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ON THE DISTRIBUTION OF SOME INTEGERS RELATED TO PERFECT AND AMICABLE NUMBERS

BY

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Abstract. Let $s'(n) = \sum_{d\mid n,\, 1<d<n} d$ be the sum of the nontrivial divisors of the natural number n, where *nontrivial* excludes both 1 and n. For example, $s'(20) = 2+4+5+10 = 21$. A natural number n is called *quasiperfect* if $s'(n) = n$, while n and m are said to form a *quasiamicable pair* if $s'(n) = m$ and $s'(m) = n$; in the latter case, both n and m are called *quasiamicable numbers*. In this paper, we prove two statistical theorems about these classes of numbers.

First, we show that the count of quasiperfect $n \le x$ is at most $x^{1/4+o(1)}$ as $x \to \infty$. In fact, we show that for each fixed a, there are at most $x^{1/4+o(1)}$ natural numbers $n \le x$ with $\sigma(n) \equiv a \pmod{n}$ and $\sigma(n)$ odd. (Quasiperfect n satisfy these conditions with $a = 1$.) For fixed $\delta \ne 0$, define the arithmetic function $s_\delta(n) := \sigma(n) - n - \delta$. Thus, $s_1 = s'$. Our second theorem says that the number of $n \le x$ which are amicable with respect to s_δ is at most $x/(\log x)^{1/2+o(1)}$.

1. Introduction. Some of the oldest problems in number theory concern the behavior of the sum-of-proper-divisors function $s(n) := \sum_{d\mid n,\, d<n} d$. In the mid-twentieth century, S. Chowla (see [19]) proposed studying the variant arithmetic function

$$s'(n) := \sum_{\substack{d\mid n\\ 1<d<n}} d,$$

whose output is the sum of the nontrivial divisors of the natural number n. Here *nontrivial* means that both 1 and n itself are excluded. With respect to this function, the analogue of a perfect number is an integer n satisfying $s'(n) = n$; these are usually called *quasiperfect* numbers, although some authors prefer the more descriptive term *reduced perfect*. Similarly, the analogue of an amicable pair—termed a *quasiamicable* or *reduced amicable pair*—is a pair n and m with $s'(n) = m$ and $s'(m) = n$.

No quasiperfect numbers are known. Any such example must be an odd square [4], must possess at least seven distinct prime factors, and must have more than than 35 decimal digits (for these last two results, see [9]). For other theoretical work on quasiperfect numbers, see [1, 23, 5, 16, 6, 15].

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About quasiamicable pairs, almost all of what we know has been gleaned from computer searches [17, 10, 3]. There are 1946 quasiamicable pairs with smaller member less than 10^{12}.

In this paper, we take a statistical approach with the goal of establishing new upper bounds for the counting functions of the quasiperfect and quasi-amicable numbers. In both cases, our results are more general, and (for instance) apply also to almost-perfect numbers and augmented amicable pairs (for these concepts, see [8, Sections B2 and B5]).

The natural number n is quasiperfect precisely when $\sigma(n) = 2n + 1$. (Here and below, $\sigma(n) := \sum_{d|n} d$ always denotes the usual sum-of-divisors function.) In particular, $\sigma(n)$ is odd and $\sigma(n) \equiv 1 \pmod{n}$. Our first theorem concerns the problem of bounding from above the number of solutions $n \le x$ to a congruence of the form

$$\sigma(n) \equiv a \pmod{n} \tag{1}$$

for which $\sigma(n)$ is odd; here a is an arbitrary (but fixed) nonzero integer.

In a recent paper [2], the present authors and Aria Anavi studied the number of $n \le x$ satisfying (1), without a restriction on the parity of $\sigma(n)$. After discarding 'trivial' solutions (see §2 below), they obtained an upper bound of roughly $x^{1/2}$ (for x large compared to $|a|$). However, the requirement that $\sigma(n)$ is odd all by itself restricts n to a set of size $O(x^{1/2})$, namely the set of squares and their doubles. Thus, the main result of [2] is trivial in the context of quasiperfect numbers. Adding a new idea to [2], we prove the following theorem.

Theorem 1.1. *As $x \to \infty$, the number of solutions $n \le x$ to the congruence* (1) *for which $\sigma(n)$ is odd is at most*

$$|a|x^{1/4} \exp(O(\log x/\log\log x)), \tag{2}$$

uniformly in integers a with $0 < |a| \le x^{1/4}$. In particular, for fixed $a \ne 0$, there are at most $x^{1/4+o(1)}$ solutions.

