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n-ARC CONNECTED SPACES

BY

BENJAMIN ESPINOZA (Greensburg, PA), PAUL GARTSIDE (Pittsburgh, PA)
and ANA MAMATELASHVILI (Pittsburgh, PA)

Abstract. A space is *n-arc connected* (n-ac) if any family of no more than n-points are contained in an arc. For graphs the following are equivalent: (i) 7-ac, (ii) n-ac for all n, (iii) continuous injective image of a closed subinterval of the real line, and (iv) one of a finite family of graphs. General continua that are $\aleph_0$-ac are characterized. The complexity of characterizing n-ac graphs for $n = 2, 3, 4, 5$ is determined to be strictly higher than that of the stated characterization of 7-ac graphs.

1. Introduction. A topological space X is called *n-arc connected* (n-ac) if for any points $p_1, \dots, p_n$ in X, there exists an arc α in X such that $p_1, \dots, p_n$ are all in α. If a space is n-ac for all $n \in \mathbb{N}$, then we will say that it is *ω-ac*. Note that this is equivalent to saying that for any finite F contained in X there is an arc α in X containing F. Call a space *$\aleph_0$-ac* if for every countable subset, S, there is an arc containing S. Evidently a space is arc connected if and only if it is 2-ac, and '$\aleph_0$-ac' implies 'ω-ac' implies '$(n+1)$-ac' implies 'n-ac' (for any fixed n).

Thus we have a family of natural strengthenings of arc connectedness, and the main aim of this paper is to characterize when 'nice' spaces have one of these strong arc connectedness properties. Secondary aims are to distinguish 'n-ac' (for each n), 'ω-ac' and '$\aleph_0$-ac', and to compare and contrast the familiar arc connectedness (i.e. 2-ac) with its strengthenings.

Observe that any Hausdorff image of an n-ac (respectively, ω-ac, $\aleph_0$-ac) space under a continuous injective map is also n-ac (respectively, ω-ac, $\aleph_0$-ac). Below, unless explicitly stated otherwise, all spaces are (metrizable) continua.

It turns out that 'sufficiently large' (in terms of dimension) arc connected spaces tend to be ω-ac. Indeed, it is not hard to see that manifolds (with or without boundary) of dimension at least 2 are ω-ac. Thus we focus on curves (1-dimensional continua) and especially on graphs (those connected spaces obtained by taking a finite family of arcs and then identifying some of the endpoints).

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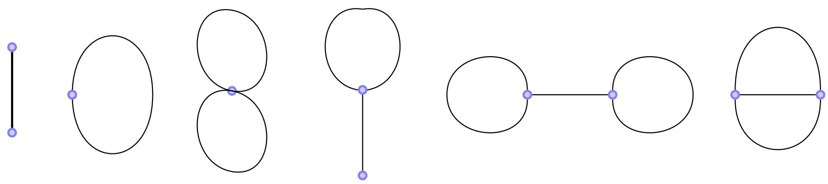
To motivate our main results consider the following examples.

Examples 1.1.

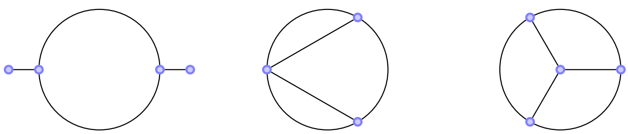
(A) The arc (the closed unit interval, $I = [0, 1]$) is $\aleph_0$-ac.
(B) The open interval, $(0, 1)$ and ray, $[0, 1)$, are ω-ac.
(C) From (A) and (B), all continua which are the continuous injective images of the arc, open interval and ray are ω-ac. It is easy to verify that these include: (a) arc, (b) circle, (c) figure eight curve, (d) lollipop, (e) dumbbell and (f) theta curve.

(a) (b) (c) (d) (e) (f)

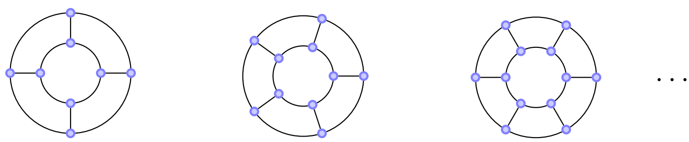


(D) The Warsaw circle, double Warsaw circle, Menger cube, and Sierpiński triangle are ω-ac.
(E) The simple triod is 2-ac but not 3-ac. It is minimal in the sense that no graph with strictly fewer edges is 2-ac not 3-ac. The graphs (a), (b) and (c) below are: 3-ac but not 4-ac, 4-ac but not 5-ac, and 5-ac but not 6-ac, respectively. All are minimal.

(a) (b) (c)



(F) The Kuratowski graph $K_{3,3}$ is 6-ac but not 7-ac. It is also minimal.
(G) The graphs below are all 6-ac and, by Theorem 1.3, none is 7-ac. Unlike $K_{3,3}$, all are planar. It is unknown if the first of these graphs (which has 12 edges) is minimal among planar graphs. A minimal example must have at least nine edges.



The diversity of examples in (D) of ω-ac curves suggests that no simple characterization of these continua is likely. The authors, together with Kovan-Bakan, prove that there is indeed no characterization of ω-ac curves any simpler than the definition (see [2] for details).

This prompts us to restrict attention to the more concrete case of graphs, and leads us to the following natural problems.

PROBLEMS 1.2.

(1) Characterize the ω-ac graphs.
(2) Characterize the $\aleph_0$-ac graphs.
(3) Characterize, for each n, the graphs which are n-ac but not $(n+1)$-ac.

In Section 2.1 below we show that the list of ω-ac graphs given in (C) is complete, answering Problem 1.2(1).

THEOREM 1.3. *For a graph G the following are equivalent:*

(1) *G is 7-ac.*
(2) *G is ω-ac.*
(3) *G is the continuous injective image of a subinterval of the real line.*
(4) *G is one of the following graphs: arc, simple closed curve, figure eight curve, lollipop, dumbbell or theta curve.*

In Section 2.2 we go on to characterize the $\aleph_0$-ac continua, giving a very strong solution to Problem 1.2(2).

THEOREM 1.4. *For any continuum K (not necessarily metrizable) the following are equivalent:*

(1) *K is $\aleph_0$-ac.*
(2) *K is a continuous injective image of a closed subinterval of the long line.*
(3) *K is one of: arc, long circle, long lollipop, long dumbbell, long figure eight curve or long theta curve.*

From Theorem 1.3 we see that there are no examples of graphs that are n-ac but not $(n+1)$-ac, for $n \geq 7$, solving Problem 1.2(3) in these cases. Of course Examples 1.1(E) and (F) show that there are n-ac non-$(n+1)$-ac graphs for $n = 2, 3, 4, 5, 6$. But the question remains: can we characterize these latter graphs? Informally our answers are as follows.

THEOREM 1.5.

(1) *The characterization of ω-ac graphs given in Theorem 1.3 is as simple as possible.*
(2) *There exist reasonably simple characterizations of n-ac non-$(n+1)$-ac graphs for $n \leq 7$.*
(3) *For $n = 2, 3, 4, 5$, there is no possible characterization of n-ac non-$(n+1)$-ac graphs which is as simple as that for ω-ac graphs.*

In Section 3 we outline some machinery from descriptive set theory which allows us to formalize and prove these claims. The situation with $n = 6$—the complexity of characterizing graphs which are 6-ac but not 7-ac—remains unclear. This, and other remaining open problems, are discussed in Section 4.

2. Characterizations. In this section we prove the characterization theorems stated in the Introduction: first, Theorem 1.3 characterizing ω-ac graphs; second, Theorem 1.4 characterizing $\aleph_0$-ac continua.

2.1. ω-ac graphs. As noted in Example 1.1(C), the graphs listed in part (4) of Theorem 1.3 are all the continuous injective images of a closed subinterval of the real line, giving (4)⇒(3), and all such images are ω-ac, yielding (3)⇒(2) of Theorem 1.3. Clearly ω-ac graphs are 7-ac, and so (2)⇒(1) in Theorem 1.3.

