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THE EXISTENCE OF RELATIVE PURE INJECTIVE ENVELOPES

BY

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Abstract. Let $\mathcal{S}$ be a class of finitely presented R-modules such that $R \in \mathcal{S}$ and $\mathcal{S}$ has a subset $\mathcal{S}^*$ with the property that for any $U \in \mathcal{S}$ there is a $U^* \in \mathcal{S}^*$ with $U^* \cong U$. We show that the class of $\mathcal{S}$-pure injective R-modules is preenveloping. As an application, we deduce that the left global $\mathcal{S}$-pure projective dimension of R is equal to its left global $\mathcal{S}$-pure injective dimension. As our main result, we prove that, in fact, the class of $\mathcal{S}$-pure injective R-modules is enveloping.

1. Introduction. Throughout this paper, R denotes a ring with identity and all modules are assumed to be left and unitary. The notion of purity plays a substantial role in algebra and model theory. It was introduced by P. M. Cohn [1] for left R-modules and by J. Łoś [12] for abelian groups; see also J. M. Maranda [13].

In 1967, R. Kiełpiński [10] introduced the notion of relative Γ-purity and proved that any R-module possesses a relative Γ-pure injective envelope. Also, he has shown that the relative Γ-pure injectivity coincides with the relative Γ-algebraic compactness.

Two years later, R. B. Warfield [18] proved that any R-module admits a pure injective envelope and the pure injectivity coincides with the algebraic compactness. Also, he introduced a notion of $\mathcal{S}$-purity for any class $\mathcal{S}$ of R-modules.

One can check that for an appropriate Γ, the Γ-purity and RD-purity coincide. However, for a general class $\mathcal{S}$ of finitely presented R-modules, the relationship between Γ-purity and $\mathcal{S}$-purity is ambiguous.

For a survey of results on various notions of purity, we refer the reader to the interesting articles [5], [6], [8]–[15] and [18], where among other things the algebraic compactness and pure homological dimensions are discussed.

We call a class $\mathcal{S}$ of R-modules *set-presentable* if it has a subset $\mathcal{S}^*$ with the property that for any $U \in \mathcal{S}$ there is a $U^* \in \mathcal{S}^*$ with $U^* \cong U$. It is

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easy to see that any class of finitely presented R-modules which is closed under isomorphism is set-presentable. So, the classes of finitely presented R-modules, cyclic cyclically-presented R-modules and cyclically-presented R-modules are set-presentable. Also, note that each of these classes contains R.

Let $\mathcal{S}$ be a set-presentable class of finitely presented R-modules containing R. Warfield [18, Proposition 1] showed that every R-module has an $\mathcal{S}$-pure projective precover. It is natural to ask whether any R-module has an $\mathcal{S}$-pure injective preenvelope. For the set-presentable classes of finitely presented R-modules, cyclic cyclically-presented R-modules and cyclically-presented R-modules, even more is proven to be true. Warfield [18, Proposition 6] has proved that every R-module has a pure injective envelope. Also, he showed that every R-module has an RD-pure injective envelope (see e.g. [4, Chapter XIII, Theorem 1.6]). More recently, Divaani-Aazar, Esmkhani and Tousi [2, Corollary 4.7, Definition 4.8 and Theorem 4.10] showed that every R-module has a cyclically pure injective envelope.

Our main aim in this paper is to prove that for any set-presentable class $\mathcal{S}$ of finitely presented R-modules containing R, the class of $\mathcal{S}$-pure injective R-modules is enveloping. We essentially use the technique and ideas introduced by Kiełpiński [10] and Warfield [18] and developed in [3], [5], [9], [2] and [11]–[15].

First in Proposition 2.4, for a general class $\mathcal{S}$ of finitely presented R-modules, we give a characterization of $\mathcal{S}$-pure exact sequences. Let $\mathcal{S}$ be a set-presentable class of finitely presented R-modules containing R. In Proposition 2.8, we show that the class of $\mathcal{S}$-pure injective R-modules is preenveloping. This, in particular, implies that the left global $\mathcal{S}$-pure projective dimension of R is equal to its left global $\mathcal{S}$-pure injective dimension (see Corollary 2.9). Finally, in Theorem 3.8, we prove that every R-module has an $\mathcal{S}$-pure injective envelope.

We continue the introduction by recalling some basic definitions and notions that we use in this paper. Let $\mathcal{S}$ be a class of R-modules. An exact sequence $0 \to A \overset{f}{\to} B \overset{g}{\to} C \to 0$ of R-modules and R-homomorphisms is called *$\mathcal{S}$-pure* if for all $U \in \mathcal{S}$ the induced homomorphism $\operatorname{Hom}_R(U, B) \to \operatorname{Hom}_R(U, C)$ is surjective. In this situation, f, g, $f(A)$ and C are called an *$\mathcal{S}$-pure monomorphism*, *$\mathcal{S}$-pure epimorphism*, *$\mathcal{S}$-pure submodule* of B, and *$\mathcal{S}$-pure homomorphic image* of B, respectively.

An R-module P (resp. E) is called *$\mathcal{S}$-pure projective* (resp. *$\mathcal{S}$-pure injective*) if for any $\mathcal{S}$-pure exact sequence $0 \to A \overset{f}{\to} B \overset{g}{\to} C \to 0$, the induced homomorphism $\operatorname{Hom}_R(P, B) \to \operatorname{Hom}_R(P, C)$ (resp. $\operatorname{Hom}_R(B, E) \to \operatorname{Hom}_R(A, E)$) is surjective. Also, a right R-module F is called *$\mathcal{S}$-pure flat* if for any $\mathcal{S}$-pure exact sequence $0 \to A \overset{f}{\to} B \overset{g}{\to} C \to 0$, the induced homomorphism $F \otimes_R A \to F \otimes_R B$ is injective.