Note that the estimate (2) would be trivial for $|a| > x^{1/4}$. When $a = 0$, the work of Hornfeck and Wirsing [14, Satz 2] gives an upper bound of $O_\epsilon(x^\epsilon)$ for the number of $n \le x$ with $\sigma(n) \equiv 0 \pmod{n}$.

Let δ be a fixed integer. For each natural number n, put $s_\delta(n) = \sigma(n) - n - \delta$. For example, s_0 is the usual sum-of-proper-divisors function, while s_1 is Chowla's function. We say that n and m form a *δ-amicable pair* if $s_\delta(n) = m$ and $s_\delta(m) = n$; in this case, both n and m are called *δ-amicable numbers*. We can now state our second theorem.

Theorem 1.2. *Fix an integer $\delta \ne 0$. For $x \ge 3$, the count of δ-amicable numbers in $[1, x]$ is*

$$\ll_\delta \frac{x}{(\log x)^{1/2}} (\log\log x)^4.$$

In previous work, the first author proved that the quasiamicable numbers have asymptotic density zero [21]. However, it would not be easy to extract a quantitative upper bound from that argument, and the final result would be very poor in comparison to Theorem 1.2.

We believe that neither of our upper bounds is very close to the truth. For example, a conjecture recorded in [2] implies that there are at most $(\log x)^{O(1)}$ quasiperfects in $[1, x]$. The quasiamicable numbers are likely to be much more dense; we expect that for $\epsilon > 0$, the number of quasiamicables in $[1, x]$ exceeds $x^{1-\epsilon}$ once $x > x_0(\epsilon)$. However, we also think that this count is at most $x/(\log x)^B$ for every B and all $x > x_0(B)$; for ordinary amicable pairs, this upper bound was shown by the second author [22].

Notation. Throughout, we use O- and o-notation, as well as the associated Vinogradov $\ll$ and $\gg$ notations, with their standard meanings. We write $P(n)$ for the largest prime factor of n, with the convention that $P(1) = 1$. We use $\tau(n)$ for the number of positive divisors of n, $\omega(n)$ for the number of distinct prime divisors of n, and $\Omega(n)$ for the number of prime divisors of n counted with multiplicity. We also write $\mathrm{rad}(n)$ for the largest squarefree divisor of n. Other notation will be introduced as necessary.

In the proof of Theorem 1.2 given in §5, all implied constants may depend on δ without further mention. Similarly, when we suppose in that section that x is sufficiently large, the notion of *large* may depend on δ.

2. Preparation for the proof of Theorem 1.1. Our first lemma, which follows from well-known results on smooth numbers, is implicit in work of Erdős, Luca and Pomerance [7] and stated explicitly as [20, Lemma 4.2].

LEMMA 2.1. *Suppose that $x \geq 3$. Let b be a natural number with $b \leq x$. The count of natural numbers $a \leq x$ for which* $\mathrm{rad}(a)$ *divides b is at most* $\exp(O(\log x/\log\log x))$.

Next, we bound the number of solutions to quadratic congruences.

LEMMA 2.2. *Let m be a natural number. Let a be an integer relatively prime to m, and let b be any integer. The number of solutions* mod m *to the quadratic congruence $ax^2 \equiv b \pmod{m}$ does not exceed*

$$\gcd(b, m)^{1/2} \cdot 2^{1+\omega(m)}.$$

Proof. We can assume that $a = 1$ by replacing b with $a'b$, where $a'a \equiv 1 \pmod{m}$.

It now suffices to show that if $p^e \,\|\, m$, then the number of solutions to $x^2 \equiv b \pmod{p^e}$ is at most

$$2\delta_p \cdot \gcd(b, p^e)^{1/2}, \tag{3}$$

where we put $\delta_2 = 2$ and $\delta_p = 1$ for $p > 2$. We consider three cases for p:

- Suppose that $p \nmid b$. If p is odd, then the congruence $x^2 \equiv b \pmod{p^e}$ has at most two solutions, since the unit group mod p^e is cyclic. If p is even, then the unit group mod p^e is either cyclic or a product of two cyclic groups, and so there are at most four square roots of b. In either case, the bound (3) holds.
- Suppose that $p^f \,\|\, b$, where $1 \le f < e$. If there is any solution to $x^2 \equiv b \pmod{m}$, then f is even and $p^{f/2} \mid x$. Putting $x = p^{f/2}x'$ and $b = p^f b'$, the number of solutions to $x^2 \equiv b \pmod{p^e}$ coincides with the number of distinct x' modulo $p^{e-f/2}$ for which $x'^2 \equiv b' \pmod{p^{e-f}}$. Since $p \nmid b'$, this latter congruence has at most $2\delta_p$ solutions modulo p^{e-f}, and so at most $2\delta_p p^{f/2} = 2\delta_p \cdot \gcd(b, p^e)^{1/2}$ solutions modulo $p^{e-f/2}$.
- If $p^e \mid b$, then $x^2 \equiv b \pmod{p^e}$ if and only if $p^{\lceil e/2 \rceil} \mid x$. So there are $p^{\lfloor e/2 \rfloor} \le \gcd(b, p^e)^{1/2}$ choices for x mod p^e. Thus, the bound (3) holds in this case also. ∎