It remains to show (1)⇒(4) in Theorem 1.3, in other words that any 7-ac graph is one of the graphs listed in (4). This is established in Theorem 2.16 below. We proceed by establishing an ever tightening sequence of restrictions on the structure of 7-ac graphs.

We note that Lelek and McAuley [6] showed that the only Peano continua which are continuous injective images of the real line are the figure eight, dumbbell and theta curve. Their proof can be modified to establish the equivalence of (3) and (4) in Theorem 1.3.

PROPOSITION 2.1. *Let G be a finite graph, and let $H \subset G$ be a subgraph of G such that $G - H$ is connected, $\overline{G-H} \cap H = \{r\}$ and r is a branch point of G. If G is n-ac, then $\overline{G-H}$ is n-ac.*

Proof. First note that $\overline{G-H} = (G-H) \cup \{r\}$. Hence every connected set intersecting $G-H$ and $H - \{r\}$ must contain r.

Let $\mathcal{P} = \{p_1, \ldots, p_n\}$ be a set of n points in $\overline{G-H}$. Then, since G is n-ac, there exists an arc α in G containing $\mathcal{P}$. If $\alpha \subset \overline{G-H}$, we are done. So assume α intersects $H - \{r\}$. Let $t_0, t_1 \in [0,1]$ be such that $\alpha(t_0) \in G - H$ and $\alpha(t_1) \in H - \{r\}$; assume without loss of generality that $t_0 < t_1$. Hence there exists $s \in [t_0, t_1]$ such that $\alpha(s) = r$. Then $\alpha([0,s])$ is an arc in $\overline{G-H}$ containing $\mathcal{P}$, otherwise $r \in \alpha((s,1])$, which is impossible since α is an injective image of $[0,1]$. This proves that $\overline{G-H}$ is n-ac. ∎

The reverse implication of Proposition 2.1 does not hold. To see this, let G be a simple triod and H be one of the edges of G. Clearly G is not 3-ac but $\overline{G-H}$ (an arc) is 3-ac.

DEFINITION 2.2. Let G be a finite graph. An edge e of G is called a *terminal edge* of G if one of the vertices of e is an end-point of G.

DEFINITION 2.3. Let G be a finite graph, and let $I = \{e_1, \ldots, e_m\}$ be the set of terminal edges of G. Let G^* be the graph given by $\overline{G-I}$. Clearly this operation can be applied to G^* as well. We perform this operation as many times as necessary until we obtain a graph G' having no terminal edges. We call G' the *reduced graph of G*.

The following is a corollary of Proposition 2.1.

COROLLARY 2.4. *Let G be an n-ac finite graph. Then the reduced graph of G is an n-ac finite graph containing no terminal edges.*

Proof. Observe that the reduced graph of G can also be obtained by removing terminal edges one at a time.

Now, from Proposition 2.1, if G is an n-ac finite graph and e is a terminal edge of G, then $\overline{G-e}$ is n-ac. This implies that each time we remove a terminal edge we obtain an n-ac graph. This and the observation prove the corollary. ■

REMARK 2.5. Note that if X is an n-ac space and $\{p_1, \ldots, p_n\}$ are n different points of X, then there is an arc α such that $\{p_1, \ldots, p_n\} \subset \alpha$ and the end-points of α belong to $\{p_1, \ldots, p_n\}$. To see this, let β be the arc containing $\{p_1, \ldots, p_n\}$, given by the fact that X is n-ac. Let $t_0 = \min\{\beta^{-1}(p_i) : i = 1, \ldots, n\}$ and $t_1 = \max\{\beta^{-1}(p_i) : i = 1, \ldots, n\}$. Then $\alpha \in \beta([t_0, t_1])$ satisfies the desired conditions.

From now on, if X is an n-ac space, $\{p_1, \ldots, p_n\}$ are n different points and α is an arc passing through $\{p_1, \ldots, p_n\}$, then we will assume that the end-points of α belong to $\{p_1, \ldots, p_n\}$.

LEMMA 2.6. *Let G be a finite graph. Assume that G contains a simple triod $T = L_1 \cup L_2 \cup L_3$ (with $\{q\} = L_i \cap L_j$ for $i \neq j$) such that for each i, $L_i - \{q\}$ contains no branch points of G. For each $i = 1, 2, 3$, let $p_i \in \operatorname{int}(L_i)$. If α is an arc containing $\{p_1, p_2, p_3\}$, then*

(1) *$q \in \operatorname{int}(\alpha)$, and*
(2) *at least one of the end-points of α lies in $[q, p_1] \cup [q, p_2] \cup [q, p_3]$.*

Proof. Let G, T and p_1, p_2, p_3 as in the hypothesis of the lemma. Let $\alpha \subset G$ be an arc containing $\{p_1, p_2, p_3\}$, and denote, for each $i = 1, 2, 3$, by $[q, l_i]$ the arc L_i.

(1) Assume, without loss of generality, that $\alpha(t_i) = p_i$ and $t_1 < t_2 < t_3$. Then $p_2 \in \operatorname{int}(\alpha)$ and $\alpha = \alpha([0, t_2]) \cup \alpha([t_2, 1])$.

We consider two cases: $q \notin \alpha([0, t_2])$ and $q \in \alpha([0, t_2])$. Assume $q \notin \alpha([0, t_2])$. Then, since $L_2 - \{q\}$ contains no branch points of G and $p_2 \in \operatorname{int}(L_2)$, we have $l_2 = \alpha(s)$ for some s with $0 < s < t_2$. Hence $[l_2, p_2] \subseteq \alpha([0, t_2])$. Therefore, since $\{p_2, p_3\} \subseteq \alpha([t_2, 1])$, $p_2 \in \operatorname{int}(L_2)$, $L_2 - q$ has no branch points of G, and α is a 1-1 function, we deduce that $[p_2, q] \subseteq \alpha([t_2, 1))$. This implies that $q \in \operatorname{int}(\alpha)$.

Now suppose that $q \in \alpha([0, t_2])$. If $q \in \alpha((0, t_2])$, then we are done. So assume that $q = \alpha(0)$, i.e. q is an end-point of α. Using the same argument as in the previous case, we can conclude that $[l_2, p_2] \subseteq \alpha([0, t_2])$. This implies, as before, that $[p_2, q] \subseteq \alpha([t_2, 1))$, which contradicts the fact that α is a 1-1 function. Hence $q \in \operatorname{int}(\alpha)$.

(2) First, assume that $\alpha(t_i) = p_i$ and $t_1 < t_2 < t_3$. We will show that one end-point of α lies on either $[q, p_1]$ or $[q, p_3]$. The other cases (rearrangements of the t_is) are done in the same way, the only difference is the conclusion: the end-point lies on $[q, p_1] \cup [q, p_2]$, or on $[q, p_2] \cup [q, p_3]$.

By (1), $q \in \operatorname{int}(\alpha)$ and if $q = \alpha(s)$, then $s < t_3$; otherwise the arc $\alpha([0, t_3])$ would contain p_1, p_2, p_3 and $q \notin \operatorname{int}(\alpha([0, t_3]))$, contrary to (1). Similarly, $t_1 < s$. Hence $t_1 < s < t_3$.

If $s < t_2$, then $p_1, q \notin \alpha([t_2, 1]) = \alpha([t_2, t_3]) \cup \alpha([t_3, 1])$. Now, since $L_3 - \{q\}$ has no branch points of G, $q \in \alpha([0, t_2])$, and $p_3 \in \operatorname{int}(L_3)$, we have $l_3 \in \alpha([t_2, t_3])$. Thus, since α is a 1-1 function, $\alpha([t_3, 1]) \subset (q, p_3]$. This shows that $\alpha(1)$ lies in $[q, p_3]$.