An R-module M is called *cyclically-presented* if it is isomorphic to a module of the form R^n/G for some $n \in \mathbb{N}$ and some cyclic submodule G of R^n. If $\mathcal{S}$ is the class of all finitely presented (resp. cyclic cyclically-presented) R-modules, then $\mathcal{S}$-purity is called *purity* (resp. RD-*purity*). If $\mathcal{S}$ is the class of all cyclically-presented R-modules, then $\mathcal{S}$-purity is called *cyclic purity.*

Let $\mathscr{X}$ be a class of R-modules and M an R-module. An R-homomorphism $\phi : M \to X$ where $X \in \mathscr{X}$ is called an $\mathscr{X}$*-preenvelope* of M if for any $X' \in \mathscr{X}$, the induced homomorphism $\mathrm{Hom}_R(X, X') \to \mathrm{Hom}_R(M, X')$ is surjective. Also, an R-homomorphism $\phi : X \to M$ where $X \in \mathscr{X}$ is called an $\mathscr{X}$*-precover* of M if for any $X' \in \mathscr{X}$, the induced homomorphism $\mathrm{Hom}_R(X', X) \to \mathrm{Hom}_R(X', M)$ is surjective.

If $\phi : M \to X$ (resp. $\phi : X \to M$) is an $\mathscr{X}$-preenvelope (resp. $\mathscr{X}$-precover) of M and any R-homomorphism $f : X \to X$ such that $f\phi = \phi$ (resp. $\phi f = \phi$) is an automorphism, then ϕ is called an $\mathscr{X}$*-envelope* (resp. $\mathscr{X}$*-cover*) of M. The class $\mathscr{X}$ is called (*pre*)*enveloping* (resp. (*pre*)*covering*) if every R-module admits an $\mathscr{X}$-(pre)envelope (resp. $\mathscr{X}$-(pre)cover).

By definition, it is clear that if $\mathscr{X}$-envelopes (resp. $\mathscr{X}$-covers) exist, then they are unique up to isomorphism. Also, it is obvious that if the class $\mathscr{X}$ contains all injective (resp. projective) R-modules, then any $\mathscr{X}$-preenvelope (resp. $\mathscr{X}$-precover) is injective (resp. surjective).

2. $\mathcal{S}$-pure exact sequences. Propositions 2.4 and 2.8 are the main results of this section. We will use them several times to prove our main result in the next section.

One can easily deduce the following result from the definitions.

LEMMA 2.1. *Let $\mathcal{S}$ be a class of R-modules and $\{M_\gamma\}_{\gamma\in\Gamma}$ an indexed family of R-modules. Also, let $\{N_\gamma\}_{\gamma\in\Gamma}$ be an indexed family of right R-modules.*

(i) *$\prod_{\gamma\in\Gamma} M_\gamma$ is $\mathcal{S}$-pure injective if and only if M_γ is $\mathcal{S}$-pure injective for all $\gamma \in \Gamma$.*
(ii) *$\bigoplus_{\gamma\in\Gamma} N_\gamma$ is $\mathcal{S}$-pure flat if and only if N_γ is $\mathcal{S}$-pure flat for all $\gamma \in \Gamma$.*

In what follows we denote the Pontryagin duality functor $\mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{Q}/\mathbb{Z})$ by $(-)^+$.

LEMMA 2.2. *Let $\mathcal{S}$ be a class of R-modules. A right R-module M is $\mathcal{S}$-pure flat if and only if M^+ is $\mathcal{S}$-pure injective.*

Proof. Let $\mathbf{X} : 0 \to X_1 \to X_2 \to X_3 \to 0$ be an $\mathcal{S}$-pure exact sequence. As $\mathbb{Q}/\mathbb{Z}$ is a faithful injective $\mathbb{Z}$-module, $M \otimes_R \mathbf{X}$ is exact if and only if $(M \otimes_R \mathbf{X})^+ \cong \mathrm{Hom}_R(\mathbf{X}, M^+)$ is exact. This implies the conclusion. ∎

Next, for any general class $\mathcal{S}$ of R-modules, we show that the class of $\mathcal{S}$-pure flat R-modules is covering.

COROLLARY 2.3. *Let $\mathcal{S}$ be a class of R-modules. Then every right R-module admits an $\mathcal{S}$-pure flat cover.*

Proof. Let M be a right $\mathcal{S}$-pure flat R-module and

$$0 \to N \xrightarrow{f} M \xrightarrow{g} L \to 0$$

a pure exact sequence of right R-modules. Then we get the split exact sequence

$$0 \to L^+ \xrightarrow{g^+} M^+ \xrightarrow{f^+} N^+ \to 0.$$

Lemma 2.2 implies that M^+ is $\mathcal{S}$-pure injective, and so by Lemma 2.1(i), one deduces that L^+ is $\mathcal{S}$-pure injective. So, using Lemma 2.2 again shows that L is $\mathcal{S}$-pure flat. Hence, the class of $\mathcal{S}$-pure flat right R-modules is closed under pure quotient modules. On the other hand, by Lemma 2.1(ii), any direct sum of $\mathcal{S}$-pure flat right R-modules is $\mathcal{S}$-pure flat. Therefore, by [7, Theorem 2.5], every right R-module has an $\mathcal{S}$-pure flat cover. ∎

For any two natural numbers n, k and any R-homomorphism $\mu : R^k \to R^n$, let $\mu^t : R^n \to R^k$ denote the R-homomorphism given by the transpose of the matrix corresponding to μ. Let U be a finitely presented R-module and $R^k \xrightarrow{\mu} R^n \xrightarrow{\pi} U \to 0$ a finite presentation of U. Then the *Auslander transpose* of U is defined by $\mathrm{tr}(U) := \mathrm{Coker}\,\mu^t$. It is unique up to projective direct summands. For further information on this notion, we refer the reader to [16, Section 11.4] and in particular to Remark in [16, p. 185].

