COLLOQUIUM MATHEMATICUM

VOL. 130 2013 NO. 2

OPTIMAL WEIGHTED HARMONIC INTERPOLATIONS BETWEEN SEIFFERT MEANS

BY

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Abstract. We provide a set of optimal estimates of the form

$$\frac{1-\mu}{\mathcal{A}(x,y)}+\frac{\mu}{\mathcal{M}(x,y)}\le\frac{1}{\mathcal{B}(x,y)}\le\frac{1-\nu}{\mathcal{A}(x,y)}+\frac{\nu}{\mathcal{M}(x,y)}$$

where $\mathcal{A}<\mathcal{B}$ are two of the Seiffert means L,P,M,T, while $\mathcal{M}$ is another mean greater than the two.

1. Introduction. The two means introduced by Seiffert, in [5]:

$$P(x,y)=\begin{cases}\dfrac{x-y}{2\arcsin\frac{x-y}{x+y}}, & x\ne y,\\ x, & x=y,\end{cases}$$

and in [6]:

$$T(x,y)=\begin{cases}\dfrac{x-y}{2\arctan\frac{x-y}{x+y}}, & x\ne y,\\ x, & x=y,\end{cases}$$

are being currently investigated by many mathematicians. Especially interesting is finding optimal bounds of Seiffert means in terms of weighted arithmetic ([1, 3, 8]), geometric ([7, 8]) or harmonic ([2, 8]) means of two other means. Interesting inequalities between P, T, arithmetic, geometric, logarithmic, identric and power means were obtained by many authors (see the references in the cited literature) using an analytic approach or properties of the Schwab–Borchardt algorithm.

It is worth recording two other Seiffert means involving the inverse hyperbolic function:

$$M(x,y)=\begin{cases}\dfrac{x-y}{2\operatorname{arsinh}\frac{x-y}{x+y}}, & x\ne y,\\ x, & x=y,\end{cases}$$

2010 *Mathematics Subject Classification*: Primary 26E60; Secondary 26D07.
Key words and phrases: Seiffert means, harmonic mean, logarithmic mean, interpolation.

DOI: 10.4064/cm130-2-8

introduced in [4] and the well known logarithmic mean

$$L(x,y)=\begin{cases}\dfrac{x-y}{2\operatorname{artanh}\frac{x-y}{x+y}}=\dfrac{x-y}{\log x-\log y}, & x\neq y,\\ x, & x=y.\end{cases}$$

The four means satisfy the inequalities $L<P<M<T$ if $x\neq y$. The goal of this paper is to provide optimal bounds of the form

$$\frac{1-\mu}{\mathcal{A}(x,y)}+\frac{\mu}{\mathcal{M}(x,y)}\le\frac{1}{\mathcal{B}(x,y)}\le\frac{1-\nu}{\mathcal{A}(x,y)}+\frac{\nu}{\mathcal{M}(x,y)} \tag{1.1}$$

where $\mathcal{A}<\mathcal{B}$ are two of the means L,P,M,T, while $\mathcal{M}$ is another mean greater than the two. Note that in this approach we find best bounds for the Seiffert mean $\mathcal{B}$ in terms of the weighted harmonic mean of $\mathcal{A}$ and $\mathcal{M}$, so we call them *harmonic interpolations*.

In [8] we developed a geometric method to obtain interpolations for means of the form $SB_{M,N}(x,y)=\frac{\sqrt{N^2(x,y)-M^2(x,y)}}{\arccos(M(x,y)/N(x,y))}$. This method proves to be very efficient in reaching our goal.

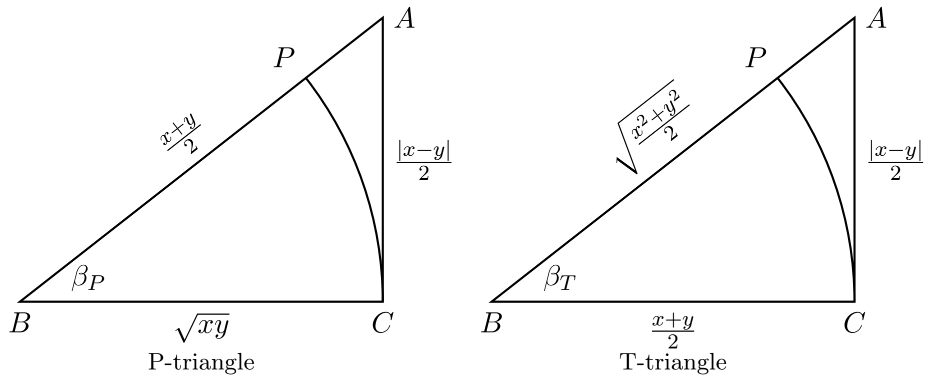
2. Notation and definitions. We shall be using the following notation: x,y are always positive. We shall be considering the geometric, arithmetic, root-mean square and contraharmonic means defined respectively by

$$G(x,y)=\sqrt{xy},\qquad A(x,y)=\frac{x+y}{2},$$
$$R(x,y)=\sqrt{\frac{x^2+y^2}{2}},\qquad C(x,y)=\frac{x^2+y^2}{x+y}.$$

In most cases we shall omit the arguments to simplify notation.

We use the symbol $a\cong b$ to indicate that a and b are of the same sign.

We denote by β_P the radial measure of the angle $\angle ABC$ in the P-triangle in the figure below, while β_T denotes the same angle in the T-triangle.



Definition of β_P and β_T

Note that as x, y vary, the angle β_P assumes all values between $(0, \pi/2)$, while β_T does not exceed $\pi/4$. This follows from the fact that in the T-triangle $|AC| < |BC|$, and makes an essential difference between the two figures.