If $t_2 < s$, then a similar argument using $-\alpha$ (α traveled in the opposite direction) shows that one of the end-points of α lies on $[q, p_1]$. ■

We obtain the following corollaries.

COROLLARY 2.7. *With the same conditions as in Lemma* 2.6*, if α is an arc containing $\{p_1, p_2, p_3\}$, and $q = \alpha(s)$, $p_i = \alpha(t_i)$ for $i = 1, 2, 3$, then $t_j < s < t_k$ for some $j, k \in \{1, 2, 3\}$.*

Proof. To see this, note that if q does not lie between two of the p_is, then either $s < t_i$ for all i, or $t_i < s$ for all i. Then either $\alpha([s, 1])$ or $\alpha([0, s])$ is an arc containing $\{p_1, p_2, p_3\}$ for which q is an end-point; this contradicts (1) of Lemma 2.6. ■

COROLLARY 2.8. *Let G be a finite graph, and let $\{p_1, \ldots, p_n\} \subset G$ be n different points. In addition, let α be an arc containing $\{p_1, \ldots, p_n\}$, with end-points belonging to $\{p_1, \ldots, p_n\}$. If there are three different indices i, j, k such that p_i, p_j and p_k belong to a triod T satisfying the conditions of Lemma* 2.6*, and such that $([q, p_i] \cup [q, p_j] \cup [q, p_k]) \cap \{p_1, \ldots, p_n\} = \{p_i, p_j, p_k\}$, then either p_i, p_j or p_k is an end-point of α.*

Proof. By (2) of Lemma 2.6, at least one of the end-points of α lies in $[q, p_i] \cup [q, p_j] \cup [q, p_k]$. Hence, since the end-points of α belong to $\{p_1, \ldots, p_n\}$ and $([q, p_i] \cup [q, p_j] \cup [q, p_k]) \cap \{p_1, \ldots, p_n\} = \{p_i, p_j, p_k\}$, one of p_i, p_j or p_k is an end-point of α. ■

PROPOSITION 2.9. *Let G be a finite graph. If G is* 5*-ac, then G has no branch point of degree greater than or equal to five.*

Proof. Assume, for contradiction, that G contains at least one branch point, q, of degree at least 5. Then, since G is a finite graph, G contains a simple 5-od, $T = L_1 \cup L_2 \cup L_3 \cup L_4 \cup L_5$, such that $\{q\} = L_i \cap L_j$ for $i \neq j$, and $L_i - \{q\}$ contains no branch points of G.

For each $i = 1, \ldots, 5$, let $p_i \in \operatorname{int}(L_i)$. Then, since G is 5-ac, there exists an arc $\alpha \subset G$ such that $\{p_1, \ldots, p_5\} \subset \alpha$. Note that T contains a triod satisfying the conditions of Lemma 2.6, hence $q \in \operatorname{int}(\alpha)$. Let $t_0 \in (0, 1)$ be the point such that $\alpha(t_0) = q$. Then $\alpha - \{q\} = \alpha([0, t_0)) \cup \alpha((t_0, 1])$, and either $\alpha([0, t_0])$ or $\alpha([t_0, 1])$ contains three points out of $\{p_1, p_2, p_3, p_4, p_5\}$; note that q is an end-point of $\alpha([0, t_0])$ and of $\alpha([t_0, 1])$. Without loss of generality, suppose that $p_1, p_2, p_3 \subset \alpha([0, t_0])$; then L_1, L_2, L_3 and the corresponding p_is satisfy the conditions of Lemma 2.6, implying that any arc containing those points contains q in its interior, a contradiction, since q is an end-point of $\alpha([0, t_0])$. This shows that G does not contain a branch point of degree greater than or equal to five. ∎

From Proposition 2.9 we obtain the following corollary.

COROLLARY 2.10. *Let G be a finite graph. If G is n-ac for $n \geq 5$, then G has no branch point of degree greater than or equal to five.*

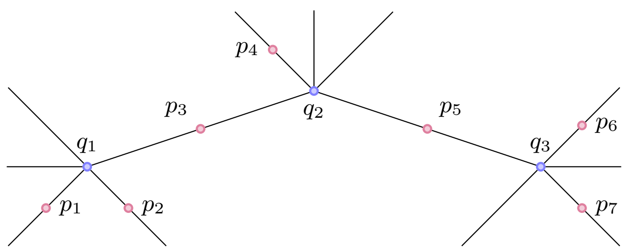
The following proposition is easy to prove.

PROPOSITION 2.11. *Let G be a finite connected graph. If G has at least three branch points, then there is an arc α such that the end-points of α are branch points of G and all the points of the interior of α, except for one, are non-branch points of G. So α contains exactly three branch points of G.*

THEOREM 2.12. *A finite graph with three or more branch points cannot be 7-ac.*

Proof. Let G be a finite graph with at least three branch points. By Proposition 2.11, there is an arc α in G containing exactly three branch points of G such that two of them are the end-points of α. Denote by q_1, q_2, and q_3 these branch points, and assume without loss of generality that q_1 and q_3 are the end-points of α.

Let p_3 be a point between q_1 and q_2, and let p_5 be a point between q_2 and q_3. Since G is a finite graph, we can find, in a neighborhood of q_1, two points p_1 and p_2 such that p_1, p_2, p_3 belong to a triod T_1 satisfying the conditions of Lemma 2.6, and q_1 is the branch point of T_1. Similarly, we can find a point p_4 in a neighborhood of q_2 such that p_3, p_4, p_5 belong to a triod T_2 satisfying the conditions of Lemma 2.6, and q_2 is the branch point of T_2. Finally, we can find two points p_6 and p_7 in a neighborhood of q_3 such that p_5, p_6, p_7 belong to a triod T_3 satisfying the conditions of Lemma 2.6, and q_3 is the branch point of T_3.



We show that there is no arc containing $\{p_1, \dots, p_7\}$. Suppose for contradiction that such an arc $\beta \subset G$ exists. Using the same argument from Remark 2.5, we can assume that the end-points of β belong to $\{p_1, \dots, p_7\}$.

Now, by Corollary 2.8, one of $\{p_1, p_2, p_3\}$ is an end-point of β. Similarly, one of $\{p_3, p_4, p_5\}$ is an end-point of β, and one of $\{p_5, p_6, p_7\}$ is an end-point of β. So, since β is an arc with end-points in $\{p_1, \dots, p_7\}$, we deduce that either

(i) p_1 or p_2 and p_5 are the end-points of β, or
(ii) p_6 or p_7 and p_3 are the end-points of β, or
(iii) p_3 and p_5 are the end-points of β,

and that these are the only possible cases. We will prove that every case leads to a contradiction.

(i) Assume that p_1 and p_5 are the end-points of β. Since the arc between q_2 and q_3 contains no branch points of G, we have either $[q_2, p_5] \subset \beta$ or $[p_5, q_3] \subset \beta$.

Assume first that $[q_2, p_5] \subset \beta$. Then, by the way p_4 was chosen and the fact that $p_4 \in \operatorname{int}(\beta)$, the arc $[p_4, q_2]$ is contained in β; similarly, since the arc $[q_1, q_2]$ contains no branch points of G and as $p_3 \in \operatorname{int}(\beta)$, we have $[p_3, q_2] \subset \beta$. Then $[p_3, q_2] \cup [p_4, q_2] \cup [q_2, p_5] \subset \beta$, which is a contradiction since $[p_3, q_2] \cup [p_4, q_2] \cup [q_2, p_5]$ is a non-degenerate simple triod.

Assume that $[p_5, q_3] \subset \beta$. Then, by the way p_6 was chosen and the fact that $p_6 \in \operatorname{int}(\beta)$, we have $[q_3, p_6] \subset \beta$. Using the same argument we can conclude that $[q_3, p_7] \subset \beta$. Hence $[p_5, q_3] \cup [p_6, q_3] \cup [q_3, p_7] \subset \beta$, which is a contradiction.

