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THE GENERALIZED MONOTONICITY CLASS AND INTEGRABILITY OF POWER SERIES

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Abstract. The aim of this paper is to obtain a generalization of W. A. Woyczyński and B. Ram results concerning integrability of power series in terms of their coefficients for the class GM of general monotonic sequences.

1. Introduction. Let Φ be a nondecreasing continuous real valued function defined on $\mathbb{R}_+ = [0; +\infty)$ and vanishing only at the origin. If Φ is convex and $\lim_{u\to 0} \Phi(u)/u = 0$, $\lim_{u\to\infty} \Phi(u)/u = \infty$, then Φ is called an *N-function* ([KR]). It is well known that every N-function Φ admits a representation $\Phi(u) = \int_0^u \phi(t)\,dt$, where $\phi(t)$ is a positive, right-continuous nondecreasing function with $\phi(0) = 0$ and $\lim_{t\to\infty} \phi(t) = +\infty$. In particular, Φ is absolutely continuous on every finite interval of $\mathbb{R}_+$ and

(a) $\Phi(x)/x$ is increasing on $(0; \infty)$.

Further we also require

(b) there exists $c > 1$ such that $\Phi(x)/x^c$ is decreasing on $(0; \infty)$.

Generalizing several results on L^p-integrability with weight $x^{-\gamma}$ (see [Bo, §4]), Y.-M. Chen [Ch1] proved

THEOREM 1.1. *Let $\{a_n\}_{n=1}^\infty$ be a sequence decreasing to zero and $g(t) = \sum_{n=1}^\infty a_n \sin nt$. If $0 < \gamma < 1$ and Φ satisfies conditions* (a) *and* (b) *above, then a necessary and sufficient condition for $x^{-\gamma}\Phi(g(x))$ to belong to $L[0;\pi]$ is the convergence of the series $\sum_{n=1}^\infty n^{\gamma-2}\Phi(na_n)$.*

Note that Lemma 3 in [Ch1], applied in the proof of Theorem 1.1, was proved with the help of integration by parts. Therefore, it seems that Theorem 1.1 also requires the assumption Φ is absolutely continuous.

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Let $L_\Phi(X, d\mu)$ be the *Orlicz class*, i.e. the set of all complex valued measurable functions f on a measure space (X, μ) such that the modular $\int_X \Phi(|f(x)|)\, d\mu$ is finite. The *Hardy–Orlicz class* $H_\Phi[0; 2\pi]$ is the closed subset of $L_\Phi([0; 2\pi], dx)$ spanned by all trigonometric polynomials of the form

$$f(t) = \sum_{n=0}^{N} a_n e^{int}.$$

W. Woyczyński [Wo] proved

THEOREM 1.2. *Let $f(z) = \sum_{k=1}^\infty a_k z^k$, $|z| < 1$, and let Φ be an N-function satisfying the Δ_2-condition. Also, suppose that for some $\alpha \in (0; 1)$ and a convex function Λ the function Φ satisfies $\Lambda(u) \le \Phi^\alpha(u) \le c\Lambda(u)$ for all $u \in [0; \infty)$. If $a_k \downarrow 0$, then the following four statements are equivalent:*

(i) $f \in L_\Phi([0; 1), d\mu)$;
(ii) $g(t) = f(e^{it}) \in H_\Phi[0; 2\pi]$;
(iii) $\sum_{n=1}^\infty n^{-2}\Phi(na_n) < \infty$;
(iv) $\sum_{n=1}^\infty n^{-2}\Phi(A_n) < \infty$,

where $A_n = \sum_{i=1}^n a_i$.

If for $\{a_n\}_{n=1}^\infty$ there exists $\tau \ge 0$ such that $\{a_n n^{-\tau}\}_{n=1}^\infty$ is decreasing, then $\{a_n\}_{n=1}^\infty$ is called *quasi monotone* (written $\{a_n\}_{n=1}^\infty \in \mathrm{QM}$). If for all $n \in \mathbb{N}$ we have

$$\sum_{k=n}^{2n-1} |a_k - a_{k+1}| \le C a_n,$$

then we write $\{a_n\}_{n=1}^\infty \in \mathrm{GM}$. Finally, if for all $n \in \mathbb{N}$ we have

$$\sum_{k=n}^{\infty} |a_k - a_{k+1}| \le C a_n,$$

then $\{a_n\}_{n=1}^\infty \in \mathrm{RBVS}$. These classes were introduced by A. A. Konyushkov [Ko], S. Tikhonov [T2] and L. Leindler [Le] respectively.

