid B) := \sum_{j=1}^{n} S^f_q(A_j \mid B_j),
\]

where

\[
S^f_q(A_j \mid B_j) = A_j^{1/2}(A_j^{-1/2}B_jA_j^{-1/2})^q f(A_j^{-1/2}B_jA_j^{-1/2})A_j^{1/2}.
\]

We recall that for \( q = 0, f(t) = \log t \) and \( A, B > 0 \), we get the relative operator entropy \( S^f_0(A \mid B) = A^{1/2} \log(A^{-1/2}BA^{-1/2})A^{1/2} = S(A \mid B) \). It is interesting to point out that \( S_q(A \mid B) = -S_{1-q}(B \mid A) \) for any real \( q \), in particular, \( S_1(A \mid B) = -S(B \mid A) \). In fact, since \( X f(X^*X) = f(XX^*)X \) for every \( X \in \mathbb{B}(\mathcal{H}) \) and every continuous function \( f \) on \( [0, \|X\|^2] \), considering \( X = B^{1/2}A^{-1/2} \) and \( f(t) = \log t \) we get

\[
S_q(A \mid B) = A^{1/2}(A^{-1/2}BA^{-1/2})^q \log(A^{-1/2}BA^{-1/2})A^{1/2}
\]

\[
= B^{1/2}B^{-1/2}A^{1/2}(A^{-1/2}BA^{-1/2})^q \log(A^{-1/2}BA^{-1/2})A^{1/2}B^{-1/2}B^{1/2}
\]

\[
= B^{1/2}X^{*-1}(X^*X)^q \log(X^*X)X^{-1}B^{1/2}
\]

\[
= B^{1/2}X^{-1}(X^*X)^{-q} \log(X^*X)X^{-1}B^{1/2}
\]

\[
= B^{1/2}(X^{-1}X^*X^{-1})^{-1}q(X^{-1}X^{-1})^{-1}X^{-1}X^{-1} \log(X^*X)X^{-1}B^{1/2}
\]

\[
= B^{1/2}(X^{-1}X^*X^{-1})^{-1}q X \log(X^*X)X^{-1}B^{1/2}
\]

\[
= B^{1/2}(X^{-1}X^*X^{-1})^{-1}q \log(XX^*)X^{-1}B^{1/2}
\]

\[
= -B^{1/2}(X^{-1}X^*X^{-1})^{-1}q \log(X^{-1}X^{-1})B^{1/2}
\]

\[
= -B^{1/2}(X^*X^{-1})^{-1}q \log(X^*X^{-1})B^{1/2}
\]

We need the following useful lemma.
Lemma 2.2 ([F1 Proposition 3.1]). If \( f \) is a continuous real function on an interval \( J \), then the following conditions are equivalent:

(i) \( f \) is operator concave.
(ii) \( f(C^*XC + t_0(I-C^*C)) \geq C^*f(X)C + f(t_0)(I-C^*C) \) for any operator \( C \) with \( \|C\| \leq 1 \) and any self-adjoint operator \( X \) with \( \text{sp}(X) \subseteq J \), and for any fixed \( t_0 \in J \).
(iii) \( f(\sum_{j=1}^n C_j^*X_jC_j + t_0(I-\sum_{j=1}^n C_j^*C_j)) \geq \sum_{j=1}^n C_j^*f(X_j)C_j + f(t_0)(I-\sum_{j=1}^n C_j^*C_j) \) for any operators \( C_j \) with \( \sum_{j=1}^n C_j^*C_j \leq I \) and self-adjoint operators \( X_j \) with \( \text{sp}(X_j) \subseteq J \) for \( j = 1, \ldots, n \), and for any fixed \( t_0 \in J \).

For other equivalent conditions the reader may consult [FMPS] and references therein. Using an idea of [F1] we prove the following result.

Theorem 2.3. Assume that \( f, A \) and \( B \) are as in Definition 2.1. Let \( \sum_{j=1}^n A_j = \sum_{j=1}^n B_j = I \) and let \( f \) be operator concave. Then

\[
\begin{align*}
&f\left[\sum_{j=1}^n (A_j z_{p+1} B_j) + t_0 \left( I - \sum_{j=1}^n A_j z_p B_j \right) \right] - f(t_0) \left( I - \sum_{j=1}^n A_j z_p B_j \right) \geq S_p^f(A | B)
\end{align*}
\]

for all \( p \in [0, 1] \) and for any fixed \( t_0 > 0 \), and

\[
\begin{align*}
&-f\left[\sum_{j=1}^n (A_j z_{p-1} B_j) + t_0 \left( I - \sum_{j=1}^n A_j z_p B_j \right) \right] + f(t_0) \left( I - \sum_{j=1}^n A_j z_p B_j \right) \leq S_p^f(A | B)
\end{align*}
\]

for all \( p \in [2, 3] \) and for any fixed \( t_0 > 0 \).