The case when p_2 and p_5 are the end-points of β is similar to the case we just proved. So (i) does not hold.

(ii) This case is equivalent to (i), therefore (ii) does not hold.

(iii) Assume that p_3 and p_5 are the end-points of β. Then, since the arc $[q_1, q_2]$ contains no branch points of G and p_3 is an end-point of β, either $[q_1, p_3] \subset \beta$ or $[p_3, q_2] \subset \beta$.

Suppose that $[q_1, p_3] \subset \beta$. As in (i), since $p_1, p_2 \in \operatorname{int}(\beta)$, we see that the arcs $[p_1, q_1]$ and $[q_1, p_2]$ are contained in β. This implies that the non-degenerate simple triod $[q_1, p_3] \cup [p_1, q_1] \cup [q_1, p_2]$ lies in β, which is a contradiction.

Now assume that $[p_3, q_2] \subset \beta$. Then $[p_5, q_3] \subset \beta$. Again, the same argument as in (i) leads to a non-degenerate simple triod being contained in β since $p_6, p_7 \in \operatorname{int}(\beta)$. Hence (iii) does not hold.

This proves that there is no arc containing $\{p_1, \dots, p_7\}$. Therefore G is not 7-ac. ∎

Since every $(n+1)$-ac space is n-ac, we have the following corollary.

Corollary 2.13. *A finite graph with three or more branch points cannot be n-ac for $n \geq 7$.*

Lemma 2.14. *If G is a finite graph with only two branch points each of degree greater than or equal to 4, then G is not 7-ac.*

Proof. Let q_1 and q_2 be the two branch points of G. Then there exists at least one edge e having q_1 and q_2 as vertices. Let $p_1 \in \operatorname{int}(e)$. Since G is a finite graph, and q_1 and q_2 have degree at least 4, we can choose three points p_2, p_3, p_4 in a neighborhood of q_1 such that $T_1 = [q_1, p_1] \cup [q_1, p_2] \cup [q_1, p_3] \cup [q_1, p_4]$ is a simple 4-od, and three points p_5, p_6, p_7 in a neighborhood of q_2 such that $T_2 = [q_2, p_1] \cup [q_2, p_5] \cup [q_2, p_6] \cup [q_2, p_7]$ is a simple 4-od, and they are such that $T_1 \cap T_2 = \{p_1\}$.

We show that there is no arc $\alpha \subset G$ containing $\{p_1, \ldots, p_7\}$. Suppose for contradiction that there exists such an arc α, and assume further that the end-points of α belong to $\{p_1, \ldots, p_7\}$. Then, since $\{p_2, p_3, p_4\}$ satisfy the conditions of Corollary 2.8, we can assume without loss of generality that p_4 is an end-point of α. Similarly for the set $\{p_5, p_6, p_7\}$, so we can assume without loss of generality that p_5 is an end-point of α. On the other hand, the set $\{p_1, p_2, p_3\}$ also satisfies the conditions of Corollary 2.8, hence p_1 or p_2 or p_3 is an end-point of α, which is impossible since α only has two end-points. This shows that there is no arc in G containing $\{p_1, \ldots, p_7\}$, and proves that G is not 7-ac. ∎

Corollary 2.15. *If G is a finite graph with only two branch points each of degree greater than or equal to 4, then G is not n-ac for $n \geq 7$.*

Theorem 2.16. *Let G be a finite graph. If G is 7-ac, then G is one of the following graphs: arc, simple closed curve, figure eight, lollipop, dumbbell or theta curve.*

Proof. Let G be a finite graph. Suppose that G is n-ac for $n \geq 7$. We will show that G is (homeomorphic to) one of the listed graphs.

Let K be the reduced graph of G. By Corollary 2.4, K is n-ac and contains no terminal edges. By Theorem 2.12, K has at most two branch points, and by Corollary 2.10, the degree of each branch point is at most 4. We consider the cases when K has no branch points, one branch point or two branch points.

Case 1: K has no branch points. In this case K is either homeomorphic to the arc, I, or to the simple closed curve, S^1.

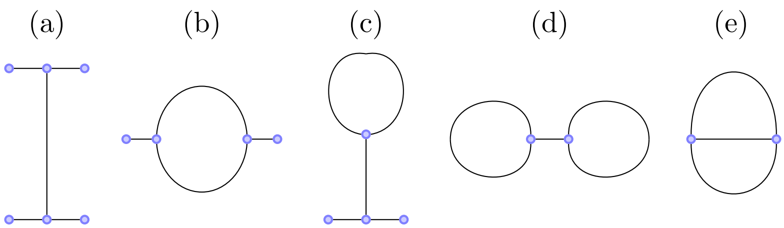
Assume first that K is homeomorphic to I; then, by the way K is obtained, $K = G$. Otherwise, reattaching the last terminal edge that was removed gives a simple triod which is not 7-ac, contrary to the hypothesis. In this case G is on the list.

Next assume K is homeomorphic to S^1. If $K = G$, then G is on the list. So assume $G \neq K$, and let e denote the last terminal edge that was removed. Then $K \cup e$ is homeomorphic to the lollipop curve. Furthermore, $G = K \cup e$, otherwise reattaching the penultimate terminal edge will give a homeomorphic copy of the graph (a) of Example 1.1(E), which is not 7-ac, or a simple closed curve with two arcs attached to it at the same point at one of their end-points, which is not 7-ac either. Hence, again, G is on the list.

Case 2: K has one branch point. Note that the only possibility for K to have a single branch point of degree 3 is for K to be homeomorphic to a simple triod or to the lollipop curve; the former is not 7-ac and the latter is not a reduced graph. Hence the degree of the branch point of K is 4. In this case K is homeomorphic either to a simple 4-od, a simple closed curve with two arcs attached to it at the same point at one of their end-points, or the figure eight curve. The first two cases are not 7-ac. Therefore K must be homeomorphic the figure eight curve. If $G = K$, then G is on the list. In fact, since attaching an arc to the figure eight curve yields a non-7-ac curve, we must have $G = K$.

Case 3: K has two branch points. Since the sum of the degrees in a graph is always even and K has no terminal edges, K cannot have one branch point of degree 3 and another of degree 4. Hence the only options are that K has two branch points of either degree 3 or degree 4. However, by Corollary 2.15, K has only two branch points of degree 3.

If K has two branch points of degree 3, then it could be homeomorphic to one of the following graphs:



However the graphs (a), (b), and (c) contain terminal edges. So K can only be homeomorphic to the dumbbell (d) or the theta curve (e); in any case, if $G = K$, then G is on the list. Note that neither curve, (d) nor (e), can be obtained from a n-ac graph ($n \geq 7$) by removing a terminal edge since by Theorem 2.12 the edge has to be attached to one of the existing branch points; it is easy to see that such a graph is not 4-ac, just take a point in the interior of each edge. Hence $G = K$. This ends the proof of the theorem. ∎

2.2. $\aleph_0$-ac continua. Call a space *κ-ac*, where κ is a cardinal, if every subset of size no more than κ is contained in an arc. Note that for finite $\kappa = n$ and $\kappa = \aleph_0$ this coincides with the earlier definitions. For infinite κ we have a complete description of κ-ac continua (not necessarily metrizable), extending Theorem 1.4. To start, let us observe that the arc is κ-ac for every cardinal κ. We will see shortly that the arc is the only separable κ-ac continuum when κ is infinite. In particular, the triod and circle are not $\aleph_0$-ac, and so any continuum containing a triod or a circle is also not $\aleph_0$-ac. This observation will be used below.