In [T2] it is proved that $\mathrm{QM} \subset \mathrm{GM}$, and the embedding $\mathrm{RBVS} \subset \mathrm{GM}$ is obvious. The example of a positive GM sequence belonging neither to QM nor to RBVS can be found in [LT]. P. Jain [Ja] generalized Theorem 1.2 to the case of weighted Orlicz classes with power weight and $\{a_n\}_{n=1}^\infty \in \mathrm{QM}$ with additional restriction $0 < B_1 \le n^\beta a_n \le B_2$ for some $\beta > 0$. A more general result was obtained by B. Ram [Ra].

Let $\psi(x)$ be a nondecreasing positive function on $(0; \infty)$ such that $\psi(x)/x^\delta$ decreases on $(0; \infty)$ for some $\delta \in (0; 1)$ (we then write $\psi \in M_1$). Under these assumptions Φ and ψ satisfy the Δ_2-condition: $\Phi(2x) \le C\Phi(x)$ and $\psi(2x) \le C\psi(x)$.

THEOREM 1.3. *Let Φ and $f(z)$ be as in Theorem* 1.2, *and ψ be as above. If $\{a_n\}_{n=1}^{\infty} \in \mathrm{QM} \cap \mathrm{RBVS}$, then the following four statements are equivalent:*

$$\Phi(|f(x)|)/\psi(1-x) \in L(0;1); \tag{1.1}$$

$$\Phi(|f(e^{ix})|)/\psi(x) \in L(0;\pi); \tag{1.2}$$

$$\sum_{n=1}^{\infty} \frac{\Phi(na_n)}{n^2\psi(1/n)} < \infty; \tag{1.3}$$

$$\sum_{n=1}^{\infty} \frac{\Phi(A_n)}{n^2\psi(1/n)} < \infty. \tag{1.4}$$

The aim of our paper is to extend Theorem 1.3 to the case $\{a_n\}_{n=1}^{\infty} \in \mathrm{GM}$ whenever Φ is an N-function satisfying condition (b).

2. Auxiliary results. A function f is called *almost increasing* on the interval P if there exists $C > 0$ such that for any $x, y \in P$, $x < y$, we have $f(x) \le Cf(y)$.

Lemma 2.1 may be found in [Ch2] without proof. For the convenience of the reader we provide a proof.

LEMMA 2.1. *Let Φ be an N-function satisfying condition* (b) *for some $c > 1$, and let $\psi \in M_1$, $a_n \ge 0$, $A_n = \sum_{i=1}^{n} a_i$ for all $n \in \mathbb{N}$. Then*

$$\sum_{n=1}^{\infty} (n^2\psi(1/n))^{-1}\Phi(A_n) \le C \sum_{n=1}^{\infty} (n^2\psi(1/n))^{-1}\Phi(na_n).$$

Proof. Let $b_n = (n^2\psi(1/n))^{-1}$, $\sigma_n = \sum_{i=n}^{\infty} b_i$, $n \in \mathbb{N}$, and $A_0 = 0$. For a continuous increasing function ω on $[0;1]$ with $\omega(0) = 0$, N. K. Bari and S. B. Stechkin [BS] established that the condition

$$\sum_{i=n}^{\infty} i^{-1}\omega(1/i) = O(\omega(1/n)), \qquad n \in \mathbb{N},$$

is equivalent to $\omega(t)/t^{\alpha}$ being almost increasing for some $\alpha \in (0;1)$. If $\omega(t) = t/\psi(t)$, then $\psi \in M_1$ implies that for all $\delta \in (0;1)$ the function $\omega(t)/t^{\alpha}$ is increasing for any $\alpha \in (0; 1-\delta)$. Therefore

$$\sigma_n = \sum_{i=n}^{\infty} (i^2\psi(1/i))^{-1} = O(n^{-1}\psi(1/n)^{-1}) = O(nb_n), \qquad n \in \mathbb{N}. \tag{2.1}$$

Using Abel's transform, we obtain, for $N \in \mathbb{N}$,

$$\sum_{n=1}^{N} \Phi(A_n)b_n = \sum_{n=1}^{N} \Phi(A_n)(\sigma_n - \sigma_{n+1}) \le \sum_{n=1}^{N} (\Phi(A_n) - \Phi(A_{n-1}))\sigma_n.$$