Thanks to this geometric interpretation we can express the means to be considered as functions of the variables β_P and β_T respectively. In the P-triangle we have

$$\begin{aligned} P &= \frac{|AC|}{\beta_P}, \quad G = \frac{|AC|}{\tan\beta_P}, \quad A = \frac{|AC|}{\sin\beta_P}, \\ L &= \frac{|AC|}{\operatorname{artanh}(\sin\beta_P)}, \quad M = \frac{|AC|}{\operatorname{arsinh}(\sin\beta_P)}. \end{aligned} \tag{2.1}$$

while in the T-triangle

$$\begin{aligned} P &= \frac{|AC|}{\beta_T}, \quad A = \frac{|AC|}{\tan\beta_T}, \quad R = \frac{|AC|}{\sin\beta_T}, \\ L &= \frac{|AC|}{\operatorname{artanh}(\tan\beta_T)}, \quad M = \frac{|AC|}{\operatorname{arsinh}(\tan\beta_T)}. \end{aligned} \tag{2.2}$$

In many cases the bounds obtained in (1.1) are absolute (i.e. valid for all arguments (x, y)), while some bounds will be trivial. For example, if $\mathcal{A} = L$, the only possible left-hand side bound is $\mu = 1$. This is a consequence of the fact that $\lim_{x\to 0} L(x, 1) = 0$, while $\lim_{x\to 0} P(x, 1) = 1/\pi$. In such a case we shall provide additional bounds assuming (x, y) vary over a restricted area.

Definition 2.1. For $0 < \alpha < \pi/2$ we say that (x, y) satisfy the P_α *condition* if

$$\frac{1 - \sin\alpha}{1 + \sin\alpha} \le \frac{x}{y} \le \frac{1 + \sin\alpha}{1 - \sin\alpha}. \tag{2.3}$$

Definition 2.2. For $0 < \alpha < \pi/2$ we say that (x, y) satisfy the T_α *condition* if

$$\frac{1 - \tan\alpha}{1 + \tan\alpha} \le \frac{x}{y} \le \frac{1 + \tan\alpha}{1 - \tan\alpha}. \tag{2.4}$$

Geometrically the P_α condition is equivalent to $\frac{|x-y|}{x+y} \le \sin\alpha$, which means the angle β in the P-triangle varies over the interval $[0, \alpha]$ only. The T_α condition means the same in the T-triangle.

For the convenience of the reader, in the Appendix we provide functions corresponding to the reciprocals of the means used (see (2.1) and (2.2)), and their respective second derivatives.

3. Main tool. If $\mathcal{A} < \mathcal{B} < \mathcal{M}$, and τ varies from 0 to 1, then the expression

$$\frac{1}{\mathcal{B}(x, y)} - \frac{1 - \tau}{\mathcal{A}(x, y)} - \frac{\tau}{\mathcal{M}(x, y)} \tag{3.1}$$

is negative at $\tau = 0$ and strictly increases, to become positive at the other end. We shall be looking for those values of τ where (3.1) is negative while the angle β_P or β_T varies over its maximal range (or over $(0, \alpha)$ if we consider the P_α/T_α condition), and for values where it is always positive. In most cases we face a situation described in the following lemma.

LEMMA 3.1. *Suppose* $u_\tau : [0, a] \to \mathbb{R}$, $\tau \in [0, 1]$, *is a family of functions satisfying the following assumptions:*

- u_τ *increases with* τ,
- $u_\tau(0) = u'_\tau(0) = 0$ *for every* τ,
- *there exists* τ_0 *such that* u_τ *is strictly concave for every* $\tau \le \tau_0$,
- *if* $\tau > \tau_0$, *then* u_τ *is strictly convex for small* x *and* u''_τ *changes sign at most once.*

Let $0 < \alpha \le a$. *Then*

- $u_\tau(x) \le 0$ *for all* $x \in [0, \alpha]$ *if and only if* $\tau \le \tau_0$,
- $u_\tau(x) \ge 0$ *for all* $x \in [0, \alpha]$ *if and only if* $u_\tau(\alpha) \ge 0$.

In particular, if $u_{\tau(\alpha)}(\alpha) = 0$, *then* u_τ *is nonnegative for all* $\tau \ge \tau(\alpha)$.

Proof. If $\tau \le \tau_0$, the function u_τ is concave, thus negative. Otherwise it is convex, thus positive for small arguments, so we are done with the first part.

In case $\tau > \tau_0$ the function u_τ is initially convex and positive, then it may reach a local maximum and become decreasing, which yields the second part. ∎

4. Harmonic interpolations with P and L. In this section we deal with approximations of the form

$$\frac{1-\mu}{L(x,y)} + \frac{\mu}{\mathcal{M}(x,y)} \le \frac{1}{P(x,y)} \le \frac{1-\nu}{L(x,y)} + \frac{\nu}{\mathcal{M}(x,y)}, \tag{4.1}$$

where $\mathcal{M}$ is a mean bounding P from above.