To state the theorem precisely we need to make a few definitions. Recall that ω_1 is the first uncountable ordinal, or equivalently the set of all countable ordinals, with the induced order topology. Note that a subset of ω_1 is bounded if and only if the set is countable. The *long ray*, R, is the lexicographic product of ω_1 with $[0,1)$ with the order topology. We can identify ω_1 (with its usual order topology) with $\omega_1 \times \{0\}$. Evidently ω_1 is cofinal in the long ray. Write R^- for R with each point x relabeled $-x$. The *long line*, L, is the space obtained by identifying 0 in the long ray, R, with -0 in R^-. The topology on the long ray and long line ensures that for any $x < y$ in R (or L) the subspace $[x,y] = \{z \in R : x \leq z \leq y\}$ is (homeomorphic to) an arc. Note that any countable subset of the long ray, or the long line, is bounded, hence both the long ray and long line are $\aleph_0$-ac. To see this for the long ray take any countable subset S; then since ω_1 is cofinal in R, the set S has an upper bound, x say, and then S is contained in $[0,x]$, which is an arc.

Let αR be the one-point compactification of R, and γL be the corresponding two-point compactification of L. The *long circle* and *long lollipop* are the spaces obtained from αR by identifying the point at infinity to 0, or any other point, respectively. The *long dumbbell*, *long figure eight* and *long theta* curves come from γL by respectively identifying the negative ($-\infty$) and positive ($+\infty$) end-points to -1 and $+1$, 0 and 0, or $+1$ and -1. As continuous injective images of the $\aleph_0$-ac spaces R and L, all the above spaces are also $\aleph_0$-ac.

Theorem 2.17. *Let K be a continuum.*

(1) *If K is separable and $\aleph_0$-ac then K is an arc.*
(2) *If K is non-separable, then the following are equivalent:*
   (i) *K is $\aleph_0$-ac,*
   (ii) *K is a continuous injective image of a closed subinterval of the long line, and*
   (iii) *K is one of: long circle, long lollipop, long dumbbell, long figure eight curve, or long theta curve.*
(3) *If K is κ-ac for some $\kappa > \aleph_0$, then K is an arc.*

For part (1) just take a dense countable set; then any arc containing the dense set is the whole space. Part (2) is proved in Propositions 2.18

((i)⇒(ii)) and 2.21 ((ii)⇒(iii)) below, while (iii)⇒(i) was observed above when defining the curves in (2)(iii). For part (3) note that all non-separable $\aleph_0$-ac spaces (as listed in part (2)(iii)) have a dense set of size $\aleph_1$, and so are not $\aleph_1$-ac. Thus κ-ac continua for $\kappa \geq \aleph_1$ are separable, hence arcs, by part (1).

It is traditional to use Greek letters (α, β et cetera) for ordinals. Consequently we will use the letter 'A' and variants for arcs, and because in Proposition 2.18 we need to construct a map, in this subsection by an 'arc' we mean any homeomorphism between the closed unit interval and a subset of a given space. If K is a space, then by 'A is an arc in K' we mean that the arc A maps into K. When A is an arc in a space K, then write $\mathrm{im}(A)$ for the image of A (it is, of course, a subspace of K homeomorphic to the closed unit interval). For any function f, we write $\mathrm{dom}(f)$ for the domain of f.

PROPOSITION 2.18. *Let K be an $\aleph_0$-ac non-separable continuum. Then there is a continuous bijection $A_\infty : J_\infty \to K$ where J_∞ is a closed unbounded subinterval of the long line L.*

We prove this by an application of Zorn's Lemma. The following lemmas help to establish that Zorn's Lemma is applicable, and that the maximal object produced is as required.

LEMMA 2.19. *Let K be an $\aleph_0$-ac non-separable continuum. If $\mathcal{K}$ is a countable collection of separable subspaces of K then there is an arc A in K such that $\bigcup \mathcal{K} \subseteq \mathrm{im}(A)$.*

Proof. Let $\mathcal{K} = \{S_n : n \in \mathbb{N}\}$ be a countable family of subspaces of K, and, for each n, let D_n be a countable dense subset of S_n. Let $D = \bigcup_n D_n$; it is countable. Since K is $\aleph_0$-ac there is an arc A in K such that $D \subseteq \mathrm{im}(A)$. As D is dense in $\bigcup \mathcal{K}$ and $\mathrm{im}(A)$ is closed, we see that $\bigcup \mathcal{K} \subseteq \mathrm{im}(A)$. ∎

LEMMA 2.20. *Let K be an $\aleph_0$-ac non-separable continuum. Suppose $[a, b]$ is a proper closed subinterval of L (or R), $A : [a, b] \to K$ is an arc in K and $y \in K \setminus \mathrm{im}(A)$. Then either*

(i) *for every $c > b$ in L there is an arc $A' : [a, c] \to K$ such that $A'\restriction_{[a,b]} = A$ and $A'(c) = y$, or*
(ii) *for every $c < a$ in L there is an arc $A' : [c, b] \to K$ such that $A'\restriction_{[a,b]} = A$ and $A'(c) = y$.*

Proof. Fix a, b, the arc A and y. Let $\mathcal{K} = \{\mathrm{im}(A), \{y\}\}$. Lemma 2.19 yields an arc $A_0 : [0, 1] \to K$ in K such that $\mathrm{im}(A_0) \supseteq \mathrm{im}(A) \cup \{y\}$. Let $J = A_0^{-1}(\mathrm{im}(A))$, $a' = \min J$, $b' = \max J$ and $c' = A_0^{-1}(y)$. Without loss of generality (replacing A_0 with $A_0 \circ \rho$, where $\rho(t) = 1 - t$, if necessary) we can suppose that $A_0(a') = A(a)$ and $A_0(b') = b$.

Since $y \notin \mathrm{im}(A)$, either $c' > b'$ or $c' < a'$. Let us suppose that $c' > b'$. This will lead to case (i) in the statement of the lemma. The other choice will give, by a very similar argument which we omit, case (ii). Take any c in L such that $c > b$. Let A_1 be a homeomorphism of the closed subinterval $[a, c]$ of L onto the subinterval $[a', c']$ of $[0, 1]$ such that $A_1(a) = a'$, $A_1(b) = b'$ and $A_1(c) = c'$. Set $A_2 = A_0 \circ A_1 : [a, c] \to K$. So A_2 is an arc in K such that $A_2(a) = A(a)$, $A_2(b) = A(b)$, $A_2(c) = y$ and $A_2([a, b]) = \mathrm{im}(A)$. The arc A_2 is almost what we require for A' but it may traverse the (set) arc $\mathrm{im}(A)$ at a 'different speed' than A. Thus we define $A' : [a, c] \to K$ to be equal to A on $[a, b]$ and equal to A_2 on $[b, c]$. Then A' is the required arc. ∎

Proof of Proposition 2.18. Let $\mathcal{A}$ be the set of all continuous injective maps $A : J \to K$ where J is a closed subinterval of L, ordered by: $A \leq A'$ if and only if $\mathrm{dom}(A) \subseteq \mathrm{dom}(A')$ and $A' \restriction_{\mathrm{dom}(A)} = A$. Then $\mathcal{A}$ is the set of all candidates for the map A_∞ we seek. We will apply Zorn's Lemma to $(\mathcal{A}, \leq)$ to extract A_∞. To do so we need to verify that $(\mathcal{A}, \leq)$ is non-empty, and all non-empty chains have upper bounds.

As K is $\aleph_0$-arc connected we know there are many arcs in K, so the set $\mathcal{A}$ is not empty. Now take any non-empty chain $\mathcal{C}$ in $\mathcal{A}$. We show that $\mathcal{C}$ has an upper bound. Let $\mathcal{J} = \{\mathrm{dom}(A') : A' \in \mathcal{C}\}$. Since $\mathcal{J}$ is a chain of subintervals in L, the set $J = \bigcup \mathcal{J}$ is also a subinterval of L. Define $A : J \to K$ by $A(x) = A'(x)$ for any A' in $\mathcal{C}$ with $x \in \mathrm{dom}(A')$. Since $\mathcal{C}$ is a chain of injections, A is well-defined and injective. Since the domains of the functions in $\mathcal{C}$ form a chain of subintervals, any point x in J is in the J-interior of some $\mathrm{dom}(A')$ (there is a set U, open in J, such that $x \in U \subseteq \mathrm{dom}(A')$), where $A' \in \mathcal{C}$, and so A coincides with A' on some J-neighborhood of x, thus, since A' is continuous at x, the map A is also continuous at x. If J is closed, then we are done: A is in $\mathcal{A}$ and $A \geq A'$ for all A' in $\mathcal{C}$.