The following result is an analogue of [18, Proposition 3] for a general class of finitely presented R-modules; see also [10, Lemma 1 and Theorem 1], [9, Proposition 1.1 and Corollary 1.2] and [15, Lemma 4.2].

PROPOSITION 2.4. *Let $\mathcal{S}$ be a class of finitely presented R-modules and*

$$\mathbf{E} :\ 0 \to A \overset{i}{\hookrightarrow} B \xrightarrow{\psi} C \to 0$$

an exact sequence of R-modules and R-homomorphisms. The following are equivalent:

(i) $\mathbf{E}$ *is $\mathcal{S}$-pure exact.*
(ii) $\mathrm{tr}(U) \otimes_R \mathbf{E}$ *is exact for all $U \in \mathcal{S}$.*
(iii) $\mu(A^k) = A^n \cap \mu(B^k)$ *for all matrices $\mu \in \mathrm{Hom}_R(R^k, R^n)$ with* $\mathrm{Coker}\,\mu^t \in \mathcal{S}$.
(iv) *For any matrix $(r_{ij}) \in \mathrm{Hom}_R(R^n, R^k)$ with* $\mathrm{Coker}\,(r_{ij}) \in \mathcal{S}$ *and any $a_1, \dots, a_n \in A$, if the linear equations $\sum_{i=1}^k r_{ij}x_i = a_j$, $1 \le j \le n$, are soluble in B, then they are also soluble in A.*

Proof. (i)⇒(iv). Let $(r_{ij}) \in \mathrm{Hom}_R(R^n, R^k)$ be a matrix with $U := \mathrm{Coker}\,(r_{ij}) \in \mathcal{S}$ and $a_1, \dots, a_n \in A$. Then U has generators $u_1, \dots, u_k$ which satisfy the relations $\sum_{i=1}^k r_{ij}u_i = 0$, $1 \le j \le n$. Assume that the linear equations

$$\sum_{i=1}^{k} r_{ij}x_i = a_j, \qquad 1 \leq j \leq n,$$

are soluble in B. We show that they are also soluble in A.

Let $y_1, \ldots, y_k \in B$ be a solution of these equations. The map $f \in \mathrm{Hom}_R(U, C)$ given by $f(u_i) := \psi(y_i)$ for all $1 \leq i \leq k$ is a well-defined R-homomorphism. As $\mathbf{E}$ is $\mathcal{S}$-pure exact, the induced homomorphism $\mathrm{Hom}_R(U, B) \to \mathrm{Hom}_R(U, C)$ is surjective, and so there exists an R-homomorphism $g \in \mathrm{Hom}_R(U, B)$ such that $f = \psi g$. Let $z_i := y_i - g(u_i)$ for all $i = 1, \ldots, k$. Then each z_i belongs to $\mathrm{Ker}\,\psi = A$ and $\sum_{i=1}^{k} r_{ij}z_i = a_j$ for all $j = 1, \ldots, n$.

(iv)⇒(i). Let U be an element of $\mathcal{S}$ which is generated by elements $u_1, \ldots, u_k$ which satisfy the relations $\sum_{i=1}^{k} r_{ij}u_i = 0$, $1 \leq j \leq n$. Let $f \in \mathrm{Hom}_R(U, C)$. For each $i = 1, \ldots, k$, choose $y_i \in B$ such that $\psi(y_i) = f(u_i)$. Then $\sum_{i=1}^{k} r_{ij}y_i \in \mathrm{Ker}\,\psi = A$ for all $j = 1, \ldots, n$. Therefore, we have a set of linear equations

$$\sum_{i=1}^{k} r_{ij}x_i = a_j, \qquad 1 \leq j \leq n,$$

with constants in A which are soluble in B. Let $z_1, \ldots, z_k$ be a solution of these equations in A. We define $g \in \mathrm{Hom}_R(U, B)$ by $g(u_i) := y_i - z_i$ for all $i = 1, \ldots, k$. Then $\psi g = f$, and so the induced homomorphism $\mathrm{Hom}_R(U, B) \to \mathrm{Hom}_R(U, C)$ is surjective.

(ii)⇔(iii). Let $\mu = (r_{ij}) \in \mathrm{Hom}_R(R^k, R^n)$ be a matrix with $U := \mathrm{Coker}\,\mu^t \in \mathcal{S}$. Tensoring the exact sequence $R^k \xrightarrow{\mu} R^n \xrightarrow{\pi} \mathrm{tr}(U) \to 0$ first by A and then by B yield the commutative diagram

$$\begin{array}{ccccccc}
A^k & \xrightarrow{\mu} & A^n & \xrightarrow{\pi_A} & \mathrm{tr}(U) \otimes_R A & \longrightarrow & 0 \\
\downarrow{\scriptstyle i^k} & & \downarrow{\scriptstyle i^n} & & \downarrow{\scriptstyle 1_{\mathrm{tr}(U)} \otimes_R i} & & \\
B^k & \xrightarrow{\mu} & B^n & \xrightarrow{\pi_B} & \mathrm{tr}(U) \otimes_R B & \longrightarrow & 0
\end{array}$$

in which all maps are natural, rows are exact and the left and middle vertical maps are injective. Clearly, $1_{\mathrm{tr}(U)} \otimes_R i$ is injective if and only if $\mathrm{Ker}\,\pi_A = \mathrm{Ker}((1_{\mathrm{tr}(U)} \otimes_R i)(\pi_A))$. On the other hand, we have

$$\mathrm{Ker}((1_{\mathrm{tr}(U)} \otimes_R i)(\pi_A)) = \mathrm{Ker}(\pi_B i^n) = A^n \cap \mathrm{Ker}\,\pi_B = A^n \cap \mu(B^k).$$

Therefore, $\mathrm{tr}(U) \otimes_R \mathbf{E}$ is exact if and only if $\mu(A^k) = A^n \cap \mu(B^k)$.