Since $\kappa(x)=\Phi(x)/x^c$ is decreasing and $c>1$, in virtue of (2.1) and Lagrange's mean value theorem we have

$$\sum_{n=1}^N \Phi(A_n)b_n \le \sum_{n=1}^N (A_n^c\kappa(A_n)-A_{n-1}^c\kappa(A_{n-1}))\sigma_n \le \sum_{n=1}^N \kappa(A_n)(A_n^c-A_{n-1}^c)\sigma_n$$
$$\le c\sum_{n=1}^N \kappa(A_n)A_n^{c-1}(A_n-A_{n-1})\sigma_n \le C_1\sum_{n=1}^N \varphi(A_n)na_nb_n,$$

where $\varphi(x)=\Phi(x)/x$. Applying the idea of H. P. Mulholland [Mu] and using the fact that $\varphi(x)$ is increasing, we obtain

$$\varphi(A_n)na_n = t^{-1}(tna_n\varphi(A_n)) \le t^{-1}\max(tna_n\varphi(tna_n), A_n\varphi(A_n))$$
$$\le t^{-1}(\Phi(tna_n)+\Phi(A_n)).$$

But $\Phi(x)/x^c$ is decreasing and for $t>1$ we have $\Phi(tx)\le t^c\Phi(x)$. Thus, $\varphi(A_n)na_n \le t^{c-1}\Phi(na_n)+t^{-1}\Phi(A_n)$ and

$$\sum_{n=1}^N \Phi(A_n)b_n \le C_1\Big(\sum_{n=1}^N t^{c-1}\Phi(na_n)b_n + t^{-1}\sum_{n=1}^N \Phi(A_n)b_n\Big),$$

whence $(1-t^{-1}C_1)\sum_{n=1}^N \Phi(A_n)b_n \le C_1t^{c-1}\sum_{n=1}^N \Phi(na_n)b_n$. Taking sufficiently large t and letting N tend to ∞, we get the inequality of the lemma. ∎

Lemma 2.2 below contains the main properties of $\{a_n\}_{n=1}^\infty \in \mathrm{GM}$ proved by S. Tikhonov [T2].

Lemma 2.2.

(i) *Suppose* $\{a_n\}_{n=1}^\infty \in \mathrm{GM}$ *and* $\sum_{i=n}^\infty i^{-1}a_i<\infty$. *Then*

$$\sum_{k=n}^\infty |a_k-a_{k+1}| \le C\Big(a_n+\sum_{k=n}^\infty a_k/k\Big), \qquad n\in\mathbb{N}.$$

(ii) *Let* $\{a_n\}_{n=1}^\infty \in \mathrm{GM}$. *Then* $a_k\le C_1a_n$ *for* $n\le k\le 2n$.

Corollary 2.3. *If* $\{a_i\}_{i=1}^\infty \in \mathrm{GM}$, *then there exists* $\beta>0$ *such that*

$$\sum_{i=kn}^{(k+1)n-1} a_i = O(k^\beta A_n), \qquad k,n\in\mathbb{N}.$$

Proof. Let $[\log_2 k]=j,\ k\ge 2$. If $i\in[kn,(k+1)n-1)$, then $i\le 2^{j+1}n$ and $a_i\le C_1^{j+1}a_n$, where C_1 is the constant from Lemma 2.2(ii). Thus,

$$\sum_{i=kn}^{(k+1)n-1} a_i \le n2^{(j+1)\log_2 C_1}a_n \le C_2\cdot 2^{2\log_2 k\cdot\log_2 C_1}\sum_{i=[n/2]}^{n} a_i \le C_2k^\beta A_n,$$

where $\beta=2\log_2 C_1$. For $k=1$ one can use similar arguments. ∎

Lemma 2.4 is a generalization of the classical Hardy inequality (see [HLP, Theorem 330]). It can be found in [Ch2] without proof and in [Ra] in the case when Φ satisfies the assumptions of Theorem B (with proof).

LEMMA 2.4. *Let Φ and ψ be as in Lemma* 2.1, *and suppose that f is a nonnegative measurable function on* $(0, a)$. *Then*

$$\int_0^a \Phi\Big(x^{-1}\int_0^x f(t)\,dt\Big)\psi(x)^{-1}\,dx \le C(a)\int_0^a \Phi(f(x))\psi(x)^{-1}\,dx.$$

3. Main result

MAIN THEOREM 3.1. *Let Φ be an N-function satisfying condition* (b), *and let $\psi \in M_1$, $\{a_n\}_{n=1}^\infty \in$ GM, and $f(z) = \sum_{n=1}^\infty a_n z^n$. Then the statements* (1.1), (1.3) *and* (1.4) *are equivalent. If, in addition,*

$$\sum_{k=n}^{\infty} a_k/k = O(a_n), \tag{3.1}$$

then all four statements (1.1)–(1.4) *are equivalent.*

Proof. We first prove (1.1)⇒(1.4), (1.4)⇒(1.1) and (1.3)⇔(1.4), and then (1.4)⇒(1.2) and (1.2)⇒(1.3).