If the interval J is not closed then it has at least one endpoint (in L) not in J. We will suppose $J = (a, \infty)$. The other cases, $J = (a, b)$ and $J = (-\infty, a)$, can be dealt with similarly. We show that we can continuously extend A to $[a, \infty)$. If so then A will be injective, hence in $\mathcal{A}$, and an upper bound for $\mathcal{C}$. Indeed, the only way the extended A could fail to be injective would be if $A(a) = A(c)$ for some $c > a$, and then $A([a, c])$ is a circle in K, contradicting the fact that K is $\aleph_0$-ac.

Evidently it suffices to continuously extend $A' = A\restriction_{(a,b]}$ to $[a, b]$. Let $\mathcal{K} = \{A((a, b])\}$ and apply Lemma 2.19 to see that A' maps the half-open interval, $(a, b]$, into I_K, a homeomorphic copy of the unit interval. Let $h : [0, 1] \to I_K$ be a homeomorphism. So we can apply some basic real analysis to get the extension. Indeed, the map $A' \circ h^{-1}$ is continuous and injective, and hence strictly monotone. By the inverse function theorem, A' has a continuous inverse, and so it is a homeomorphism of $(a, b]$ with some half-open interval, $(c, d]$ or $[d, c)$ in the closed unit interval. Defining $A(a) = h(c)$ gives the desired continuous extension.

Let A_∞ be a maximal element of $\mathcal{A}$. Then its domain is a closed subinterval of the long line L. We first check that $\mathrm{dom}(A_\infty)$ is not bounded. Then we prove that A_∞ maps onto K.

If A_∞ had a bounded domain, then it is an arc. So it has separable image. As K is not separable we can pick a point y in $K \setminus \mathrm{im}(A_\infty)$. Applying Lemma 2.20 we can properly extend A_∞ to an arc A'. But then A' is in $\mathcal{A}$, $A_\infty \leq A'$ and $A_\infty \neq A'$, contradicting maximality of A_∞.

We complete the proof by showing that A_∞ is surjective. We go for a contradiction and suppose that instead there is a point y in $K \setminus \mathrm{im}(A_\infty)$. Two cases arise depending on the domain of A_∞.

Suppose first that $\mathrm{dom}(A_\infty) = L$. Pick a point x in $\mathrm{im}(A_\infty)$. Pick an arc A from x to y. Taking a subarc if necessary, we can suppose $A : [0, 1] \to K$, $A(0) = x$ and $A(t) \notin \mathrm{im}(A_\infty)$ for all $t > 0$. Let $x' = A_\infty^{-1}(x)$. Pick any a', b' from L such that $a' < x' < b'$. Then the subspace $A_\infty([a', b']) \cup A([0, 1])$ is a triod in K, which contradicts K being $\aleph_0$-ac.

Now suppose that $\mathrm{dom}(A_\infty)$ is a proper subset of L. Let us assume that $\mathrm{dom}(A_\infty) = [a, \infty)$. (The other case, $\mathrm{dom}(A_\infty) = (-\infty, a]$, follows similarly.) Pick any $b > a$, and apply Lemma 2.20 to $A = A_\infty\restriction_{[a,b]}$ and y. If case (ii) holds then pick any $c < a$ and A can be extended 'to the left' to an arc A' with domain $[c, b]$. This gives a proper extension of A_∞ defined on $[c, \infty)$ (which is A' on $[c, a]$ and A_∞ on $[a, \infty)$), contradicting maximality of A_∞.

So case (i) must hold. Pick any $c > b$, and get an arc $A' : [a, c] \to K$ in K extending A. Let $T = A_\infty([a, c]) \cup A'([a, c])$. Observe that T has at least three non-cut-points, namely $A'(a) = A_\infty(a)$, $A_\infty(c)$ and $A'(c)$. So T is not an arc, but it is a separable subcontinuum of the $\aleph_0$-ac continuum K, which is the desired contradiction. ∎

To complete the proof of Theorem 2.17 it remains to identify the continuous injective images of closed subintervals of the long line. We recall some basic definitions and facts connected with the space of countable ordinals, ω_1 (see [5], for example). A subset of ω_1 is *closed and unbounded* if it is cofinal in ω_1 and closed in the order topology. A countable intersection of closed and unbounded sets is closed and unbounded. The set Λ of all limit ordinals in ω_1 is a closed and unbounded set. A subset of ω_1 is *non-stationary* if it is contained in the complement of a closed and unbounded set. A subset of ω_1 is *stationary* if it is not non-stationary, or equivalently if it meets every closed and unbounded set. The Pressing Down Lemma (also known as Fodor's Lemma) states than if S is a stationary set and $f : S \to \omega_1$ is *regressive* (for every α in S we have $f(\alpha) < \alpha$) then there is a β in ω_1 such that $f^{-1}(\beta)$ is cofinal in ω_1.

PROPOSITION 2.21. *If K is a non-separable continuum and is the continuous injective image of a closed subinterval of long line, then K is one of: long circle, long lollipop, long dumbbell, long figure eight, or long theta curve.*

Proof. The closed non-separable subintervals of the long line are (up to homeomorphism) just the long ray and the long line itself.

Let us suppose for the moment that K is the continuous injective image of the long ray R. We may identify points of K with points in R. Note that on any closed subinterval, $[a, b]$ say, of R, (by compactness of $[a, b]$ in R, and Hausdorffness of K) the standard order topology and the K-topology coincide. It follows that at any point with a bounded K-open neighborhood the standard topology and K-topology agree. We will show that there is a point x in R such that every K-open U containing x contains a tail, (t, ∞), for some t. Assuming this, by Hausdorffness of K, every point distinct from x has bounded neighborhoods, and so x is the only point where the K-topology differs from the usual topology. Then K is either the long circle or the long lollipop depending on where x is in R (in particular, if it equals 0). The corresponding result for continuous injective images of the long line follows immediately.

Suppose, for a contradiction, that for every x in R, there is a K-open set U_x containing x such that U_x contains no tail. By compactness of K, some finite collection, $U_{x_1}, \ldots, U_{x_n}$, covers K. Let $S_i = U_{x_i} \cap \Lambda$, where Λ is the set of limits in ω_1. Then (since the finitely many S_i cover the closed unbounded set Λ) at least one of the S_i is stationary. Take any α in S_i, and consider it as a point of the closed subinterval $[0, \alpha]$ of R, where we know the standard topology and the K-topology agree. Since α is a limit point which is in $U_{x_i} \cap [0, \alpha]$, and this latter set is open, we know there is an ordinal $f(\alpha) < \alpha$ such that $(f(\alpha), \alpha] \subseteq U_{x_i}$. Thus we have a regressive map, f, defined on the stationary set S_i, so by the Pressing Down Lemma there is a β such that $f^{-1}(\beta)$ is cofinal in ω_1. Hence U_{x_i} contains $\bigcup\{(f(\alpha), \alpha] : \alpha \in f^{-1}(\beta)\} = (\beta, \infty)$, and so U_{x_i} does indeed contain a tail. ∎

3. Complexity of characterizations. Theorem 1.5 from the Introduction makes certain claims about the complexity of characterizing, for various n, the n-ac graphs which are not $(n+1)$-ac. We introduce the necessary technology from descriptive set theory to make these claims precise. Then Theorem 3.1 is the formalized version of Theorem 1.5.