(iii)⇒(iv). Assume that $(r_{ij}) \in \mathrm{Hom}_R(R^n, R^k)$ is a matrix with $\mathrm{Coker}\,(r_{ij}) \in \mathcal{S}$. Consider the linear equations

$$\sum_{i=1}^{k} r_{ij}x_i = a_j, \quad 1 \leq j \leq n,$$

with constants in A. Let $b_1, \dots, b_k$ be a solution of these equations in B. Set $\mu := (r_{ij})^t$. Then the hypothesis yields $\mu(A^k) = A^n \cap \mu(B^k)$. As $(a_1, \dots, a_n) \in A^n \cap \mu(B^k)$, there exists $(a'_1, \dots, a'_k) \in A^k$ such that $\mu((a'_1, \dots, a'_k)) = (a_1, \dots, a_n)$. Consequently, $a'_1, \dots, a'_k$ is a solution of the above equations in A.

(iv)⇒(iii). Let $\mu = (r_{ij}) \in \mathrm{Hom}_R(R^k, R^n)$ be a matrix with $U := \mathrm{Coker}\, \mu^t \in \mathcal{S}$. Let $(a_1, \dots, a_n) \in A^n \cap \mu(B^k)$. Then $\mu(b_1, \dots, b_k) = (a_1, \dots, a_n)$ for some $b_1, \dots, b_k \in B$. Hence, $b_1, \dots, b_k$ is a solution of the equations

$$\sum_{i=1}^{k} r_{ji}x_i = a_j, \quad 1 \leq j \leq n.$$

Let $a'_1, \dots, a'_k \in A$ be a solution of the above equations. Then $\mu(a'_1, \dots, a'_k) = (a_1, \dots, a_n)$, and so $(a_1, \dots, a_n) \in \mu(A^k)$. ∎

Now, we deduce a couple of corollaries of Proposition 2.4.

COROLLARY 2.5. *Let $\mathcal{S}$ be a class of finitely presented R-modules and*

$$\mathbf{X} : \ 0 \to X_1 \to X_2 \to X_3 \to 0$$

an exact sequence of R-modules and R-homomorphisms. Then the following conditions are equivalent:

(i) $\mathbf{X}$ *is $\mathcal{S}$-pure exact.*
(ii) $\mathrm{Hom}_R(P, \mathbf{X})$ *is exact for all $\mathcal{S}$-pure projective R-modules P.*
(iii) $\mathrm{Hom}_R(\mathbf{X}, E)$ *is exact for all $\mathcal{S}$-pure injective R-modules E.*
(iv) $F \otimes_R \mathbf{X}$ *is exact for all $\mathcal{S}$-pure flat R-modules F.*

Proof. (i)⇒(ii) and (i)⇒(iii) are clear. (ii)⇒(i) comes from the fact that every $U \in \mathcal{S}$ is $\mathcal{S}$-pure projective.

(iii)⇒(iv). Let F be an $\mathcal{S}$-pure flat R-module. Then, by Lemma 2.2, F^+ is $\mathcal{S}$-pure injective. So,

$$\mathrm{Hom}_R(\mathbf{X}, F^+) \cong \mathrm{Hom}_{\mathbb{Z}}(F \otimes_R \mathbf{X}, \mathbb{Q}/\mathbb{Z}) = (F \otimes_R \mathbf{X})^+$$

is exact. Since $\mathbb{Q}/\mathbb{Z}$ is a faithful injective $\mathbb{Z}$-module, it follows that $F \otimes_R \mathbf{X}$ is exact.

(iv)⇒(i). By Proposition 2.4, $\mathrm{tr}(U)$ is $\mathcal{S}$-pure flat for all $U \in \mathcal{S}$. Hence, Proposition 2.4 implies that the sequence $\mathbf{X}$ is $\mathcal{S}$-pure exact. ∎

In what follows, for a class $\mathcal{S}$ of finitely presented R-modules, we denote the class $\{\mathrm{tr}(U) \mid U \in \mathcal{S}\}$ by $\mathrm{tr}(\mathcal{S})$.

COROLLARY 2.6. *Assume that R is commutative and $\mathcal{S}$ is a set-presentable class of finitely presented R-modules containing R. If $\mathcal{S} \subseteq \mathrm{tr}(\mathcal{S})$, then every $\mathcal{S}$-pure projective R-module is $\mathcal{S}$-pure flat.*

Proof. Assume that $\mathcal{S} \subseteq \operatorname{tr}(\mathcal{S})$. Then, by Proposition 2.4, any element of $\mathcal{S}$ is $\mathcal{S}$-pure flat. By [18, Proposition 1], an R-module M is $\mathcal{S}$-pure projective if and only if it is a summand of a direct sum of copies of modules in $\mathcal{S}$. Thus, by Lemma 2.1(ii), every $\mathcal{S}$-pure projective R-module is $\mathcal{S}$-pure flat. ∎

EXAMPLE 2.7. Let $\mathcal{S}$ be a class of finitely presented R-modules.