For the arithmetic mean we have

THEOREM 4.1. *The inequalities*

$$\frac{1-\mu}{L(x,y)} + \frac{\mu}{A(x,y)} \le \frac{1}{P(x,y)} \le \frac{1-\nu}{L(x,y)} + \frac{\nu}{A(x,y)}$$

hold if and only if $\nu \le 1/2$ *and* $\mu = 1$. *If the* P_α *condition is satisfied, then*

$$\frac{1-\mu}{L(x,y)} + \frac{\mu}{A(x,y)} \le \frac{1}{P(x,y)} \;\Leftrightarrow\; \mu \ge \frac{\operatorname{artanh}(\sin\alpha) - \alpha}{\operatorname{artanh}(\sin\alpha) - \sin\alpha}.$$

Proof. Using (2.1) we can write

$$\frac{1}{P}-\frac{1-\tau}{L}-\frac{\tau}{A}\cong\beta_P-(1-\tau)\operatorname{artanh}(\sin\beta_P)-\tau\sin\beta_P.$$

The functions $u_\tau(x)=x-(1-\tau)\operatorname{artanh}(\sin x)-\tau\sin x$ satisfy $u_\tau(0)=u_\tau'(0)=0$ and

$$u_\tau''(x)=-\sin x\left(\frac{1-\tau}{\cos^2 x}-\tau\right).$$

The expression in brackets increases from $1-2\tau$ to infinity, so Lemma 3.1 applies with $\tau_0=1/2$. Since for $\tau<1$ we have $\lim_{x\to\pi/2}u_\tau(x)=-\infty$, our functions cannot be globally positive. Thus the only global left bound is $\mu=1$. Solving in τ the inequality $u_\tau(\alpha)\geq 0$ leads us to optimal μ in case the variables satisfy the P_α condition. ∎

Note: the concluding statement in the above proof applies to all theorems in all sections where the logarithmic mean is involved.

For the M mean we have

THEOREM 4.2. *The inequalities*

$$\frac{1-\mu}{L(x,y)}+\frac{\mu}{M(x,y)}\leq\frac{1}{P(x,y)}\leq\frac{1-\nu}{L(x,y)}+\frac{\nu}{M(x,y)}$$

hold if and only if $\nu\leq 1/3$ and $\mu=1$. If the P_α condition is satisfied, then

$$\frac{1-\mu}{L(x,y)}+\frac{\mu}{M(x,y)}\leq\frac{1}{P(x,y)}\ \Leftrightarrow\ \mu\geq\frac{\operatorname{artanh}(\sin\alpha)-\alpha}{\operatorname{artanh}(\sin\alpha)-\operatorname{arsinh}(\sin\alpha)}.$$

Proof. We have

$$\frac{1}{P}-\frac{1-\tau}{L}-\frac{\tau}{M}\cong\beta_P-(1-\tau)\operatorname{artanh}(\sin\beta_P)-\tau\operatorname{arsinh}(\sin\beta_P).$$

The functions $u_\tau(x)=x-(1-\tau)\operatorname{artanh}(\sin x)-\tau\operatorname{arsinh}(\sin x)$ satisfy $u_\tau(0)=u_\tau'(0)=0$ and

$$u_\tau''(x)=-\frac{\sin x}{\cos^2 x}\left(1-\tau-2\tau\frac{\cos^2 x}{(2-\cos^2 x)^{3/2}}\right).$$

The expression in brackets increases from $1-3\tau$ to $1-\tau$, so Lemma 3.1 applies with $\tau_0=1/3$. ∎

THEOREM 4.3. *The inequalities*

$$\frac{1-\mu}{L(x,y)}+\frac{\mu}{T(x,y)}\leq\frac{1}{P(x,y)}\leq\frac{1-\nu}{L(x,y)}+\frac{\nu}{T(x,y)}$$

hold if and only if $\nu\leq 1/4$ and $\mu=1$. If the P_α condition is satisfied, then

$$\frac{1-\mu}{L(x,y)}+\frac{\mu}{T(x,y)}\leq\frac{1}{P(x,y)}\ \Leftrightarrow\ \mu\geq\frac{\operatorname{artanh}(\sin\alpha)-\alpha}{\operatorname{artanh}(\sin\alpha)-\arctan(\sin\alpha)}.$$

Proof. We have

$$\frac{1}{P}-\frac{1-\tau}{L}-\frac{\tau}{T}\cong\beta_P-(1-\tau)\operatorname{artanh}(\sin\beta_P)-\tau\arctan(\sin\beta_P).$$

The functions $u_\tau(x)=x-(1-\tau)\operatorname{artanh}(\sin x)-\tau\arctan(\sin x)$ satisfy $u_\tau(0)=u_\tau'(0)=0$ and

$$u_\tau''(x)=-\sin x\,\frac{4(1-\tau)-2(2-\tau)\cos^2x+(1-2\tau)\cos^4x}{\cos^2x\,(1+\sin^2x)^2}. \tag{4.2}$$

The critical point of the function $p(z)=4(1-\tau)-2(2-\tau)z+(1-2\tau)z^2$ lies outside the interval $(0,1)$ thus we conclude the numerator in (4.2) increases from $1-4\tau$ to $4-4\tau$. Again Lemma 3.1 applies with $\tau_0=1/4$. ∎

The root-mean square mean can be written in the language of the P-triangle as

$$R=\sqrt{2A^2-G^2}=\sqrt{|AB|^2+|AC|^2}=|AC|\frac{\sqrt{1+\sin^2\beta_P}}{\sin\beta_P}.$$

THEOREM 4.4. *The inequalities*

$$\frac{1-\mu}{L(x,y)}+\frac{\mu}{R(x,y)}\le\frac{1}{P(x,y)}\le\frac{1-\nu}{L(x,y)}+\frac{\nu}{R(x,y)}$$

hold if and only if $\nu\le 1/5$ and $\mu=1$. If the P_α condition is satisfied, then

$$\frac{1-\mu}{L(x,y)}+\frac{\mu}{R(x,y)}\le\frac{1}{P(x,y)}\iff\mu\ge\frac{\operatorname{artanh}(\sin\alpha)-\alpha}{\operatorname{artanh}(\sin\alpha)-\frac{\sin\alpha}{\sqrt{1+\sin^2\alpha}}}.$$