Recall (see [4]) that the Borel subsets of a space ramify into a hierarchy, $\Pi_\alpha, \Sigma_\alpha$, indexed by countable ordinals. Sets lower in the hierarchy are less complex than those found higher up. Most relevant here are: Π_3, the set of $F_{\sigma\delta}$ subsets; Σ_3, the set of all $G_{\delta\sigma}$ subsets; and $D_2(\Sigma_3)$, the set of intersections of one Π_3 and one Σ_3 set.

The complexity of a set in terms of its position in the Borel hierarchy is precisely correlated to the complexity of the logical formulae needed to define it. A Π_3 set, S, can be defined by a formula, ϕ (via $S = \{x : \phi(x) \text{ is true}\}$), of the form $\forall p\ \exists q\ \forall r$ (something simple), where the quantifiers run over *countable* sets, and 'something simple' is boolean. A Σ_3 set, T, can be defined by a formula, ψ, of the form $\exists p\ \forall q\ \exists r$ (something simple). Furthermore a $D_2(\Sigma_3)$ set can be defined by a formula of the form $\phi \wedge \psi$, where ϕ and ψ are as above.

For example, let $S_3^* = \{\alpha \in 2^{\mathbb{N}\times\mathbb{N}} : \exists J\ \forall j > J\ \exists k\ \alpha(j,k) = 0\}$, and $P_3 = \{\beta \in 2^{\mathbb{N}\times\mathbb{N}} : \forall j\ \exists K\ \forall k \geq K\ \beta(j,k) = 0\}$. Then S_3^* is Σ_3, and P_3 is Π_3 in $2^{\mathbb{N}\times\mathbb{N}}$. And $S_3^* \times P_3$ is a $D_2(\Sigma_3)$ subset of $(2^{\mathbb{N}\times\mathbb{N}})^2$.

For a class Γ of subsets, a set A is *Γ-hard* if A is not in any proper subclass, while it is *Γ-complete* if it Γ-hard and in Γ. In other words, A is Γ-complete if and only if it has complexity precisely Γ. It is known [4] that S_3^* is Σ_3-complete, P_3 is Π_3-complete, and $S_3^* \times P_3$ is $D_2(\Sigma_3)$-complete. We can rephrase these last two statements as follows: there is a formula characterizing P_3 of the form $\forall\exists\forall$, but we can be *certain* that *no logically simpler characterizing formula exists*, and there is a formula characterizing $S_3^* \times P_3$ of the form $(\exists\forall\exists) \wedge (\forall\exists\forall)$, but no logically simpler characterizing formula exists.

Let $A \subseteq X$, $B \subseteq Y$ and f a continuous map of X to Y such that $f^{-1}(B) = A$ (such an f is a *Wadge reduction*). Note that if B is in some Borel class Γ, then by continuity so is $A = f^{-1}(B)$. Hence if A is Γ-hard, then so is B.

We work inside the hyperspace $C(I^N)$ of all subcontinua of I^N with the Vietoris topology, which makes it a continuum. In light of the remarks above, it should now be clear that the following is indeed a formal version of Theorem 1.5.

THEOREM 3.1. *Fix $N \geq 2$. Inside the space $C(I^N)$:*

(1) *the set of graphs which are ω-ac is Π_3-complete,*
(2) *any family of homeomorphism classes of graphs is Π_3-hard and always $D_2(\Sigma_3)$, and*
(3) *the set of n-ac non-$(n+1)$-ac graphs is $D_2(\Sigma_3)$-complete for $n = 2, 3, 4, 5$.*

Claims (1)–(3) are the contents of Lemmas 3.3, 3.2 and Proposition 3.4, respectively.

Lemma 3.2. *Let $\mathcal{C}$ be any collection of graphs. Then $H(\mathcal{C})$, the set of all subcontinua of I^N homeomorphic to some member of $\mathcal{C}$, is Π_3-hard and in $D_2(\Sigma_3)$.*

Proof. That $H(\mathcal{C})$ is Π_3-hard is immediate from Theorem 7.3 of [1]. It remains to show it is in $D_2(\Sigma_3)$.

For spaces X and Y, write $X \le Y$ if X is Y-like, $X < Y$ if $X \le Y$ but $Y \not\le X$, and $X \sim Y$ if $X \le Y$ and $Y \le X$. Further write $\mathcal{L}_X = \{Y : Y \le Y\}$ and $Q(X) = \{Y : X \sim Y\}$.

Let $\mathcal{C}_0$ be a maximal family of pairwise non-homeomorphic members of $\mathcal{C}$. Up to homeomorphism there are only countably many graphs. So enumerate $\mathcal{C}_0 = \{G_m : m \in \mathbb{N}\}$. According to Theorem 1.7 of [1], for a graph G and a Peano continuum P, we observe that P is G-like if and only if P is a graph obtained from G by identifying disjoint (connected) subgraphs to points. For a fixed graph G, then, there are, up to homeomorphism, only finitely many G-like graphs. For each G_m in $\mathcal{C}$ pick graphs $G_{m,i}$ for $i = 1, \dots, k_m$ such that $G_{m,i} < G$ for each i, and if G' is a graph such that $G' < G$ then for some i we have $H(G') = H(G_{m,i})$.

For a graph G, $H(G) = Q(G)$ (see [3]). Hence, writing $\mathcal{P}$ for the class of Peano continua, we have $H(\mathcal{C}) = \bigcup_m Q(G_m) = \mathcal{P} \cap \bigcup_m R_m$, where

$$R_m = \mathcal{L}_{G_m} \setminus \bigcup_{i=1}^{k_m} \mathcal{L}_{G_{m,i}} = \mathcal{L}_{G_m} \cap \Big(C(I^N) \setminus \bigcup_{i=1}^{k_m} \mathcal{L}_{G_{m,i}}\Big).$$

By Corollary 5.4 of [1], for a graph G, the set $\mathcal{L}_G$ is Π_2. Hence each R_m, as the intersection of a Π_2 and a Σ_2, is Σ_3, and so is their countable union. Since $\mathcal{P}$ is Π_3, we see that $H(\mathcal{C})$ is indeed the intersection of a Π_3 set and a Σ_3 set. ∎

Lemma 3.3. *The set AC_ω of all subcontinua of I^N which are ω-ac graphs is Π_3-complete.*

Proof. For a graph G, $H(G)$ is Π_3. By Theorem 1.3, AC_ω is a finite union of $H(G)$ for graphs G, and so is also Π_3. Hence by Lemma 3.2, AC_ω is Π_3-complete. ∎

Proposition 3.4. *For any n, let AC_n be the set of subcontinua of I^N which are n-ac but not $(n+1)$-ac graphs. Then for $n = 2, 3, 4, 5$ the sets AC_n are $D_2(\Sigma_3)$-complete.*

Proof. According to Lemma 3.2, AC_n is $D_2(\Sigma_3)$, so it suffices to show that AC_n is $D_2(\Sigma_3)$-hard.

To show that AC_n is $D_2(\Sigma_3)$-hard it suffices to show that there is a continuous map $F : (2^{\mathbb{N}\times\mathbb{N}})^2 \to C(I^N)$ such that $F^{-1}(AC_n) = S_3^* \times P_3$. We do the construction for $N = 2$. Since $\mathbb{R}^2$ embeds naturally in general $\mathbb{R}^N$, the proof obviously extends to all $N \ge 2$.

We first deal with the case when $n = 5$. Then we will explain how to make the minor modifications needed for the other cases, $n = 2, 3, 4$.

For x, y in $\mathbb{R}^2$, let $\overline{xy}$ be the straight line segment from x to y. Set $O = (0, 0)$, $T = (3, 1)$, $B_1 = (1, 0)$, $B_2 = (4/3, 0)$, $B_3 = (5/3, 0)$, $B_4 = (2, 0)$ and $T_1 = (1, 1)$, $T_2 = (4/3, 1)$, $T_3 = (5/3, 1)$, $T_4 = (2, 1)$. Let $K_0 = \overline{OB_4} \cup \overline{B_4T} \cup \overline{TT_1} \cup \overline{T_1O} \cup \overline{B_2T_2} \cup \overline{B_3T_3}$. Then K_0 is a 5-ac non-6-ac graph. Define $b_j = (1/j, 0)$, $t_j = (1/j, 1/j)$, $t_j^k = (1/j, 1/j - 1/(kj))$ and $s_j^k = (1/j - 1/(kj(j+1)), 0)$. Then $K_J = K_0 \cup \bigcup_{j=1}^{J} \overline{b_jt_j}$—for each J—is also a 5-ac non-6-ac graph.