(i) If $\mathcal{S}$ is the class of all cyclic free R-modules, then $\mathcal{S}$-pure exact sequences are the usual exact sequences. So, $\mathcal{S}$-pure projective, $\mathcal{S}$-pure injective and $\mathcal{S}$-pure flat R-modules are the usual projective, injective and flat R-modules, respectively.
(ii) If $\mathcal{S}$ is the class of all finitely presented R-modules, then $\mathcal{S}$-purity coincides with the usual purity.
(iii) If $\mathcal{S}$ is the class of all cyclic cyclically-presented R-modules, then $\mathcal{S}$-purity coincides with RD-purity.
(iv) If $\mathcal{S}$ is the class of all cyclically-presented R-modules, then $\mathcal{S}$-purity coincides with cyclic purity.
(v) Assume that R is commutative. Obliviously, if $R \in \mathcal{S}$, then $R \in \operatorname{tr}(\mathcal{S})$. It is easy to see that if $\mathcal{S}$ is set-presentable, then $\operatorname{tr}(\mathcal{S})$ has a subclass $\widetilde{\mathcal{S}}$, which is a set, and $\operatorname{tr}(\mathcal{S})$-purity coincides with $\widetilde{\mathcal{S}}$-purity. In cases (i)–(iii) above, one can easily verify that $\mathcal{S} = \operatorname{tr}(\mathcal{S})$. In (iv), $\operatorname{tr}(\mathcal{S})$-purity coincides with $\widetilde{\mathcal{S}}$-purity, where $\widetilde{\mathcal{S}}$ is the set

$$\{R/I \mid I \text{ is a finitely generated ideal of } R\}.$$

PROPOSITION 2.8. *Let $\mathcal{S}$ be a set-presentable class of finitely presented R-modules containing R. Then every R-module M admits an $\mathcal{S}$-pure injective preenvelope.*

Proof. Since $\mathcal{S}$ is set-presentable, it has a subclass $\mathcal{S}^*$, which is a set, with the property that for any $U \in \mathcal{S}$ there is a $U^* \in \mathcal{S}^*$ with $U^* \cong U$. Let Γ be the set of all pairs (U, f) with $U \in \mathcal{S}^*$ and $f \in \operatorname{Hom}_R(M, \operatorname{tr}(U)^+)$, and for each $\gamma \in \Gamma$ denote the corresponding U and f by U_γ and f_γ. Let $E := \prod_{\gamma \in \Gamma} \operatorname{tr}(U_\gamma)^+$ and let $\phi : M \to E$ be the R-homomorphism defined by $\phi(x) = (f_\gamma(x))_\gamma$. Then, by Proposition 2.4 and Lemmas 2.2 and 2.1(i), E is an $\mathcal{S}$-pure injective R-module. As $R \in \mathcal{S}$, it is easy to see that ϕ is injective.

We show that ϕ is our desired $\mathcal{S}$-pure injective preenvelope. By Corollary 2.5, it is enough to check that ϕ is an $\mathcal{S}$-pure monomorphism. For any $U \in \mathcal{S}^*$, the homomorphism

$$1_{\operatorname{tr}(U)} \otimes_R \phi : \operatorname{tr}(U) \otimes_R M \to \operatorname{tr}(U) \otimes_R E$$

is injective if and only if

$$(1_{\operatorname{tr}(U)} \otimes_R \phi)^+ : (\operatorname{tr}(U) \otimes_R E)^+ \to (\operatorname{tr}(U) \otimes_R M)^+$$

is surjective. Now, consider the following commutative diagram:

$$\begin{array}{ccc} (\mathrm{tr}(U)\otimes_R E)^+ & \xrightarrow{(1_{\mathrm{tr}(U)}\otimes_R \phi)^+} & (\mathrm{tr}(U)\otimes_R M)^+ \\ \cong\downarrow & & \downarrow\cong \\ \mathrm{Hom}_R(E,\mathrm{tr}(U)^+) & \xrightarrow{\mathrm{Hom}_R(\phi,1_{\mathrm{tr}(U)^+})} & \mathrm{Hom}_R(M,\mathrm{tr}(U)^+) \end{array}$$

Since the vertical maps are isomorphisms and, by our construction, the bottom map is surjective, we deduce that $1_{\mathrm{tr}(U)} \otimes_R \phi$ is injective. Thus, by Proposition 2.4, ϕ is an $\mathcal{S}$-pure monomorphism. ∎

Let $\mathcal{F}$ and $\mathcal{G}$ be two classes of R-modules. The functor $\mathrm{Hom}_R(-,\sim)$ is said to be *right balanced* by $\mathcal{F}\times\mathcal{G}$ if for any R-module M, there are complexes

$$\mathbf{F}_\bullet :\ \cdots \to F_n \to F_{n-1} \to \cdots \to F_0 \to M \to 0$$

and

$$\mathbf{G}^\bullet :\ 0 \to M \to G^0 \to \cdots \to G^n \to G^{n+1} \to \cdots$$

in which $F_n \in \mathcal{F}, G^n \in \mathcal{G}$ for all $n \geq 0$, and for any $F \in \mathcal{F}$ and any $G \in \mathcal{G}$, the complexes $\mathrm{Hom}_R(\mathbf{F}_\bullet, G)$ and $\mathrm{Hom}_R(F, \mathbf{G}^\bullet)$ are exact.

The concept of pure homological dimensions was introduced in a special case by Griffith [6], and in a general setting by Kiełpiński and Simson [11]. For an R-module M, we define the *$\mathcal{S}$-pure projective dimension* of M as the infimum of the lengths of left $\mathcal{S}$-pure exact resolutions of M which consist of $\mathcal{S}$-pure projective R-modules. Then the *left global $\mathcal{S}$-pure projective dimension* of R is defined to be the supremum of the $\mathcal{S}$-pure projective dimensions of all R-modules. The *$\mathcal{S}$-pure injective dimension* of R-modules and the *left global $\mathcal{S}$-pure injective dimension* of R are defined dually.

We end this section by recording the following useful application.