Proof. We have

$$\frac{1}{P}-\frac{1-\tau}{L}-\frac{\tau}{R}\cong\beta_P-(1-\tau)\operatorname{artanh}(\sin\beta_P)-\tau\frac{\sin\beta_p}{\sqrt{1+\sin^2\beta_P}}.$$

The functions $u_\tau(x)=x-(1-\tau)\operatorname{artanh}(\sin x)-\tau\frac{\sin x}{\sqrt{1+\sin^2x}}$ satisfy $u_\tau(0)=u_\tau'(0)=0$ and

$$u_\tau''(x)=-\frac{\sin x}{\cos^2x}\left(1-\tau-2\tau\frac{\cos^2x(\cos^2x+1)}{(1+\sin^2x)^{5/2}}\right).$$

The expression in brackets strictly increases from $1-5\tau$ to $1-\tau$ and the reasoning as before permits us to end the proof. ∎

The contraharmonic mean can be written as

$$C=2A-\frac{G^2}{A}=|AC|\frac{1+\sin^2\beta_P}{\sin\beta_P}.$$

THEOREM 4.5. *The inequalities*

$$\frac{1-\mu}{L(x,y)}+\frac{\mu}{C(x,y)}\le\frac{1}{P(x,y)}\le\frac{1-\nu}{L(x,y)}+\frac{\nu}{C(x,y)}$$

hold if and only if $\nu \le 1/8$ *and* $\mu = 1$. *If the* P_α *condition is satisfied, then*

$$\frac{1-\mu}{L(x,y)} + \frac{\mu}{C(x,y)} \le \frac{1}{P(x,y)} \quad \Leftrightarrow \quad \mu \ge \frac{\operatorname{artanh}(\sin\alpha) - \alpha}{\operatorname{artanh}(\sin\alpha) - \frac{\sin\alpha}{1+\sin^2\alpha}}.$$

Proof. We have

$$\frac{1}{P} - \frac{1-\tau}{L} - \frac{\tau}{C} \cong \beta_P - (1-\tau)\operatorname{artanh}(\sin\beta_P) - \tau\frac{\sin\beta_p}{1+\sin^2\beta_P}.$$

The functions $u_\tau(x) = x - (1-\tau)\operatorname{artanh}(\sin x) - \tau\frac{\sin x}{1+\sin^2 x}$ satisfy $u_\tau(0) = u_\tau'(0) = 0$ and

$$u_\tau''(x) = -\frac{\sin x}{\cos^2 x}\left(1 - \tau - \tau\frac{\cos^4 x\,(\cos^2 x + 6)}{(1+\sin^2 x)^3}\right).$$

The expression in brackets strictly increases (the numerator decreases, the denominator increases) from $1-8\tau$ to $1-\tau$ etc. ∎

5. Harmonic interpolations with M and L. In this section we deal with approximations of the form

$$\frac{1-\mu}{L(x,y)} + \frac{\mu}{\mathcal{M}(x,y)} \le \frac{1}{M(x,y)} \le \frac{1-\nu}{L(x,y)} + \frac{\nu}{\mathcal{M}(x,y)}, \tag{5.1}$$

where $\mathcal{M}$ is a mean bounding M from above. Let us begin with the T mean.

THEOREM 5.1. *The inequalities*

$$\frac{1-\mu}{L(x,y)} + \frac{\mu}{T(x,y)} \le \frac{1}{M(x,y)} \le \frac{1-\nu}{L(x,y)} + \frac{\nu}{T(x,y)}$$

hold if and only if $\nu \le 3/4$ *and* $\mu = 1$. *If the* P_α *condition is satisfied, then*

$$\frac{1-\mu}{L(x,y)} + \frac{\mu}{T(x,y)} \le \frac{1}{M(x,y)} \quad \Leftrightarrow \quad \mu \ge \frac{\operatorname{artanh}(\sin\alpha) - \operatorname{arsinh}(\sin\alpha)}{\operatorname{artanh}(\sin\alpha) - \arctan(\sin\alpha)}.$$

Proof. We have

$$\begin{aligned}\frac{1}{M} - \frac{1-\tau}{L} - \frac{\tau}{T} &\cong \operatorname{arsinh}(\sin\beta_P) - (1-\tau)\operatorname{artanh}(\sin\beta_P) - \tau\arctan(\sin\beta_P)\\ &\triangleq u_\tau(\sin\beta_P).\end{aligned}$$

The functions $u_\tau(x)$ satisfy $u_\tau(0) = u_\tau'(0) = 0$ and

$$\begin{aligned}u_\tau''(x) &= x\left(\frac{2(\tau-1)}{(1-x^2)^2} + \frac{2\tau}{(1+x^2)^2} - \frac{1}{(1+x^2)^{3/2}}\right)\\ &\cong 2(\tau-1)(1+x^2)^2 + 2\tau(1-x^2)^2 - (1-x^2)^2\sqrt{1+x^2} \triangleq p(x^2).\end{aligned} \tag{5.2}$$

We shall show that p strictly decreases. Indeed,

$$p'(z) = (8\tau - 4)z - 4 + \frac{3 + 2z - 5z^2}{2\sqrt{z+1}} \le 4z - 4 + \frac{3 + 2z - 5z^2}{2\sqrt{z+1}} = h(z).$$

The function h is concave, attains its maximum at $z_0 \approx 1.631$ and $h(1) = 0$, which means it is negative for $z < 1$. Therefore p decreases and so does $p(x^2)$ (or (5.2)). Since $p(0) = 4\tau - 3$, we are able to apply Lemma 3.1 with $\tau_0 = 3/4$. ■

Now we turn to the R mean.