The next two lemmas are taken from [2] (see that paper's Lemma 2, Lemma 5, and the remark following Lemma 7). To understand their statements, we recall from [2] that the solutions to (1) can be divided into regular solutions and sporadic solutions. A *regular* (or *trivial*) solution is a natural number n of the form

$$n = pm, \quad \text{where} \quad p \nmid m,\ m \mid \sigma(m),\ \text{and } \sigma(m) = a. \tag{4}$$

It is straightforward to check that all these n really do satisfy (1). The remaining solutions to (1) are called *sporadic*.

If n is a regular solution to (1) for which $\sigma(n)$ is odd, then $p = 2$ in (4) (otherwise $2 \mid p+1 \mid \sigma(n)$). Also, a is positive and $m \le a$. Thus, $n = pm \le 2a$. So if $n > 2|a|$ is an odd solution to (1), then n is sporadic. This observation will be important in what follows.

Lemma 2.3. *Let a be a nonzero integer. Suppose that n is a sporadic solution to the congruence* (1) *for which $6a^2 \log(6|a|) < n \le x$. If we write*

$$\sigma(n) = kn + a,$$

then the integer k satisfies $2 \le k \le 2 + \log x$.

Lemma 2.4. *Let a be a nonzero integer. Suppose that n is a sporadic solution to the congruence* (1) *for which*

$$\max\{6a^2 \log(6|a|), x^{1/2}\} < n \le x.$$

For every real number y with $1 \le y \le x^{1/2}$, there is a divisor d of n with

$$\frac{y}{64(\log x)^4} < d \le y.$$

3. Proof of Theorem 1.1. Since $\sigma(n)$ is odd, either $n = m^2$ or $n = 2m^2$. We give a complete proof of the upper bound (2) for the count of solutions of the form $n = m^2$, and at the end of the proof we make some comments about the (very similar) case when $n = 2m^2$.

We may assume that

$$\max\{6a^2 \log(6|a|), x^{1/2}\} < m^2 \leq x. \tag{5}$$

Indeed, since $|a| \leq x^{1/4}$, the failure of (5) implies that $m \ll x^{1/4}\sqrt{\log x}$, and this upper bound is negligible in comparison to (2).

Since $m^2 > 2|a|$, our remarks in the last section show that m^2 is a sporadic solution to (1). Write $\sigma(m^2) = km^2 + a$, where k is an integer. By Lemma 2.3, we have $2 \leq k \leq 2 + \log x$; in particular, there are only $O(\log x)$ possibilities for k. Since $\log x$ is dwarfed by the factor of $\exp(O(\log x/\log\log x))$ appearing in (2), we may assume for the remainder of the proof that k is fixed.

By Lemma 2.4 with $y = x^{1/4}$, we can choose a divisor d of m^2 with

$$\frac{x^{1/4}}{64(\log x)^4} < d \leq x^{1/4}.$$

There is a unique unitary divisor e of m (that is, $\gcd(e, m/e) = 1$) that is supported on the primes dividing d. Since $d \mid m^2$, it must be that $\prod_{p^v \| d} p^{\lceil v/2 \rceil} \mid e$, so that

$$e \geq d^{1/2} > \frac{x^{1/8}}{8(\log x)^2}. \tag{6}$$

Put $m = ef$. Since e and f are relatively prime, we find that

$$ke^2f^2 + a = km^2 + a = \sigma(m^2) = \sigma(e^2f^2) = \sigma(e^2)\sigma(f^2),$$

and thus

$$ke^2f^2 \equiv -a \pmod{\sigma(e^2)}. \tag{7}$$

Put $D := \gcd(ke^2, \sigma(e^2))$. ($D$ depends on e, but we suppress this in what follows.) From (7), we see that $D \mid a$. Dividing the congruence (7) through by D and applying Lemma 2.2, we find that f is restricted to at most