COROLLARY 2.9. *Let $\mathcal{S}$ be a set-presentable class of finitely presented R-modules containing R. Denote the class of all $\mathcal{S}$-pure projective* (*resp. $\mathcal{S}$-pure injective*) *R-modules by $\mathcal{SP}$* (*resp. $\mathcal{SI}$*). *Then the functor* $\mathrm{Hom}_R(-,\sim)$ *is right balanced by $\mathcal{SP} \times \mathcal{SI}$. In particular, the left global $\mathcal{S}$-pure projective dimension of R is equal to its left global $\mathcal{S}$-pure injective dimension.*

Proof. Let M and N be two R-modules. In view of [18, Proposition 1] and Corollary 2.5, we can construct an exact complex

$$\mathbf{P}_\bullet :\ \cdots \to P_n \to P_{n-1} \to \cdots \to P_0 \to M \to 0$$

such that each P_n is $\mathcal{S}$-pure projective, and for any $\mathcal{S}$-pure projective R-module P and any $\mathcal{S}$-pure injective R-module I, the complexes $\mathrm{Hom}_R(P, \mathbf{P}_\bullet)$ and $\mathrm{Hom}_R(\mathbf{P}_\bullet, I)$ are exact. Also, by Proposition 2.8 and Corollary 2.5, we can construct an exact complex

$$\mathbf{I}^\bullet :\ 0 \to N \to I^0 \to \cdots \to I^n \to I^{n+1} \to \cdots$$

such that each I^n is $\mathcal{S}$-pure injective, and for any $\mathcal{S}$-pure injective R-module

I and any $\mathcal{S}$-pure projective R-module P, the complexes $\mathrm{Hom}_R(\mathbf{I}^\bullet, I)$ and $\mathrm{Hom}_R(P, I^\bullet)$ are exact. Thus, $\mathrm{Hom}_R(-, \sim)$ is right balanced by $\mathcal{SP} \times \mathcal{SI}$.

Denote the complexes

$$\cdots \to P_n \to P_{n-1} \to \cdots \to P_0 \to 0$$

and

$$0 \to I^0 \to \cdots \to I^n \to I^{n+1} \to \cdots$$

by $\mathbf{P}_\circ$ and $\mathbf{I}^\circ$, respectively. Then [3, Theorem 8.2.14] implies that the complexes $\mathrm{Hom}_R(\mathbf{P}_\circ, N)$ and $\mathrm{Hom}_R(M, \mathbf{I}^\circ)$ have isomorphic homology modules.

Let n be a non-negative integer. In view of [3, Theorem 8.2.3(2) and Corollary 8.2.4(2)], it is straightforward to check that the $\mathcal{S}$-pure projective dimension of M is less than or equal to n if and only if $H^{n+1}(\mathrm{Hom}_R(\mathbf{P}_\circ, L)) = 0$ for all R-modules L. Also, by [3, Theorem 8.2.5(1) and Corollary 8.2.6(1)], the $\mathcal{S}$-pure injective dimension of N is less than or equal to n if and only if $H^{n+1}(\mathrm{Hom}_R(L, \mathbf{I}^\circ)) = 0$ for all R-modules L. These facts show that the left global $\mathcal{S}$-pure projective dimension of R is equal to its left global $\mathcal{S}$-pure injective dimension. ■

3. $\mathcal{S}$-pure injective envelopes. To prove Theorem 3.8, which is our main result, we need to prove five preliminary lemmas. We begin with the following definition (compare with [3], [5], [10], [11], [15] and [18]).

DEFINITION 3.1. Let $\mathcal{S}$ be a class of R-modules and N an $\mathcal{S}$-pure submodule of an R-module M.

(i) We say M is an *$\mathcal{S}$-pure essential extension* of N if any R-homomorphism $\varphi : M \to L$ with $\varphi|_N$ an $\mathcal{S}$-pure monomorphism is injective.
(ii) We say M is a *maximal $\mathcal{S}$-pure essential extension* of N if M is an $\mathcal{S}$-pure essential extension of N and no proper extension of M is an $\mathcal{S}$-pure essential extension of N.
(iii) We say M is a *minimal $\mathcal{S}$-pure injective extension* of N if M is $\mathcal{S}$-pure injective and no proper $\mathcal{S}$-pure injective submodule of M contains N.

LEMMA 3.2. *Let $\mathcal{S}$ be a class of R-modules. Let M and M' be R-modules and $f : M \to M'$ an R-isomorphism. Let N be a submodule of M and $N' := f(N)$.*

(i) *N is an $\mathcal{S}$-pure submodule of M if and only if N' is an $\mathcal{S}$-pure submodule of M'.*
(ii) *M is an $\mathcal{S}$-pure essential extension of N if and only if M' is an $\mathcal{S}$-pure essential extension of N'.*
(iii) *M is a maximal $\mathcal{S}$-pure essential extension of N if and only if M' is a maximal $\mathcal{S}$-pure essential extension of N'.*

Proof. (i) is clear.

(ii) Assume that M is an $\mathcal{S}$-pure essential extension of N. By (i), N' is an $\mathcal{S}$-pure submodule of M'. Let $\varphi : M' \to L$ be an R-homomorphism such that $\varphi|_{N'}$ is an $\mathcal{S}$-pure monomorphism. Then $\varphi f : M \to L$ is an R-homomorphism such that $(\varphi f)|_N$ is an $\mathcal{S}$-pure monomorphism. Now, as M is an $\mathcal{S}$-pure essential extension of N, it follows that φf is injective, and so φ is also injective. The converse follows by symmetry. Note that $f^{-1} : M' \to M$ is an R-isomorphism with $f^{-1}(N') = N$.

(iii) By symmetry, it is enough to show the "only if" part. Suppose that M is a maximal $\mathcal{S}$-pure essential extension of N. By (ii), M' is an $\mathcal{S}$-pure essential extension of N'. Let L' be an extension of M' which is an $\mathcal{S}$-pure essential extension of N'. By [17, Proposition 1.1], there are an extension L of M and an R-isomorphism $g : L \to L'$ such that the following diagram commutes:

$$\begin{array}{ccccc} N & \hookrightarrow & M & \hookrightarrow & L \\ \downarrow{\scriptstyle f|_N} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} \\ N' & \hookrightarrow & M' & \hookrightarrow & L' \end{array}$$

It follows by (ii) that L is an $\mathcal{S}$-pure essential extension of N. Hence, by the maximality assumption on M, we obtain $L = M$. Therefore $L' = M'$, as required. ∎

LEMMA 3.3. *Let $\mathcal{S}$ be a class of finitely presented R-modules and N an $\mathcal{S}$-pure submodule of an R-module M. Then there exists a submodule K of M such that $K \cap N = 0$ and M/K is an $\mathcal{S}$-pure essential extension of $(K + N)/K$.*

Proof. Let Σ denote the set of all submodules U of M which satisfy the following conditions:

(i) $U \cap N = 0$; and
(ii) $(U + N)/U$ is an $\mathcal{S}$-pure submodule of M/U.