(1.1)⇒(1.4). If $1 - x = y$ and $y \in ((n+1)^{-1}; n^{-1}] = I_n$, $n \in \{2, 3, \ldots\}$, then

$$f(1-y) \ge \sum_{k=0}^{n} a_k(1-y)^k \ge (1-1/n)^n \sum_{k=1}^{n} a_k \ge 4^{-1}A_n.$$

Since f is increasing, and since Φ and ψ satisfy the Δ_2-condition and $\Phi(A_n) \le \Phi(4f(1-u))$ for $u \in I_n$, $n \ge 2$, we have

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{\Phi(A_n)}{n^2\psi(1/n)} &\le C_1 \sum_{n=1}^{\infty} \int_n^{n+1} \frac{t^{-2}\Phi(A_n)}{\psi(1/t)}\,dt \\ &= C_1 \sum_{n=1}^{\infty} \int_{1/(n+1)}^{1/n} \frac{\Phi(A_n)}{\psi(u)}\,du \\ &\le C_1 \int_{1/2}^{1} \frac{\Phi(a_1)}{\psi(u)}\,du + C_2 \sum_{n=2}^{\infty} \int_{1/(n+1)}^{1/n} \frac{\Phi(f(1-u))}{\psi(u)}\,du \\ &\le C_3\Big(1 + \int_0^1 \frac{\Phi(f(x))}{\psi(1-x)}\,dx\Big) < \infty.\end{aligned}$$

(1.4)⇒(1.1). It is easy to see that $(1-1/n)^i \le (1-1/n)^{nk} \le e^{-k}$ for $nk \le i \le n(k+1) - 1$. Using Corollary 2.3 and the Δ_2-condition on Φ, we have

$$\int_0^1 \frac{\Phi(f(x))}{\psi(1-x)}\,dx = \sum_{n=2}^{\infty} \int_{1/n}^{1/(n-1)} \frac{\Phi(f(1-x))}{\psi(x)}\,dx$$
$$\le \sum_{n=2}^{\infty} \int_{1/n}^{1/(n-1)} \frac{\Phi(\sum_{k=1}^{\infty} a_k(1-1/n)^k)}{\psi(x)}\,dx$$
$$\le \sum_{n=2}^{\infty} \int_{1/n}^{1/(n-1)} \Phi\Big(\sum_{k=1}^{n-1} a_k + \sum_{k=1}^{\infty} \sum_{i=kn}^{(k+1)n-1} a_k e^{-k}\Big)\psi(1/n)^{-1}\,dx$$
$$\le C_4 \sum_{n=2}^{\infty} n^{-2}\Phi\Big(A_n + C_5 \sum_{k=1}^{\infty} k^{\beta} e^{-k} A_n\Big)\psi(n)^{-1}$$
$$\le C_6 \sum_{n=2}^{\infty} \frac{\Phi(A_n)}{n^2\psi(1/n)} < \infty.$$

(1.4)⇔(1.3). By Lemma 2.2(ii) we have

$$na_n \le C_7 \sum_{k=[n/2]}^{n} a_k \le C_7 A_n, \tag{3.2}$$

proving (1.4)⇒(1.3). The inverse implication follows from Lemma 2.1.

(1.4)⇒(1.2). If $D_k(x) = 1/2 + \sum_{i=1}^{k} \cos ix$, then it is well known that $|D_k(x)| \le \pi/x$. Using Abel's transform and Lemma 2.2(i), we obtain

$$|\Re f(e^{ix})| = \Big|\sum_{k=1}^{\infty} a_k \cos kx\Big| \tag{3.3}$$
$$\le \sum_{k=1}^{n} a_k + \Big|\sum_{k=n}^{\infty}(a_k - a_{k+1})D_k(x)\Big| + |a_n D_n(x)|$$
$$\le A_n + \pi a_n x^{-1} + \pi x^{-1} \sum_{k=n}^{\infty} |a_k - a_{k+1}|$$
$$\le A_n + C_8 x^{-1}\Big(a_n + \sum_{k=n}^{\infty} a_k/k\Big)$$
$$\le A_n + C_9 n a_n \le C_{10} A_n, \qquad x \in (\pi/(n+1); \pi/n],$$

by (3.1) and (3.2). Using (3.3) and the Δ_2-condition on Φ and ψ, we conclude

$$\int_0^{\pi} \frac{\Phi(|\Re f(e^{ix})|)}{\psi(x)}\,dx \le \sum_{n=1}^{\infty} \int_{\pi/n+1}^{\pi/n} \frac{\Phi(C_{10}A_n)}{\psi(x)}\,dx \le C_{11} \sum_{n=1}^{\infty} \frac{\Phi(A_n)}{n^2\psi(1/n)} < \infty.$$

Similarly we can prove that $\int_0^{\pi} \Phi(|\Im f(e^{ix})|)\psi(x)^{-1}\,dx < \infty$.