Proof. Since \( \sum_{j=1}^n A_j z_q B_j \leq (\sum_{j=1}^n A_j) z_q (\sum_{j=1}^n B_j) \) (see [FMPS Theorem 5.7]) for every \( q \in [0, 1] \), and \( \sum_{j=1}^n A_j = \sum_{j=1}^n B_j = I \), we have

\[
\sum_{j=1}^n A_j z_p B_j \leq I.
\]

Fix a positive real number \( t_0 \). Since \( f \) is operator concave, we get

\[
\begin{align*}
&f\left[\sum_{j=1}^n (A_j z_{p+1} B_j) + t_0 \left( I - \sum_{j=1}^n A_j z_p B_j \right) \right] \\
= &f\left[\sum_{j=1}^n ((A_j^{-1/2} B_j A_j^{-1/2})^{p/2} A_j^{1/2}) (A_j^{-1/2} B_j A_j^{-1/2}) ((A_j^{-1/2} B_j A_j^{-1/2})^{p/2} A_j^{1/2}) \right] \\
&+ t_0 \left( I - \sum_{j=1}^n A_j z_p B_j \right)
\end{align*}
\]
Following a similar argument, we obtain
\[
\frac{1}{n} \sum_{j=1}^{n} S^f_p(A_j | B_j) \geq \frac{1}{n} \sum_{j=1}^{n} S^f_{p-2}(A_j | B_j) + f(t_0) \left( I - \sum_{j=1}^{n} A_j \sharp_p B_j \right) \]

Thus
\[
-f \left[ \sum_{j=1}^{n} (A_j \sharp_{p-1} B_j) + t_0 \left( I - \sum_{j=1}^{n} A_j \sharp_p B_j \right) \right] + f(t_0) \left( I - \sum_{j=1}^{n} A_j \sharp_p B_j \right) \leq -S^f_{p-2}(A | B).
\]

Since \( f \) is a continuous nonnegative function, \( X^q f(X) \geq 0 \) for every \( X \geq 0 \) and \( q \in \mathbb{R} \). Hence
\[
(A^{-1/2}_j B_j A^{-1/2}_j)^q f(A^{-1/2}_j B_j A^{-1/2}_j) \geq 0.
\]
Consequently, \( S^f_q(A_j | B_j) \geq 0 \). Thus
\[
S^f_p(A_j | B_j) + S^f_{p-2}(A_j | B_j) \geq 0 \quad (j = 1, \ldots, n),
\]
whence \( -S^f_{p-2}(A | B) \leq S^f_p(A | B) \), which yields the required result.

**Remark 2.4.** By taking \( f(t) = \log t \) in Theorem 2.3, we get (1.1).

**Corollary 2.5.** Let \( A = (A_1, \ldots, A_n) \) and \( B = (B_1, \ldots, B_n) \) be two sequences of strictly positive operators on a Hilbert space \( \mathcal{H} \) such that
\[
\sum_{j=1}^{n} A_j = \sum_{j=1}^{n} B_j = I. \text{ If } f : (0, \infty) \to [0, \infty) \text{ is a function which is both operator monotone and operator concave, then }
\]

(i) \( f(\sum_{j=1}^{n} B_j A_j^{-1} B_j) \geq S_1^f(A \mid B) \),

(ii) \( f(I) \geq S_0^f(A \mid B) \).

**Proof.** (i) Setting \( p = 1 \) in Theorem 2.3 and applying \( \sum_{j=1}^{n} A_j z_1 B_j = \sum_{j=1}^{n} B_j = I \), we obtain

\[
f\left(\sum_{j=1}^{n} B_j A_j^{-1} B_j\right) = f\left(\sum_{j=1}^{n} A_j z_2 B_j\right) \geq S_1^f(A \mid B).
\]

(ii) Putting \( p = 0 \) in Theorem 2.3 and using \( \sum_{j=1}^{n} A_j z_0 B_j = \sum_{j=1}^{n} A_j = I \), we get

\[
f(I) = f\left(\sum_{j=1}^{n} B_j\right) = f\left(\sum_{j=1}^{n} A_j z_1 B_j\right) \geq S_0^f(A \mid B). \]

Next we extend the operator entropy to \( n \) strictly positive operators \( A_1, \ldots, A_n \in \mathbb{B}(\mathcal{H}) \) and refine the operator entropy inequality.