THEOREM 5.2. *The inequalities*

$$\frac{1-\mu}{L(x,y)} + \frac{\mu}{R(x,y)} \le \frac{1}{M(x,y)} \le \frac{1-\nu}{L(x,y)} + \frac{\nu}{R(x,y)}$$

hold if and only if $\nu \le 3/5$ and $\mu = 1$. If the P_α condition is satisfied, then

$$\frac{1-\mu}{L(x,y)} + \frac{\mu}{R(x,y)} \le \frac{1}{M(x,y)} \;\Leftrightarrow\; \mu \ge \frac{\operatorname{artanh}(\sin\alpha) - \operatorname{arsinh}(\sin\alpha)}{\operatorname{artanh}(\sin\alpha) - \frac{\sin\alpha}{\sqrt{1+\sin^2\alpha}}}.$$

Proof. We have

$$\frac{1}{M} - \frac{1-\tau}{L} - \frac{\tau}{R} \cong \operatorname{arsinh}(\sin\beta_P) - (1-\tau)\operatorname{artanh}(\sin\beta_P) - \frac{\tau\sin\beta_p}{\sqrt{1+\sin^2\beta_P}}$$
$$\triangleq u_\tau(\sin\beta_P).$$

The functions $u_\tau(x)$ satisfy $u_\tau(0) = u'_\tau(0) = 0$ and

$$u''_\tau(x) = x\left(\frac{2(\tau-1)}{(1-x^2)^2} - \frac{1}{(1+x^2)^{3/2}} + \frac{3\tau}{(1+x^2)^{5/2}}\right)$$
$$\cong 2(\tau-1)(1+x^2)^{5/2} - (1-x^2)^2(1+x^2) + 3\tau(1-x^2)^2 \triangleq p(x^2).$$

We shall show that p decreases in $(0,1)$. We have $-(3z^2 - 2z - 1) \le 4/3$, $5(z+1)^{3/2} \ge 5$ and

$$p'(z) = \tau(5(z+1)^{3/2} + 6z - 6) - (3z^2 - 2z - 1) - 5(z+1)^{3/2}.$$

If $5(z+1)^{3/2} + 6z - 6 < 0$, then $p'(z) < -(3z^2 - 2z - 1) - 5(z+1)^{3/2} < -11/5 < 0$. Otherwise $p'(z) \le 5(z+1)^{3/2} + 6z - 6 - (3z^2 - 2z - 1) - 5(z+1)^{3/2} = -3z^2 + 8z - 5 < 0$ in $(0,1)$. Since $p(0) = 5\tau - 3$, Lemma 3.1 applies with $\tau_0 = 3/5$. ■

For the contraharmonic mean we obtain

THEOREM 5.3. *The inequalities*

$$\frac{1-\mu}{L(x,y)} + \frac{\mu}{C(x,y)} \le \frac{1}{M(x,y)} \le \frac{1-\nu}{L(x,y)} + \frac{\nu}{C(x,y)}$$

hold if and only if $\nu \le 3/8$ *and* $\mu = 1$. *If the* P_α *condition is satisfied, then*

$$\frac{1-\mu}{L(x,y)} + \frac{\mu}{C(x,y)} \le \frac{1}{M(x,y)} \quad \Leftrightarrow \quad \mu \ge \frac{\operatorname{artanh}(\sin\alpha) - \operatorname{arsinh}(\sin\alpha)}{\operatorname{artanh}(\sin\alpha) - \frac{\sin\alpha}{1+\sin^2\alpha}}.$$

Proof. We have

$$\begin{aligned}\frac{1}{M} - \frac{1-\tau}{L} - \frac{\tau}{C} &\cong \operatorname{arsinh}(\sin\beta_P) - (1-\tau)\operatorname{artanh}(\sin\beta_P) - \tau\frac{\sin\beta_p}{1+\sin^2\beta_P}\\ &\triangleq u_\tau(\sin\beta_P).\end{aligned}$$

The functions $u_\tau(x)$ satisfy $u_\tau(0) = u'_\tau(0) = 0$ and

$$\begin{aligned}u''_\tau(x) &= x\left(\frac{2(\tau-1)}{(1-x^2)^2} - \frac{1}{(1+x^2)^{3/2}} - \frac{2\tau(x^2-3)}{(1+x^2)^3}\right)\\ &\cong 2(\tau-1)(1+x^2)^3 - (1-x^2)^2(1+x^2)^{3/2} - 2\tau(1-x^2)^2(x^2-3)\\ &\triangleq p(x^2).\end{aligned}$$

We shall show that p decreases in $(0,1)$. We have $-(7z^2-6z-1)\sqrt{z+1} \le 25\sqrt{2}/7$ and $12(z+1)^2 \ge 5$ and

$$2p'(z) = 16\tau(4z-1) - \sqrt{z+1}(7z^2-6z-1) - 12(z+1)^2.$$

If $4z-1<0$, then $p'(z) < 25\sqrt{2}/7 - 12 < 0$. Otherwise $p'(z) \le 48z - 16 + 25\sqrt{2}/7 - 12(z+1)^2 = -12(z-1)^2 - 16 + 25\sqrt{2}/7 < 0$. Since $p(0) = 8\tau - 3$, Lemma 3.1 applies with $\tau_0 = 3/8$. ∎