$$\begin{aligned} \gcd(a/D, \sigma(e^2)/D)^{1/2} \cdot 2^{1+\omega(\sigma(e^2)/D)} &\leq (|a|/D)^{1/2} \cdot 2\tau(\sigma(e^2)/D) \\ &\leq 2|a|^{1/2} D^{-1/2} \cdot \tau(\sigma(e^2)) \end{aligned}$$

residue classes modulo $\sigma(e^2)/D$. So given e, the number of corresponding $f \leq x^{1/2}/e$ is

$$\begin{aligned} &\ll \left(\frac{x^{1/2}/e}{\sigma(e^2)/D} + 1\right)(|a|^{1/2} D^{-1/2} \cdot \tau(\sigma(e^2))) \\ &\leq |a|^{1/2} D^{1/2} x^{1/2} \frac{\tau(\sigma(e^2))}{e^3} + |a|^{1/2}\tau(\sigma(e^2)). \end{aligned}$$

To bound the number of possible values of m, we sum over admissible values of d and e. Put

$$\sum\nolimits^{(1)} := \sum_{d,e} |a|^{1/2} D^{1/2} x^{1/2} \frac{\tau(\sigma(e^2))}{e^3} \quad \text{and} \quad \sum\nolimits^{(2)} := \sum_{d,e} |a|^{1/2} \tau(\sigma(e^2)).$$

By the maximal order of the divisor function (see, e.g., [13, Theorem 317]),

$$\tau(\sigma(e^2)) \le \exp(O(\log x/\log\log x)). \tag{8}$$

As noted above, D divides a, and so

$$D \le |a|. \tag{9}$$

Since $d \mid e^2$, once e is given, d is restricted to at most $\exp(O(\log x/\log\log x))$ possible values. Now using (6), (8), and (9) with the definition of $\sum^{(1)}$, we find that

$$\begin{aligned}\sum\nolimits^{(1)} &\le |a| x^{1/2} \exp(O(\log x/\log\log x)) \sum_{e > \frac{1}{8}x^{1/8}/(\log x)^2} \frac{1}{e^3} \\ &= |a| x^{1/4} \exp(O(\log x/\log\log x)).\end{aligned}$$

Inserting (8) into the definition of $\sum^{(2)}$, we obtain

$$\begin{aligned}\sum\nolimits^{(2)} &\le |a|^{1/2} \exp(O(\log x/\log\log x)) \sum_{d,e} 1 \\ &\le |a|^{1/2} x^{1/4} \exp(O(\log x/\log\log x));\end{aligned}$$

here we used the fact that $d \le x^{1/4}$ and that $\mathrm{rad}(e) \mid d$, so that by Lemma 2.1, e is determined from d in at most $\exp(O(\log x/\log\log x))$ ways. The desired upper bound (2) follows upon combining our estimates for $\sum^{(1)}$ and $\sum^{(2)}$.

Now suppose that $n = 2m^2$. The proof for this case is similar to the one given above, and so we sketch it quickly. Using Lemma 2.4, choose a divisor d_0 of $2m^2$ with

$$\frac{x^{1/4}}{64(\log x)^4} < d_0 \le x^{1/4}.$$

Put $d = d_0$ if d_0 is odd and $d = \frac{1}{2}d_0$ if d_0 is even. Then $d \mid m^2$ and

$$\frac{x^{1/4}}{128(\log x)^4} < d \le x^{1/4}.$$

Let e be the unitary divisor of m supported on the primes dividing d, and write $m = ef$. If e is even, the relation $\sigma(2m^2) = 2ke^2f^2 + a$ implies that

$$2ke^2f^2 \equiv -a \pmod{\sigma(2e^2)},$$

while if e is odd, we find that

$$2ke^2f^2 \equiv -a \pmod{\sigma(e^2)}.$$

In either case, obvious changes to our previous arguments allow us to count the number of solutions to this congruence with $f \le (x/2)^{1/2}/e$. After summing over d and e, we eventually again obtain the desired upper bound (2). This completes the proof of Theorem 1.1. ∎

4. Preparation for the proof of Theorem 1.2. We begin by quoting three lemmas taken from the study of the "anatomy of integers". The first is a consequence of Brun's sieve; compare, for example, with [11, Theorem 2.2].

Lemma 4.1. *Suppose that $3/2 \le y \le z \le x$. The count of natural numbers $n \le x$ with no prime factor from the interval $(y, z]$ is*

$$\ll x \frac{\log y}{\log z}.$$

Say that the natural number n is *y-smooth* (an alternative term is *y-friable*) if $P(n) \le y$. Let $\Psi(x, y) := \#\{n \le x : P(n) \le y\}$ denote the counting function of the y-smooth numbers. The following upper bound on $\Psi(x, y)$ appears as [24, Theorem 1, p. 359].