**Corollary 2.6.** Let \( A_1, \ldots, A_n \in \mathbb{B}(\mathcal{H}) \) be a sequence of strictly positive operators on a Hilbert space \( \mathcal{H} \) such that \( \sum_{j=1}^{n} A_j = I \). Then

\[
\log\left(\sum_{j=1}^{n} A_j^{-1}\right) \geq (\log n) I - \frac{1}{n} \sum_{j=1}^{n} \log A_j.
\]

**Proof.** Taking \( A = (A_1, \ldots, A_n) \) and \( B = (\frac{1}{n} I, \ldots, \frac{1}{n} I) \) and \( f(t) = \log t \) in Corollary 2.5 (i), we get

\[
-2(\log n) I + \log\left(\sum_{j=1}^{n} A_j^{-1}\right) = \log\left(\frac{1}{n^2} \sum_{j=1}^{n} A_j^{-1}\right) \geq S_1^{\log}(A \mid B)
\]

\[
= \sum_{j=1}^{n} \frac{1}{n} A_j^{-1/2} \log\left(\frac{1}{n^2} A_j^{-1}\right) A_j^{1/2} = \sum_{j=1}^{n} \frac{1}{n} \log\left(\frac{1}{n} A_j^{-1}\right)
\]

\[
= -\sum_{j=1}^{n} \frac{1}{n} ((\log n) I + \log A_j) = -(\log n) I - \frac{1}{n} \sum_{j=1}^{n} \log A_j,
\]

which yields (2.1). ■

**Corollary 2.7 (Operator entropy inequality).** Assume that \( A_1, \ldots, A_n \in \mathbb{B}(\mathcal{H}) \) are positive invertible operators satisfying \( \sum_{j=1}^{n} A_j = I \). Then

\[
-\sum_{j=1}^{n} A_j \log A_j \leq (\log n) I.
\]
**Proof.** Letting $A = (A_1, \ldots, A_n)$, $B = (\frac{1}{n} I, \ldots, \frac{1}{n} I)$ and $f(t) = \log t$ in Corollary 2.5(ii), we get
\[
0 = \log I \geq c_0^{\log}(A | B)
\]
\[
= \sum_{j=1}^{n} A_j^{1/2} \log \left( \frac{1}{n} A_j^{-1} \right) A_j^{1/2} = \sum_{j=1}^{n} A_j^{1/2} (- (\log n) I - \log A_j) A_j^{1/2}
\]
\[
= - (\log n) \sum_{j=1}^{n} A_j - \sum_{j=1}^{n} A_j^{1/2} (\log A_j) A_j^{1/2}. \]

**Remark 2.8.** Let $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$ be $n$-tuples of positive numbers such that $\sum_{j=1}^{n} a_j = \sum_{j=1}^{n} b_j = 1$. Put $A_i = [a_i]_{1 \times 1} \in M_1(\mathbb{C})$ and $B_i = [b_i]_{1 \times 1} \in M_1(\mathbb{C})$. It follows from Corollary 2.5(ii) that $0 \geq \sum_{j=1}^{n} a_j \log (b_j/a_j)$, which is an entropy inequality related to the Kullback–Leibler relative entropy or information divergence $S(p, q) = \sum_{j=1}^{n} p_j \log (p_j/q_j)$ with the convention $x \log x = 0$ if $x = 0$, and $x \log y = +\infty$ if $y = 0$ and $x \neq 0$ (cf. [KL]).

**Theorem 2.9.** Let $p \in [0, 1]$ and let $A, B$ be strictly positive operators on a Hilbert space $\mathcal{H}$ such that $A \leq B \leq I$ and $B^2 \leq A^2$. If $f : (0, \infty) \to [0, \infty)$ is both operator monotone and operator concave, then
\[
f(A \hat{\circ} p+1 B + t_0 (I - A \hat{\circ} p B)) - f(t_0)(I - A \hat{\circ} p B)
\]
\[
\geq S_p^f(A | B) = - f(A \hat{\circ} p+1 B + t_0 (I - A \hat{\circ} p B)) + f(t_0)(I - A \hat{\circ} p B)
\]
for any fixed real number $t_0 > 0$.