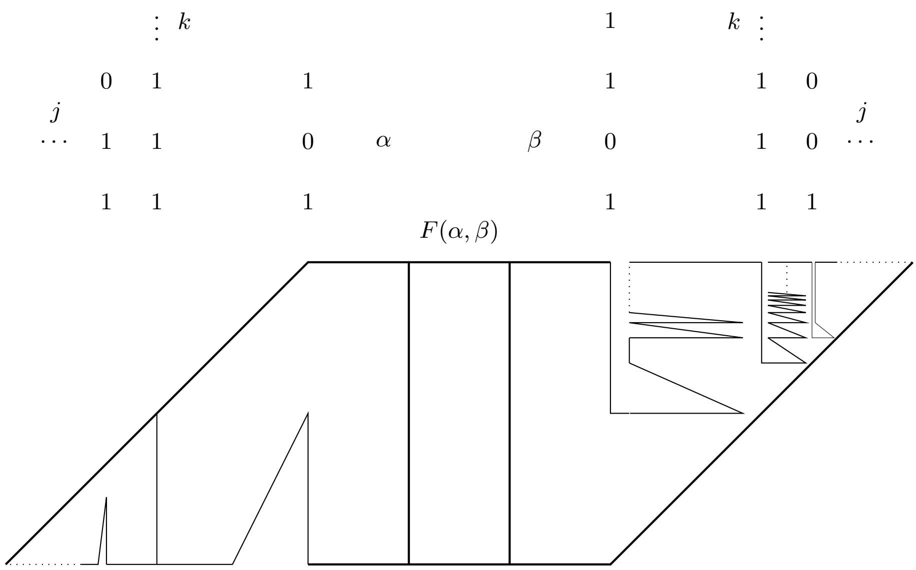
Let K_0' be K_0 with the interior of the line from O to B_1, and the interior of the line from T_4 to T, deleted.

We now define F at some α and β in $2^{\mathbb{N}\times\mathbb{N}}$. Fix j. If $\alpha(j, k) = 1$ for all k, then let $R_j = \overline{b_jt_j} \cup \overline{b_jb_{j+1}}$. Otherwise, let $k_0 = \min\{k : \alpha(j, k) = 0\}$, and let $R_j = \overline{b_jt_j^{k_0}} \cup \overline{t_j^{k_0}s_j^{k_0}} \cup \overline{s_j^{k_0}b_{j+1}}$.

For any j, k set $p_j = 3 - 1/j$, $q_j^k = 1 - 1/(j+k)$, $l_j = p_j + (1/8)(p_{j+1} - p_j)$ and $r_j = p_j + (7/8)(p_{j+1} - p_j)$. Fix j. Define

$$\begin{aligned} s_j = {} & \overline{(p_j, 1)(p_j, q_j^1)} \cup \overline{(p_j, q_j^1)(l_j, q_j^1)} \cup \overline{(l_j, 1)(p_{j+1}, 1)} \\ & \cup \bigcup \{\overline{(l_j, q_j^k)(l_j, q_j^{k+1})} : \beta(j, k) = 0\} \\ & \cup \bigcup \{\overline{(l_j, q_j^k)(r_j, q_j^k)} \cup \overline{(r_j, q_j^k)(l_j, q_j^{k+1})} : \beta(j, k) = 1\}. \end{aligned}$$

Let $F(\alpha, \beta) = K_0' \cup \bigcup_j (R_j \cup S_j)$. Then it is straightforward that F maps $(2^{\mathbb{N}\times\mathbb{N}})^2$ continuously into $C([0, 4]^2)$.



Take any α. For any j, the set R_j connects the bottom edge $\overline{OB_1}$ to the diagonal edge $\overline{OT_1}$ if $\alpha(j,k) = 1$ for all k, and otherwise it is an arc from b_j to b_{j+1}. Hence $\bigcup_{j>J} R_j$ is a free arc from B_{J+1} to O if α is in S_3^*, and otherwise cannot be a subspace of a graph (because it contains infinitely many points of order 3).

Take any β. For any j, S_j is an arc from $(p_j, 1)$ to $(p_{j+1}, 1)$ if $\beta(j,k) = 0$ for all but finitely many k, but contains a 'topologist's sine curve' if $\beta(j,k) = 1$ for infinitely many k. Thus $\bigcup_j S_j$ is a free arc from T_4 to T if β is in P_3, and otherwise it cannot be a subspace of a graph (because it contains a topologist's sine curve).

Hence if (α, β) is in $S_3^* \times P_3$, $F(\alpha, \beta)$ is homeomorphic to some K_J, which in turn means it is a graph which is 5-ac but not 6-ac. On the other hand, if either α is not in S_3^* or β is not in P_3, then $F(\alpha, \beta)$ contains subspaces which cannot be subspaces of a graph—and so is not a graph. Thus $F^{-1}(AC_5) = S_3^* \times P_3$ as required.

Let $T_1^+ = (1, 2)$, $T_2^+ = (4/3, 2)$ and $T_3^+ = (5/3, 2)$. Suppose now that $n = 4$. Modify K_0 by adding the line segment from T_1 to T_1^+. Then this modified K_0 is 4-ac but not 3-ac. Further, for any J, the modified K_J obtained by taking the modified K_0 as a base is also 4-ac but not 5-ac. Thus we get the desired reduction in the case when $n = 4$.

Similarly, for $n = 3$, modify K_0 by adding both the line segments $\overline{T_1T_1^+}$ and $\overline{T_2T_2^+}$. This gives a base graph, and a family of K_J, which are all 3-ac but not 4-ac. Finally, by adding the three line segments $\overline{T_1T_1^+}$, $\overline{T_2T_2^+}$ and $\overline{T_3T_3^+}$ to K_0 we get 2-ac non-3-ac graphs. The desired reductions for $n = 3$ and $n = 2$ follow. ∎

4. Open problems. The main theorems, Theorems 1.3, 1.4 and 1.5, raise some natural problems.

PROBLEMS 4.1.

- Find examples of continua which are n-ac but not $(n+1)$-ac for $n \geq 7$. Theorem 1.3 implies that no graph is an example. Are there regular examples?
- Find a 'simple' (i.e. Π_3) characterization of 6-ac graphs which are non-7-ac. Alternatively, prove that no such characterization is possible, and show that the set of 6-ac non-7-ac graphs is $D_2(\Sigma_3)$. Note (Example 1.1(G)) that there are infinitely many 6-ac non-7-ac graphs—rather than the only finite family of 7-ac graphs—but this does not rule out a 'simple' characterization.
- Characterize the ω-ac regular continua. The Sierpiński triangle is a regular ω-ac continuum. The authors, with Kovan-Bakan, show in [2]

that there is no Borel characterization of *rational* ω-ac continua. However the examples used in that argument are far from regular (not even locally connected).

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Benjamin Espinoza
Department of Mathematics
University of Pittsburgh at Greensburg
236 Frank A. Cassell Hall
150 Finoli Drive
Greensburg, PA 15601, U.S.A.
E-mail: bee1@pitt.edu

Paul Gartside
Department of Mathematics
University of Pittsburgh
508 Thackeray Hall
Pittsburgh, PA 15260, U.S.A.
E-mail: gartside@math.pitt.edu

Ana Mamatelashvili
Department of Mathematics
University of Pittsburgh
301 Thackeray Hall
Pittsburgh, PA 15260, U.S.A.
E-mail: anm137@pitt.edu@math.pitt.edu

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