Lemma 4.2. *Suppose that $x \ge y \ge 2$. Then*

$$\Psi(x, y) \ll x \exp(-u/2), \quad \text{where} \quad u := \frac{\log x}{\log y}.$$

The next lemma, which bounds from above the count of n with extraordinarily many prime factors, appears as [12, Exercise 05, p. 12]. The proof is worked out explicitly in [18, Lemmas 12, 13].

Lemma 4.3. *Let $x \ge 2$. Let k be a natural number. The count of natural numbers $n \le x$ with $\Omega(n) \ge k$ is*

$$\ll \frac{k}{2^k} x \log x.$$

In addition to these three results, we need the following simple consequence of Lemma 4.1.

Lemma 4.4. *Let $\mathscr{E}$ be the set of positive integers m for which $m/P(m) \le \exp((\log m)^{1/2})$, and let $E(x) := \#\mathscr{E} \cap [1, x]$. For $x \ge 2$,*

$$E(x) \ll x/(\log x)^{1/2}. \tag{10}$$

Proof. It suffices to prove the upper bound (10) for large x. Summing dyadically, this reduces to showing that for large v, the size of $\mathscr{E} \cap (v, 2v]$ is $O(v/(\log v)^{1/2})$. So suppose that $m \in (v, 2v]$ and $m/P(m) \le \exp((\log m)^{1/2})$. Then m has no prime factors between $y := \exp((\log 2v)^{1/2})$ and $z := v/\exp((\log 2v)^{1/2})$. By Lemma 4.1, the number of such $m \le 2v$ is $\ll v \log y/\log z \ll v/(\log v)^{1/2}$, as desired. ∎

5. Proof of Theorem 1.2. The proof is a variant of the argument of [22].

5.1. Preliminary pruning. Recall that

$$\limsup \frac{\sigma(n)}{n \log\log n} = e^{\gamma} < 2$$

(see, e.g., [13, Theorem 323]). So if $n \le x$ is δ-amicable and x is sufficiently large, then $s_\delta(n) = \sigma(n) - n - \delta < 2x \log\log x$. For the rest of this section, we assume X is defined in terms of x by

$$X := 2x \log\log x.$$

For large x, the count of δ-amicable $n \le x$ is at most double the count of δ-amicable pairs $\{n_1, n_2\}$ with $n_1, n_2 \le X$. We now show that we may ignore all δ-amicable pairs except those satisfying a certain list of technical conditions.

Lemma 5.1. *Suppose that x is sufficiently large. Among all δ-amicable pairs $\{n_1, n_2\} \subset [1, X]$, all but*

$$\ll x(\log\log x)^4/(\log x)^{1/2} \tag{11}$$

satisfy each of the following conditions:

(i) *each $n_i > x/\log x$,*
(ii) *each $P(n_i) > \exp\big(\frac{1}{2}\log x/\log\log x\big)$,*
(iii) *each n_i has a prime factor from the interval*

$$\big(\exp((\log\log x)^{5/2}), \exp\big(\tfrac{1}{4}\sqrt{\log x/\log\log x}\big)\big],$$

(iv) *the largest squarefull divisor of each n_i is bounded above by $(\log x)^2$,*
(v) *if we write $n_i = P(n_i)m_i$, then $m_i > \exp\big(\frac{1}{4}\log x/\log\log x\big)$,*
(vi) *for every prime p dividing each n_i, we have $\Omega(p+1) < 5\log\log x$,*
(vii) *with the m_i defined as in (v),*

$$\frac{m_i - \delta}{P(m_i - \delta)} > \exp\big(\tfrac{1}{3}\sqrt{\log x/\log\log x}\big).$$

Proof. The number of δ-amicable pairs for which condition (i) fails is at most $x/\log x$, which is within the allowable bound (11). So such pairs may be ignored. We can similarly ignore those pairs failing (ii), since the number of these is at most

$$\Psi\big(X, \exp\big(\tfrac{1}{2}\log x/\log\log x\big)\big) \le X/(\log x)^{1+o(1)} \quad \text{as } x \to \infty,$$

by Lemma 4.2. Applying Lemma 4.1 with

$$y = \exp((\log\log x)^{5/2}) \quad \text{and} \quad z = \exp\big(\tfrac{1}{4}\sqrt{\log x/\log\log x}\big),$$

we see that the number of pairs where (iii) fails is

$$\ll X \frac{(\log\log x)^{5/2}}{\sqrt{\log x/\log\log x}} \ll x(\log\log x)^4/(\log x)^{1/2},$$

which is acceptable. Since the count of squarefull numbers in $[1,t]$ is $O(t^{1/2})$ for all $t \geq 1$, the number of failures of (iv) does not exceed

$$X \sum_{\substack{d>(\log x)^2 \\ d \text{ squarefull}}} \frac{1}{d} \ll X/\log x,$$

by partial summation. Again, this is allowable.