**Proof.** It follows from $A \leq B \leq I$ that
\[
A^{1/2} (A^{-1/2} BA^{-1/2})^{p-2} A^{1/2} \leq I,
\]
\[
(A^{-1/2} BA^{-1/2})^{p-2} \leq A^{-1},
\]
\[
A^{-1/2} BA^{-1/2} \leq (A^{-1/2} BA^{-1/2}) A^{-1} (A^{-1/2} BA^{-1/2}),
\]
\[
A^{1/2} (A^{-1/2} BA^{-1/2})^p A^{1/2} \leq BA^{-2} B.
\]
Since $B^2 \leq A^2$ and the map $t \mapsto -1/t$ is operator monotone, we have
\[
A^{1/2} (A^{-1/2} BA^{-1/2})^p A^{1/2} \leq I,
\]
so that $A \hat{\circ} p B \leq I$. Now the same reasoning as in the proof of Theorem 2.3 (with $n = 1$ and using Lemma 2.2(ii)) yields the desired inequalities. ■

Recall that a map $\Phi : \mathbb{B}(\mathcal{H}) \to \mathbb{B}(\mathcal{H})$, where $\mathcal{H}$ and $\mathcal{K}$ are Hilbert spaces, is called positive if $\Phi(A) \geq 0$ whenever $A \geq 0$, and it is said to be normalized if it preserves the identity. The paper [MMM, Lemma 5.2] includes the following refinement of the Jensen inequality for Hilbert space operators: Let $\mu = (\mu_1, \ldots, \mu_m)$ and $\lambda = (\lambda_1, \ldots, \lambda_n)$ be probability vectors. By a (discrete) weight function (with respect to $\mu$ and $\lambda$) we mean a mapping
\[ \omega : \{(i, j) : 1 \leq i \leq m, 1 \leq j \leq n\} \to [0, \infty) \] such that \( \sum_{i=1}^{m} \omega(i, j) \mu_i = 1 \) \( (j = 1, \ldots, n) \) and \( \sum_{j=1}^{n} \omega(i, j) \lambda_j = 1 \) \( (i = 1, \ldots, m) \). If \( f \) is a real-valued operator concave function on an interval \( J \), \( A_1, \ldots, A_n \) are self-adjoint operators with spectra in \( J \) and \( \Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K}) \) is a normalized positive map, then

\[
(2.2) \quad f\left( \sum_{j=1}^{n} \lambda_j \Phi(A_j) \right) \geq \sum_{i=1}^{m} \mu_i f\left( \sum_{j=1}^{n} \omega(i, j) \lambda_j \Phi(A_j) \right) \geq \sum_{j=1}^{n} \lambda_j \Phi(f(A_j)).
\]

A matrix \( A = [a_{ij}] \in \mathcal{M}_n(\mathbb{C}) \) is said to be doubly stochastic if \( a_{ij} \geq 0 \) \( (i, j = 1, \ldots, n) \) and \( \sum_{i=1}^{n} a_{ij} = \sum_{j=1}^{n} a_{ij} = 1 \). Now we introduce a refinement of the operator Jensen inequality.

**Theorem 2.10.** Suppose that \( f \) is a real-valued operator concave function on an interval \( J \) and \( A_1, \ldots, A_n \) are self-adjoint operators with spectra in \( J \). Assume that \( B = [b_{ij}] \) and \( C = [c_{ij}] \) are \( n \times n \) doubly stochastic matrices, \( \omega_1 \) and \( \omega_2 \) are weight functions with respect to the same probability vector, and \( \Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K}) \) is a normalized positive map. If the operator-valued functions \( F_{\omega_1, \omega_2} \) and \( F_{B, C} \) are defined by

\[
F_{\omega_1, \omega_2}(t) := \sum_{i=1}^{m} \mu_i f\left( \sum_{j=1}^{n}\omega_1(i, j)(1-t)\lambda_j \Phi(A_j) + t \omega_2(i, j) \lambda_j \Phi(A_j) \right) \quad (0 \leq t \leq 1)
\]

and

\[
F_{B, C}(t) := \frac{1}{n} \sum_{i=1}^{n} f\left( \sum_{j=1}^{n}\omega_1(i, j)(1-t)b_{ij} + t c_{ij} \lambda_j \Phi(A_j) \right) \quad (0 \leq t \leq 1),
\]

then

(i)

\[
(2.4) \quad f\left( \sum_{j=1}^{n} \lambda_j \Phi(A_j) \right) = F_{\omega_1, \omega_2}(t) \geq \sum_{j=1}^{n} \lambda_j \Phi(f(A_j)) \quad (0 \leq t \leq 1).
\]