6. Harmonic interpolations with T and L

THEOREM 6.1. *The inequalities*

$$\frac{1-\mu}{L(x,y)} + \frac{\mu}{R(x,y)} \le \frac{1}{T(x,y)} \le \frac{1-\nu}{L(x,y)} + \frac{\nu}{R(x,y)}$$

hold if and only if $\nu \le 4/5$ *and* $\mu = 1$. *If the* T_α *condition is satisfied, then*

$$\frac{1-\mu}{L(x,y)} + \frac{\mu}{R(x,y)} \le \frac{1}{T(x,y)} \quad \Leftrightarrow \quad \mu \ge \frac{\operatorname{artanh}(\tan\alpha) - \alpha}{\operatorname{artanh}(\tan\alpha) - \sin\alpha}.$$

Proof. We have

$$\frac{1}{T} - \frac{1-\tau}{L} - \frac{\tau}{R} \cong \beta_T - (1-\tau)\operatorname{artanh}(\tan\beta_T) - \tau\sin\beta_T.$$

The functions $u_\tau(x) = x - (1-\tau)\operatorname{artanh}(\tan x) - \tau\sin x$ satisfy $u_\tau(0) = u'_\tau(0) = 0$ and

$$u''_\tau(x) = -\frac{\sin x}{\cos^2 x}p(\cos x),$$

where $p(z) = 4(1-\tau)z - \tau(2z^2-1)^2$. In the interval $(\sqrt{2}/2, 1)$ the function p is concave, $p(\sqrt{2}/2) = 2\sqrt{2}(1-\tau) \ge 0$, $p(1) = 4 - 5\tau$, so it is positive for

$\tau \le 4/5$ and changes sign once otherwise, thus the already known argument applies. ∎

In the T-triangle the contraharmonic mean is represented as

$$C = \frac{R^2}{A} = |AC|\frac{2}{\sin 2\beta_T}.$$

THEOREM 6.2. *The inequalities*

$$\frac{1-\mu}{L(x,y)} + \frac{\mu}{C(x,y)} \le \frac{1}{T(x,y)} \le \frac{1-\nu}{L(x,y)} + \frac{\nu}{C(x,y)}$$

hold if and only if $\nu \le 1/2$ *and* $\mu = 1$. *If the* T_α *condition is satisfied, then*

$$\frac{1-\mu}{L(x,y)} + \frac{\mu}{R(x,y)} \le \frac{1}{T(x,y)} \quad\Leftrightarrow\quad \mu \ge \frac{\operatorname{artanh}(\tan\alpha) - \alpha}{\operatorname{artanh}(\tan\alpha) - \frac{1}{2}\sin 2\alpha}.$$

Proof. We have

$$\frac{1}{T} - \frac{1-\tau}{L} - \frac{\tau}{C} \cong \beta_T - (1-\tau)\operatorname{artanh}(\tan\beta_T) - \frac{\tau}{2}\sin 2\beta_T.$$

The functions $u_\tau(x) = x - (1-\tau)\operatorname{artanh}(\tan x) - \frac{\tau}{2}\sin 2x$ satisfy $u_\tau(0) = u_\tau'(0) = 0$ and

$$u_\tau''(x) = -2\sin 2x\left(\frac{1-\tau}{\cos^2 2x} - \tau\right).$$

The expression in brackets increases from $1-2\tau$ to infinity, which allows us to complete the proof as usual. ∎

7. Harmonic interpolations with M and P. In this section we deal with approximations of the form

$$\frac{1-\mu}{P(x,y)} + \frac{\mu}{\mathcal{M}(x,y)} \le \frac{1}{M(x,y)} \le \frac{1-\nu}{P(x,y)} + \frac{\nu}{\mathcal{M}(x,y)}, \tag{7.1}$$

where $\mathcal{M}$ is a mean bounding M from above. We go back to the P-triangle and begin with the T mean. From now on we obtain absolute bounds.

THEOREM 7.1. *The inequalities*

$$\frac{1-\mu}{P(x,y)} + \frac{\mu}{T(x,y)} \le \frac{1}{M(x,y)} \le \frac{1-\nu}{P(x,y)} + \frac{\nu}{T(x,y)}$$

hold if and only if $\nu \le 2/3$ *and* $\mu \ge (2\pi - 4\operatorname{arsinh} 1)/\pi$.

Proof. We have

$$\begin{aligned}\frac{1}{M} - \frac{1-\tau}{P} - \frac{\tau}{T} &\cong \operatorname{arsinh}(\sin\beta_P) - (1-\tau)\beta_P - \tau\arctan(\sin\beta_P)\\ &\triangleq u_\tau(\beta_P).\end{aligned}$$

The functions $u_\tau(x)$ satisfy $u_\tau(0) = u'_\tau(0) = 0$ and

$$\begin{aligned} u''_\tau(x) &= -\sin x \left(\frac{2}{(1+\sin^2 x)^{3/2}} - \tau \frac{3-\sin^2 x}{(1+\sin^2 x)^2} \right) \\ &\cong -\sin x \left(\frac{2(1+\sin^2 x)^{1/2}}{3-\sin^2 x} - \tau \right). \end{aligned}$$

The expression in brackets increases from $2/3-\tau$, thus is positive for $\tau \le 2/3$. Applying Lemma 3.1 we see that u_τ is positive on $(0, \pi/2)$ if and only if $u_\tau(\pi/2) \ge 0$, which completes the proof. ■

Now we turn to the R mean.