We have to work harder to deal with condition (v). Suppose that all of (i)–(iv) hold but that (v) fails. For notational convenience, put $P_i = P(n_i)$. Since each $P_i > \exp\big(\frac{1}{2}\log x/\log\log x\big)$ while $n_i = P_i m_i \leq X$, each $m_i \leq X/\exp\big(\frac{1}{2}\log x/\log\log x\big)$. So if (v) fails to hold, then the number of possibilities for the pair $\{m_1, m_2\}$ is at most

(12)
$$\frac{X}{\exp\big(\frac{1}{2}\log x/\log\log x\big)} \cdot \exp\big(\tfrac{1}{4}\log x/\log\log x\big) = X/\exp\big(\tfrac{1}{4}\log x/\log\log x\big).$$

We claim that m_1 and m_2 completely determine P_1 and P_2, and so also determine $n_1 = P_1 m_1$ and $n_2 = P_2 m_2$. Since the right-hand side of (12) satisfies the upper bound (11) with much room to spare, this shows that those pairs where (v) fails are indeed negligible. To prove the claim, observe that

$$\begin{aligned} P_1 m_1 = s'(P_2 m_2) &= \sigma(P_2 m_2) - P_2 m_2 - \delta \\ &= (P_2+1)\sigma(m_2) - P_2 m_2 - \delta = P_2 s(m_2) + \sigma(m_2) - \delta. \end{aligned}$$

(To simplify $\sigma(P_2 m_2)$, we used the fact that $P_2 \nmid m_2$; this follows from conditions (ii) and (iv) above.) By symmetry, we also have

$$P_2 m_2 = P_1 s(m_1) + \sigma(m_1) - \delta.$$

Rearranging, we obtain the following system of equations in P_1 and P_2:

$$\begin{aligned} P_1 m_1 + P_2(-s(m_2)) &= \sigma(m_2) - \delta, \\ P_1(-s(m_1)) + P_2 m_2 &= \sigma(m_1) - \delta. \end{aligned}$$

To show that P_1 and P_2 are uniquely determined by m_1 and m_2, it suffices to show that the determinant $\Delta := m_1 m_2 - s(m_1)s(m_2)$ is not zero. If we multiply the first equation by $s(m_1)$, the second by m_1, and add, we find that

$$\Delta \cdot P_2 = s(m_1)(\sigma(m_2) - \delta) + m_1(\sigma(m_1) - \delta). \tag{13}$$

We can assume that m_1 and m_2 both exceed δ. For example, if $m_1 \le \delta$ (so that $\delta > 0$), then there are at most $\delta \cdot \pi(X) \ll X/\log x$ choices for $n_1 = m_1 p$, and similarly if $m_2 \le \delta$. This is negligible. But if each $m_i > \delta$, then the right-hand side of (13) is positive, implying that $\Delta > 0$. This completes the proof that we can assume (v).

We now turn to condition (vi). Suppose that n_i has a prime divisor p with $\Omega(p+1) \ge 5 \log\log x$. Putting $Z := \lceil 5 \log\log x \rceil$, the number of possibilities for n_i is at most

$$\sum_{\substack{p \le X \\ \Omega(p+1) \ge Z}} \frac{X}{p} \le \sum_{\substack{p \le X \\ \Omega(p+1) \ge Z}} \frac{2X}{p+1} \le 2 \sum_{\substack{d \le X+1 \\ \Omega(d) \ge Z}} \frac{X}{d}$$
$$\ll X \frac{Z}{2^Z} \int_2^{X+1} \frac{\log t}{t}\, dt \ll X \frac{Z(\log X)^2}{2^Z} \ll \frac{X}{\log x},$$

where we use partial summation and Lemma 4.3 in the second line. So the number of pairs where (vi) fails is negligible.