Then Σ is not empty, because $0 \in \Sigma$. Let $\{K_\alpha\}_{\alpha\in\Omega}$ be a totally ordered subset of Σ and set $\widetilde{K} := \bigcup_{\alpha\in\Omega} K_\alpha$. We show that $\widetilde{K}$ satisfies the conditions (i) and (ii).

Clearly, $\widetilde{K}\cap N=0$. Let $(r_{ij})\in\operatorname{Hom}_R(R^n, R^k)$ be a matrix with $\operatorname{Coker}(r_{ij}) \in \mathcal{S}$. Let

$$(*) \qquad \sum_{i=1}^{k} r_{ij}x_i = a_j + \widetilde{K}, \quad 1 \le j \le n,$$

be a set of linear equations with constants in $(\widetilde{K} + N)/\widetilde{K}$. Let $y_1 + \widetilde{K}, \dots, y_k + \widetilde{K}$ be a solution of these equations in $M/\widetilde{K}$. Then $\sum_{i=1}^k r_{ij}y_i - a_j \in \widetilde{K}$ for all $j = 1, \dots, n$. There exists $\beta \in \Omega$ such that $\sum_{i=1}^k r_{ij}y_i - a_j \in K_\beta$

for all $j = 1, \ldots, n$. So, $y_1 + K_\beta, \ldots, y_k + K_\beta$ is a solution of the equations $\sum_{i=1}^{k} r_{ij}x_i = a_j + K_\beta$, $1 \le j \le n$, in M/K_β. Now, as $(K_\beta + N)/K_\beta$ is an $\mathcal{S}$-pure submodule of M/K_β, there exist $z_1, \ldots, z_k \in N$ such that

$$\sum_{i=1}^{k} r_{ij}z_i - a_j \in K_\beta \subseteq \widetilde{K}$$

for all $j = 1, \ldots, n$. Hence, $z_1 + \widetilde{K}, \ldots, z_k + \widetilde{K}$ is a solution of $(*)$ in $(\widetilde{K} + N)/\widetilde{K}$. So, by Proposition 2.4, $(\widetilde{K} + N)/\widetilde{K}$ is an $\mathcal{S}$-pure submodule of $M/\widetilde{K}$. Thus, by Zorn's Lemma, Σ has a maximal element K.

Suppose that $\varphi : M/K \to L$ is an R-homomorphism such that the restriction $\varphi|_{(N+K)/K}$ is an $\mathcal{S}$-pure monomorphism. Let $\operatorname{Ker}\varphi = K'/K$. Then φ induces an R-monomorphism

$$\varphi^* : (M/K)/(K'/K) \to L.$$

Set $P := ((N + K)/K + K'/K)/(K'/K)$. Since $\varphi((N + K)/K)$ is an $\mathcal{S}$-pure submodule of L and

$$\varphi((N + K)/K) = \varphi^*(P) \le \varphi^*((M/K)/(K'/K)) \le L,$$

it follows that $\varphi^*(P)$ is an $\mathcal{S}$-pure submodule of $\varphi^*((M/K)/(K'/K))$, and so by Lemma 3.2(i), $(N + K')/K'$ is an $\mathcal{S}$-pure submodule of M/K'.

Now, K' is a submodule of M containing K and satisfying (ii). We can easily check that K' also satisfies (i), i.e. $K' \cap N = 0$. Hence, by the maximality of K, we obtain $K' = K$, and so φ is injective. ∎

Next, as an application of the above lemma, we present a characterization of $\mathcal{S}$-pure injective R-modules.

Corollary 3.4. *Let $\mathcal{S}$ be a set-presentable class of finitely presented R-modules containing R. Then for an R-module E, the following are equivalent:*

(i) *E is $\mathcal{S}$-pure injective.*
(ii) *E has no proper $\mathcal{S}$-pure essential extension.*

Proof. (i)⇒(ii). Let M be an $\mathcal{S}$-pure essential extension of E. Then $0 \to E \overset{i}{\hookrightarrow} M \to M/E \to 0$ is an $\mathcal{S}$-pure exact sequence. Since E is $\mathcal{S}$-pure injective, there is an R-homomorphism $f : M \to E$ such that $fi = 1_E$. Then $M = E + \operatorname{Ker} f$ and $E \cap \operatorname{Ker} f = 0$. Denote the R-homomorphism $if : M \to M$ by φ. Then $\varphi|_E = i$, and so $\varphi|_E$ is an $\mathcal{S}$-pure monomorphism. Hence φ is injective, because M is an $\mathcal{S}$-pure essential extension of E. This implies that $\operatorname{Ker} f = \operatorname{Ker}\varphi = 0$, and so $M = E$.