THEOREM 7.2. *The inequalities*

$$\frac{1-\mu}{P(x,y)} + \frac{\mu}{R(x,y)} \le \frac{1}{M(x,y)} \le \frac{1-\nu}{P(x,y)} + \frac{\nu}{R(x,y)}$$

hold if and only if $\nu \le 1/2$ *and* $\mu \ge (\pi - 2\operatorname{arsinh} 1)/(\pi - \sqrt{2})$.

Proof. We have

$$\frac{1}{M} - \frac{1-\tau}{P} - \frac{\tau}{R} \cong \operatorname{arsinh}(\sin \beta_P) - (1-\tau)\beta_P - \tau \frac{\sin \beta_P}{\sqrt{1+\sin^2 \beta_P}} \triangleq u_\tau(\beta_P).$$

The functions $u_\tau(x)$ satisfy $u_\tau(0) = u'_\tau(0) = 0$ and

$$\begin{aligned} u''_\tau(x) &= -2\sin x \left(\frac{1}{(1+\sin^2 x)^{3/2}} - \tau \frac{2-\sin^2 x}{(1+\sin^2 x)^{5/2}} \right) \\ &\cong -\sin x \left(\frac{1+\sin^2 x}{2-\sin^2 x} - \tau \right). \end{aligned}$$

The expression in brackets increases from $1/2-\tau$, and application of Lemma 3.1 completes the proof. ■

For the contraharmonic mean we obtain

THEOREM 7.3. *The inequalities*

$$\frac{1-\mu}{P(x,y)} + \frac{\mu}{C(x,y)} \le \frac{1}{M(x,y)} \le \frac{1-\nu}{P(x,y)} + \frac{\nu}{C(x,y)}$$

hold if and only if $\nu \le 2/7$ *and* $\mu \ge (\pi - 2\operatorname{arsinh} 1)/(\pi - 1)$.

Proof. We have

$$\frac{1}{M} - \frac{1-\tau}{P} - \frac{\tau}{C} \cong \operatorname{arsinh}(\sin \beta_P) - (1-\tau)\beta_P - \tau \frac{\sin \beta_P}{1+\sin^2 \beta_P} \triangleq u_\tau(\beta_P).$$

The functions $u_\tau(x)$ satisfy $u_\tau(0) = u'_\tau(0) = 0$ and

$$\begin{aligned} u''_\tau(x) &= -\sin x \left(\frac{2}{(1+\sin^2 x)^{3/2}} - \tau \frac{\cos^4 x + 6\cos^2 x}{(1+\sin^2 x)^2} \right) \\ &\cong -\sin x \left(\frac{2(1+\sin^2 x)^{1/2}}{\cos^4 x + 6\cos^2 x} - \tau \right). \end{aligned}$$

The expression in brackets increases from $2/7 - \tau$, and again application of Lemma 3.1 completes the proof. ■

8. Harmonic interpolations with T and M. In this section we deal with approximations of the form

$$\frac{1-\mu}{M(x,y)} + \frac{\mu}{\mathcal{M}(x,y)} \le \frac{1}{T(x,y)} \le \frac{1-\nu}{M(x,y)} + \frac{\nu}{\mathcal{M}(x,y)}, \tag{8.1}$$

where $\mathcal{M}$ is a mean bounding T from above. We switch back to the T-triangle.

The first upper bound for T is the root-mean square mean.

THEOREM 8.1. *The inequalities*

$$\frac{1-\mu}{M(x,y)} + \frac{\mu}{R(x,y)} \le \frac{1}{T(x,y)} \le \frac{1-\nu}{M(x,y)} + \frac{\nu}{R(x,y)}$$

hold if and only if $\nu \le 1/2$ *and* $\mu \ge \pi/(2(\pi - \sqrt{2}))$.

Proof. We have

$$\begin{aligned} \frac{1}{T} - \frac{1-\tau}{M} - \frac{\tau}{R} &\cong \beta_T - (1-\tau)\operatorname{arsinh}(\tan\beta_T) - \tau \sin\beta_T \\ &\triangleq u_\tau(\beta_T). \end{aligned}$$

The functions $u_\tau(x)$ satisfy $u_\tau(0) = u'_\tau(0) = 0$ and

$$\begin{aligned} u''_\tau(x) &= -(1-\tau)\frac{\sin x}{\cos^2 x} + \tau \sin x \\ &\cong -(1-\tau)\sin x \left(\frac{1}{\cos^2 x} - \frac{\tau}{1-\tau} \right). \end{aligned}$$

The expression in brackets increases from $1 - \frac{\tau}{1-\tau}$, and once more application of Lemma 3.1 completes the proof. ■

For the contraharmonic mean we obtain

THEOREM 8.2. *The inequalities*

$$\frac{1-\mu}{M(x,y)} + \frac{\mu}{C(x,y)} \le \frac{1}{T(x,y)} \le \frac{1-\nu}{M(x,y)} + \frac{\nu}{C(x,y)}$$

hold if and only if $\nu \le 1/5$ *and* $\mu \ge \pi/(2(\pi - 1))$.