In particular,

\[
f\left( \frac{1}{n} \sum_{j=1}^{n} \Phi(A_j) \right) \geq F_{B, C}(t) \geq \frac{1}{n} \sum_{j=1}^{n} \Phi(f(A_j)) \quad (0 \leq t \leq 1).
\]

(ii) For any \( i = 1, \ldots, n \), the maps

\[
t \mapsto f\left( \sum_{j=1}^{n}\omega_1(i, j)(1-t)\lambda_j \Phi(A_j) + t \omega_2(i, j) \lambda_j \Phi(A_j) \right) \quad (0 \leq t \leq 1),
\]

as well as the function \( F_{\omega_1, \omega_2} \), are operator concave. In particular, \( F_{B, C} \) is concave on \([0, 1] \).
Proof. (i) Since for every \( t \) in \([0, 1]\), the map
\[
(i, j) \mapsto (1 - t)\omega_1(i, j) + t\omega_2(i, j) \quad (1 \leq i \leq m, 1 \leq j \leq n)
\]
is a weight function, (2.4) follows from (2.2). By taking \( m = n, \lambda_j = \mu_i = 1/n, \omega_1(i, j) = nb_{ij}, \omega_2(i, j) = nc_{ij} \) in \( F_{\omega_1, \omega_2}(t) \), we obtain the second part.

(ii) Let \( \eta_1, \eta_2 \geq 0 \) with \( \eta_1 + \eta_2 = 1 \) and let \( t_1, t_2 \in [0, 1] \). For every \( i \) with \( 1 \leq i \leq m \), we have
\[
f\left(\sum_{j=1}^{n}[(1 - \eta_1 t_1 - \eta_2 t_2)\omega_1(i, j) + (\eta_1 t_1 + \eta_2 t_2)\omega_2(i, j)]\lambda_j \Phi(A_j)\right)
= f\left(\sum_{j=1}^{n}[(1 - t_1)\omega_1(i, j) + t_1\omega_2(i, j)]\lambda_j \Phi(A_j)\right)
+ \eta_2\sum_{j=1}^{n}[(1 - t_2)\omega_1(i, j) + t_2\omega_2(i, j)]\lambda_j \Phi(A_j)
\geq \eta_1 f\left(\sum_{j=1}^{n}[(1 - t_1)\omega_1(i, j) + t_1\omega_2(i, j)]\lambda_j \Phi(A_j)\right)
+ \eta_2 f\left(\sum_{j=1}^{n}[(1 - t_2)\omega_1(i, j) + t_2\omega_2(i, j)]\lambda_j \Phi(A_j)\right) \quad \text{(by concavity of } f),
\]
which implies (ii). The concavity of \( F_{B,C} \) over \([0, 1]\) is clear. □

By taking \( f(t) = -t \log t \) and \( \Phi(t) = t \) in (2.3) and by using Theorem 2.10, we obtain the following result:

**Corollary 2.11 (Refinement of an operator entropy inequality).** Assume that \( A_1, \ldots, A_n \) are positive self-adjoint invertible operators with spectra in an interval \( J \) and \( \sum_{j=1}^{n} A_j = I \). If \( B = [b_{ij}] \) and \( C = [c_{ij}] \) are \( n \times n \) doubly stochastic matrices, then
\[
(\log n)I \geq \sum_{i=1}^{n} \left[ -\left( \sum_{j=1}^{n} [(1 - t)b_{ij} + tc_{ij}]A_j \right) \log \left( \sum_{j=1}^{n} [(1 - t)b_{ij} + tc_{ij}]A_j \right) \right]
\geq -\sum_{j=1}^{n} A_j \log A_j \quad (0 \leq t \leq 1).
\]

**Acknowledgements.** The authors would like to sincerely thank the anonymous referee for several useful comments improving the paper. The first author was supported by a grant from Ferdowsi University of Mashhad (No. MP90242MOS). The third author would like to thank the Tusi Mathematical Research Group (TMRG).
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Received 7 May 2012;
revised 10 August 2012