(ii)⇒(i). By Proposition 2.8, there exists an $\mathcal{S}$-pure injective extension L of E. By Lemma 3.3, there is a submodule K of L such that L/K is an $\mathcal{S}$-pure essential extension of $(E + K)/K$ and $E \cap K = 0$. But E has no proper $\mathcal{S}$-pure essential extension, and so $E + K = L$. This implies that $L = E \oplus K$. Hence, by Lemma 2.1(i), E is $\mathcal{S}$-pure injective. ∎

LEMMA 3.5. *Let $\mathcal{S}$ be a class of finitely presented R-modules, E an $\mathcal{S}$-pure injective R-module and N an $\mathcal{S}$-pure submodule of E. There is a submodule M of E which is a maximal $\mathcal{S}$-pure essential extension of N.*

Proof. Denote the inclusion map $N \hookrightarrow E$ by i. Let L be an $\mathcal{S}$-pure essential extension of N. Since E is an $\mathcal{S}$-pure injective R-module and i is an $\mathcal{S}$-pure monomorphism, there exists an R-monomorphism $\psi : L \to E$ such that $\psi|_N = i$. So, $|L| = |\psi(L)| \leq |E|$. If L is a maximal $\mathcal{S}$-pure essential extension of N, then by Lemma 3.2(iii), $\psi(L)$ is also a maximal $\mathcal{S}$-pure essential extension of N. Hence, the proof will be completed if we show that N has a maximal $\mathcal{S}$-pure essential extension.

Suppose that the contrary is true. Then, by using transfinite induction, we show that for any ordinal β, there is an $\mathcal{S}$-pure essential extension M_β of N. Set $M_0 := N$. Let β be an ordinal and assume that M_α is defined for all $\alpha < \beta$. Assume that β has a predecessor $\beta - 1$. As $M_{\beta-1}$ is not a maximal $\mathcal{S}$-pure essential extension of N, there is a proper extension M_β of $M_{\beta-1}$ such that M_β is an $\mathcal{S}$-pure essential extension of N.

If β is a limit ordinal, then in view of Proposition 2.4, it is easy to see that $M_\beta := \bigcup_{\alpha<\beta} M_\alpha$ is an $\mathcal{S}$-pure essential extension of N.

Now, let β be an ordinal with $|\beta| > |E|$. Then $|\beta| \leq |M_\beta| \leq |E|$, which is a contradiction. ∎

LEMMA 3.6. *Let $\mathcal{S}$ be a set-presentable class of finitely presented R-modules containing R. Let M be an R-module and E a maximal $\mathcal{S}$-pure essential extension of M. Then E is an $\mathcal{S}$-pure injective R-module.*

Proof. In view of Proposition 2.8, Corollary 2.5 and Lemma 2.1(i), it is enough to show that E is a direct summand of every R-module L which contains E as an $\mathcal{S}$-pure submodule.

Since L is also an $\mathcal{S}$-pure extension of M, by Lemma 3.3, there exists a submodule K of L such that $K \cap M = 0$ and L/K is an $\mathcal{S}$-pure essential extension of $(K + M)/K$. We will show that $L \cong K \oplus E$.

First, we show that $K_1 := K \cap E = 0$. Let $\pi : E \to E/K_1$ denote the natural epimorphism. As $K_1 \cap M = 0$, we see that $\pi|_M$ is an $\mathcal{S}$-pure monomorphism. Hence, π is injective, and so $K_1 = 0$.

Now, let $f : E \to (K + E)/K$ denote the natural isomorphism. Then $f(M) = (K + M)/K$. Thus, by Lemma 3.2(iii), $(K + E)/K$ is a maximal $\mathcal{S}$-pure essential extension of $(K + M)/K$. But L/K is an $\mathcal{S}$-pure essential extension of $(K + M)/K$ and $(K + E)/K \subseteq L/K$. Thus $L = K + E$, and so $L = K \oplus E$, as required. ∎

LEMMA 3.7. *Let $\mathcal{S}$ be a set-presentable class of finitely presented R-modules containing R. Let E be an R-module and M a submodule of E. The following are equivalent:*

(i) *E is a maximal $\mathcal{S}$-pure essential extension of M.*
(ii) *E is an $\mathcal{S}$-pure essential extension of M which is $\mathcal{S}$-pure injective.*
(iii) *E is a minimal $\mathcal{S}$-pure injective extension of M.*

Proof. (i)⇒(ii) is clear by Lemma 3.6.

(ii)⇒(iii). Suppose E_1 is a submodule of E containing M such that E_1 is $\mathcal{S}$-pure injective. By Lemma 3.5, there exists a submodule E_2 of E_1 which is a maximal $\mathcal{S}$-pure essential extension of M. Since E is an $\mathcal{S}$-pure essential extension of M, it follows that $E_2 = E$. Hence $E_1 = E$.

(iii)⇒(i). By Lemma 3.5, there is a submodule E_1 of E such that E_1 is a maximal $\mathcal{S}$-pure essential extension of M. Now, Lemma 3.6 shows that E_1 is $\mathcal{S}$-pure injective. Thus $E_1 = E$ by the minimality assumption. ∎

Finally, we are ready to prove our main result.

THEOREM 3.8. *Let $\mathcal{S}$ be a set-presentable class of finitely presented R-modules containing R. Then every R-module M has an $\mathcal{S}$-pure injective envelope.*

Proof. By Proposition 2.8 and Lemma 3.5, there exists a maximal $\mathcal{S}$-pure essential extension E of M. Let $\phi : M \hookrightarrow E$ denote the inclusion R-homomorphism. Let E' be an $\mathcal{S}$-pure injective R-module and $\psi : M \to E'$ an R-homomorphism. Since E' is $\mathcal{S}$-pure injective, there exists an R-homomorphism $f : E \to E'$ such that $f\phi = \psi$.

Now, suppose $f : E \to E$ is an R-homomorphism such that $f\phi = \phi$. Since $f|_M = \phi$ is an $\mathcal{S}$-pure monomorphism and E is an $\mathcal{S}$-pure essential extension of M, we see that f is injective. By Lemma 3.2(iii), $f(E)$ is also a maximal $\mathcal{S}$-pure essential extension of M. Hence, by Lemma 3.6, $f(E)$ is $\mathcal{S}$-pure injective. Now, as by Lemma 3.7, E is a minimal $\mathcal{S}$-pure injective extension of M, we deduce that $f(E) = E$. So, f is an automorphism. ∎

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