Finally, we turn to condition (vii). We can suppose that all of (i)–(vi) hold. Note that (v) implies that $m_i - \delta > 0$ for each i. We now show that, ignoring a negligible set of pairs, each $m_i - \delta$ falls outside of the set $\mathscr{E}$ of Lemma 4.4. Indeed, suppose that some $m_i - \delta$ is in $\mathscr{E}$. Since $n_i = P_i m_i \le X$, the prime number theorem shows that the number of possibilities for P_i is

$$\ll \frac{X/m_i}{\log(X/m_i)} \ll \frac{X \log\log x}{m_i \log x},$$

using in the final step the inequality

$$\exp\left(\tfrac{1}{2} \log x / \log\log x\right) \le P_i \le X/m_i.$$

So summing over the possibilities for m_i, we find that the total number of corresponding numbers n_i is (again, for large x)

$$\ll \frac{X \log\log x}{\log x} \sum_{\substack{m \le X \\ m-\delta \in \mathscr{E}}} \frac{1}{m} \ll \frac{X \log\log x}{\log x} \sum_{\substack{m \le X \\ m-\delta \in \mathscr{E}}} \frac{1}{m-\delta}$$
$$\le \frac{X \log\log x}{\log x} \sum_{e \in \mathscr{E} \cap [1,2X]} \frac{1}{e} \ll \frac{X \log\log x}{\sqrt{\log x}},$$

where the final step follows by partial summation and the estimate of Lemma 4.4. This count of possible numbers n_i is negligible, and so we may assume that each $m_i - \delta$ is outside $\mathscr{E}$. But then (v) implies that

$$\frac{m_i - \delta}{P(m_i - \delta)} > \exp((\log(m_i - \delta))^{1/2}) > \exp\left(\tfrac{1}{3} \sqrt{\log x / \log\log x}\right)$$

for large x, so that (vii) holds. This completes the proof of Lemma 5.1. ∎

5.2. Completion of the proof of Theorem 1.2. We finish off the proof of Theorem 1.2 by establishing the following result.

PROPOSITION 5.2. *Let x be sufficiently large. The number of δ-amicable pairs $\{n_1, n_2\} \subset [1, X]$ satisfying all of the conditions of Lemma* 5.1 *is*

$$\ll x/\exp\big(\tfrac{1}{10}(\log\log x)^{3/2}\big).$$

The precise form of the upper bound is not essential; what is important is that it is (much) smaller than the upper estimate asserted in Theorem 1.2.

Proof of Proposition 5.2. Suppose that $\{n_1, n_2\} \subset [1, X]$ is a δ-amicable pair. We choose the labeling so that $P(n_1) \geq P(n_2)$. As above, we adopt the notation $P_i = P(n_i)$, and we write $n_i = P_i m_i$. Since condition (iii) of Lemma 5.1 is satisfied, we can choose a prime $\ell_1 \mid n_1$ with

$$\exp((\log\log x)^{5/2}) < \ell_1 \leq \exp\big(\tfrac{1}{4}\sqrt{\log x/\log\log x}\big).$$

By (ii), we have $\ell_1 < P_1$, so that

$$\ell_1 \mid m_1 = n_1/P_1. \tag{14}$$

Moreover, $\ell_1^2 \nmid n_1$ by condition (iv), so that $\ell_1 + 1 = \sigma(\ell_1) \mid \sigma(n_1)$. Set

$$r := P(\ell_1 + 1); \tag{15}$$

recalling condition (vi), we find that

$$r \geq (\ell_1 + 1)^{1/\Omega(\ell_1+1)} \geq (\ell_1 + 1)^{1/(5\log\log x)} \geq \exp\big(\tfrac{1}{5}(\log\log x)^{3/2}\big).$$

In what follows, we set

$$y := \exp\big(\tfrac{1}{5}(\log\log x)^{3/2}\big).$$

Since r divides $\sigma(n_1) = n_1 + n_2 + \delta = \sigma(n_2)$, there is a prime power $\ell_2^e \,\|\, n_2$ for which $r \mid \sigma(\ell_2^e)$. Moreover,

$$2\ell_2^e > \ell_2^e + \ell_2^{e-1} + \cdots + 1 = \sigma(\ell_2^e) \geq r,$$

so that $\ell_2^e > r/2 \geq y/2$. So if $e > 1$, then n_2 has the squarefull divisor $\ell_2^e > y/2$, contradicting condition (iv). Hence $e = 1$, so that $\ell_2^e = \ell_2$ and

$$r \mid \sigma(\ell_2) = \ell_2 + 1. \tag{16}$$

Observing that

$$\ell_2 \mid n_2 = s_\delta(n_1) = \sigma(P_1 m_1) - P_1 m_1 - \delta = P_1 s(m_1) + \sigma(m_1) - \delta,$$

we obtain the congruence

$$P_1 s(m_1) \equiv \delta - \sigma(m_1) \pmod{\ell_2}. \tag{17}$$

We now consider two cases, according to whether or not $\ell_2 \mid s(m_1)$.