(1.2)⇒(1.3). Since $\psi(x)^{-1}$ is decreasing and not vanishing on $(0;\pi)$, we have $\int_0^\pi \Phi(|f(e^{ix})|)\,dx < \infty$, whence $f(e^{ix}) \in L[0;\pi]$. Using the standard method of double integration and (3.2), we obtain for $r(t) = \Re f(e^{it})$, $r_1(t) = \int_0^t r(u)\,du$ and $r_2(t) = \int_0^t r_1(u)\,du$,

$$\begin{aligned} r_2(t) &= \sum_{j=1}^{\infty} a_j(1-\cos jt)j^{-2} \geq 2\sum_{j=1}^{n} j^{-2}a_j\sin^2(jt/2) \\ &\geq 2\sum_{j=1}^{n} j^{-2}a_j(jt/\pi)^2 \geq t^2C_{12}\sum_{j=1}^{n} a_j \\ &\geq C_{13}t^2na_n, \qquad t\in(\pi/(n+1);\pi/n]. \end{aligned}$$

Using Lemma 2.4 similarly to the paper [AW], where $\Phi(x) = x^p$, $p > 1$, $\psi(x) = x^c$, we obtain

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{\Phi(na_n)}{n^2\psi(1/n)} &\leq C_{14}\sum_{n=1}^{\infty}\int_{\pi/(n+1)}^{\pi/n} \frac{\Phi(x^{-2}r_2(x))}{\psi(x)}\,dx \\ &= C_{14}\int_0^\pi \frac{\Phi(x^{-2}r_2(x))}{\psi(x)}\,dx \leq C_{14}\int_0^\pi \frac{\Phi(x^{-1}\int_0^x |r_1(t)|t^{-1}\,dt)}{\psi(x)}\,dx \\ &\leq C_{15}\int_0^\pi \frac{\Phi(x^{-1}\int_0^x |r(t)|\,dt)}{\psi(x)}\,dx \leq C_{16}\int_0^\pi \frac{\Phi(|f(e^{ix})|)}{\psi(x)}\,dx < \infty. \ \blacksquare \end{aligned}$$

COROLLARY 3.2. *If the conditions of Theorem* 3.1 *hold and, in addition,* $\{a_n\}_{n=1}^\infty \in$ RBVS, *then all statements* (1.1)–(1.4) *are equivalent.*

Proof. The inequality (3.1) was used in the implication (1.4)⇒(1.2) only. In this case instead of (3.3) we have

$$|\Re f(e^{ix})| \leq A_n + C_1x^{-1}\sum_{k=n}^{\infty}|a_k - a_{k+1}| \leq A_n + C_2x^{-1}a_n$$

and further we proceed as in the proof of Theorem 3.1. ∎

REMARK. Let us note that the part (1.3)⇒(1.4)⇒(1.2) of Corollary 3.2 was shown in [T1] for a very general function Φ. A necessary and sufficient conditions for (1.2) in terms of summability properties of $\{a_n\}$ in the case $\Phi(t) = t^p$, $\psi(t) = t^\alpha$, can be found in [Bo], and the sharp result for $\{a_n\} \in$ GM in [T2]. Many interesting results for (1.1) in the case $\Phi(t) = t^p$, $\psi(t) = t^\alpha$ are in [MP].

COROLLARY 3.3. *If the conditions of Theorem* 3.1 *hold and* $\{a_n\}_{n=1}^\infty \in$ QM, *then* (1.1), (1.3), (1.4) *are equivalent. If, in addition,* (3.1) *holds, then all statements* (1.1)–(1.4) *are equivalent.*

REMARK. It would be of interest to prove the equivalence of (1.1)–(1.4) without condition (3.1). Note that the inequality

$$\sum_{n=1}^{\infty} \varPhi\Big(\sum_{k=n}^{\infty} a_k\Big)\psi(n)^{-1} \le C\sum_{n=1}^{\infty} \psi(n)^{-1}\varPhi(na_n) \tag{3.4}$$

in the classical case $\varPhi(x) = x^p$, $1 < p < \infty$, $\psi(x) = x^c$, is valid only for $c < 1$ [HLP, Th. 346]. Since $n^2\psi(1/n)/n \to \infty$ as $n \to \infty$, the inequality (3.4) with $(n^2\psi(1/n))^{-1}$ instead of $\psi(n)^{-1}$ seems to be false.

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