Case I: $\ell_2 \nmid s(m_1)$. In this case, for a given m_1, (17) places P_1 in a uniquely determined residue class modulo ℓ_2, say $P_1 \equiv a(m_1) \pmod{\ell_2}$. So the number of possibilities for $n_1 = P_1 m_1$ is at most

$$\sum_{r>y} \sum_{\substack{\ell_1 \equiv -1 \pmod r \\ \ell_1 \le X}} \sum_{\substack{\ell_2 \equiv -1 \pmod r \\ \ell_2 \le X}} \sum_{\substack{m_1 \equiv 0 \pmod{\ell_1} \\ m_1 \le X}} \sum_{\substack{P_1 \equiv a(m_1) \pmod{\ell_2} \\ \ell_2 \le P_1 \le X/m_1}} 1.$$

Here r, ℓ_1, ℓ_2, and P_1 are understood to be prime; the congruence conditions on ℓ_1, ℓ_2 follow from (15) and (16), the congruence $m_1 \equiv 0 \pmod{\ell_1}$ comes from (14), and the inequality $P_1 \ge \ell_2$ follows from our initial assumption that $P(n_1) \ge P(n_2)$. Making crude upper estimates at each step, we find that our quintuple sum is

$$\begin{aligned} \text{(18)} \qquad &\ll \sum_r \sum_{\ell_1} \sum_{\ell_2} \sum_{m_1} \frac{X}{m_1 \ell_2} \ll X \log X \sum_r \sum_{\ell_1} \sum_{\ell_2} \frac{1}{\ell_1 \ell_2} \\ &\ll X(\log X)^3 \sum_r \frac{1}{r^2} \ll X(\log X)^3/y \ll x/\exp\big(\tfrac{1}{10}(\log\log x)^{3/2}\big). \end{aligned}$$

This completes the proof in this case.

Case II: $\ell_2 \mid s(m_1)$. In this case, the congruence (17) implies that $\ell_2 \mid \sigma(m_1) - \delta$. So $\ell_2 \mid (\sigma(m_1) - \delta) - s(m_1) = m_1 - \delta$, and thus

$$m_1 \equiv \delta \pmod{\ell_2}.$$

By condition (vii) and the selection of ℓ_1,

$$\frac{m_1 - \delta}{\ell_2} \ge \frac{m_1 - \delta}{P(m_1 - \delta)} \ge \exp\big(\tfrac{1}{3}\sqrt{\log x/\log\log x}\big) > \ell_1 + |\delta|,$$

so that $m_1 > \ell_1\ell_2 + \ell_2|\delta| + \delta > \ell_1\ell_2$. Hence, the number of possibilities for $n = P_1 m_1$ is at most

$$\sum_{r>y} \sum_{\substack{\ell_1 \equiv -1 \pmod r \\ \ell_1 \le X}} \sum_{\substack{\ell_2 \equiv -1 \pmod r \\ \ell_2 \le X}} \sum_{\substack{m_1 \equiv 0 \pmod{\ell_1} \\ m_1 \equiv \delta \pmod{\ell_2} \\ \ell_1\ell_2 < m_1 \le X}} \sum_{P_1 \le X/m_1} 1.$$

Now consider the fourth sum appearing above. If $\ell_1 \ne \ell_2$, then the simultaneous congruences $m_1 \equiv 0 \pmod{\ell_1}$ and $m_1 \equiv \delta \pmod{\ell_2}$ have a unique solution modulo $\ell_1\ell_2$. Otherwise, $\ell_1 = \ell_2 > r > |\delta|$, so that $\ell_2 \nmid \delta$ and the simultaneous congruences have no solution. So making crude estimates, we find that our quintuple sum is

$$\ll \sum_r \sum_{\ell_1} \sum_{\ell_2} \sum_{m_1} \frac{X}{m_1} \ll X \log X \sum_r \sum_{\ell_1} \sum_{\ell_2} \frac{1}{\ell_1 \ell_2}.$$

Continuing as in (18), we get an upper bound of $O\big(x/\exp\big(\tfrac{1}{10}(\log\log x)^{3/2}\big)\big)$. This completes the proof of the proposition, and also that of Theorem 1.2 for large x. But for bounded values of x, Theorem 1.2 is vacuous. ∎

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