Proof. We have

$$\begin{aligned}\frac{1}{T}-\frac{1-\tau}{M}-\frac{\tau}{C}&\cong\beta_T-(1-\tau)\operatorname{arsinh}(\tan\beta_T)-\tau\frac{\sin 2\beta_T}{2}\\&\triangleq u_\tau(\beta_T).\end{aligned}$$

The functions $u_\tau(x)$ satisfy $u_\tau(0)=u'_\tau(0)=0$ and

$$\begin{aligned}u''_\tau(x)&=-(1-\tau)\frac{\sin x}{\cos^2 x}+4\tau\sin x\cos x\\&=-4(1-\tau)\sin x\cos x\left(\frac{1}{4\cos^3 x}-\frac{\tau}{1-\tau}\right).\end{aligned}$$

The expression in brackets increases from $\frac{1}{4}-\frac{\tau}{1-\tau}$ and once more application of Lemma 3.1 completes the proof. ∎

9. Harmonic interpolations with T and P. In this section we deal with approximations of the form

$$\frac{1-\mu}{P(x,y)}+\frac{\mu}{\mathcal{M}(x,y)}\le\frac{1}{T(x,y)}\le\frac{1-\nu}{P(x,y)}+\frac{\nu}{\mathcal{M}(x,y)},\tag{9.1}$$

where $\mathcal{M}$ is a mean bounding T from above. We switch back to the P-triangle.

The first upper bound for T is the root-mean square mean.

THEOREM 9.1. *The inequalities*

$$\frac{1-\mu}{P(x,y)}+\frac{\mu}{R(x,y)}\le\frac{1}{T(x,y)}\le\frac{1-\nu}{P(x,y)}+\frac{\nu}{R(x,y)}$$

hold if and only if $\nu\le 3/4$ *and* $\mu\ge\pi/(2(\pi-\sqrt{2}))$.

Proof. We have

$$\begin{aligned}\frac{1}{T}-\frac{1-\tau}{P}-\frac{\tau}{R}&\cong\arctan(\sin\beta_P)-(1-\tau)\beta_P-\tau\frac{\sin\beta_P}{(1+\sin^2\beta_P)^{1/2}}\\&\triangleq u_\tau(\beta_P).\end{aligned}$$

The functions $u_\tau(x)$ satisfy $u_\tau(0)=u'_\tau(0)=0$ and

$$\begin{aligned}u''_\tau(x)&=-\frac{\sin x\,(\cos^2 x+2)}{(1+\sin^2 x)^2}+\tau\frac{2\sin x(\cos^2 x+1)}{(1+\sin^2 x)^{5/2}}\\&=-\frac{2\sin x\,(\cos^2 x+1)}{(1+\sin^2 x)^{5/2}}\left(\frac{(\cos^2 x+2)(1+\sin^2 x)^{1/2}}{2(\cos^2 x+1)}-\tau\right).\end{aligned}$$

The expression in brackets increases from $3/4-\tau$ (because both $1+\frac{1}{1+\cos^2 x}$ and $\sqrt{1+\sin^2 x}$ increase), and once more application of Lemma 3.1 completes the proof. ∎

For the contraharmonic mean we obtain

THEOREM 9.2. *The inequalities*

$$\frac{1-\mu}{P(x,y)}+\frac{\mu}{C(x,y)}\le\frac{1}{T(x,y)}\le\frac{1-\nu}{P(x,y)}+\frac{\nu}{C(x,y)}$$

hold if and only if $\nu\le 3/7$ *and* $\mu\ge\pi/(2(\pi-1))$.

Proof. We have

$$\begin{aligned}\frac{1}{T}-\frac{1-\tau}{P}-\frac{\tau}{C}&\cong\arctan(\sin\beta_P)-(1-\tau)\beta_P-\tau\frac{\sin\beta_P}{1+\sin^2\beta_P}\\&\triangleq u_\tau(\beta_P).\end{aligned}$$

The functions $u_\tau(x)$ satisfy $u_\tau(0)=u'_\tau(0)=0$ and

$$\begin{aligned}u''_\tau(x)&=-\frac{\sin x\,(\cos^2 x+2)}{(1+\sin^2 x)^2}+\tau\frac{\sin x\,(\cos^4 x+6\cos^2 x)}{(1+\sin^2 x)^3}\\&=-\frac{\sin x\,(\cos^4 x+6\cos^2 x)}{(1+\sin^2 x)^3}\left(\frac{4-\cos^4 x}{\cos^4 x+6\cos^2 x}-\tau\right).\end{aligned}$$

The expression in brackets increases from $3/7-\tau$ etc. ∎

Appendix

	P-Triangle		T-Triangle	
Mean	f	f''	f	f''
$\frac{1}{L}$	$\operatorname{artanh}(\sin x)$	$\frac{\sin x}{\cos^2 x}$	$\operatorname{artanh}(\tan x)$	$\frac{2\sin 2x}{\cos^2 2x}$
$\frac{1}{P}$	x	0		
$\frac{1}{A}$	$\sin x$	$-\sin x$	$\tan x$	$\frac{2\sin x}{\cos^3 x}$
$\frac{1}{M}$	$\operatorname{arsinh}(\sin x)$	$-\frac{2\sin x}{(1+\sin^2 x)^{3/2}}$	$\operatorname{arsinh}(\tan x)$	$\frac{\sin x}{\cos^2 x}$
$\frac{1}{T}$	$\arctan(\sin x)$	$-\frac{\sin x\,(\cos^2 x+2)}{(1+\sin^2 x)^2}$	x	0
$\frac{1}{R}$	$\frac{\sin x}{(1+\sin^2 x)^{1/2}}$	$-\frac{2\sin x\,(\cos^2 x+1)}{(1+\sin^2 x)^{5/2}}$	$\sin x$	$-\sin x$
$\frac{1}{C}$	$\frac{\sin x}{1+\sin^2 x}$	$-\frac{\sin x\,(\cos^4 x+6\cos^2 x)}{(1+\sin^2 x)^3}$	$\frac{1}{2}\sin 2x$	$-2\sin 2x$

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Received 22 December 2011;
revised 1 February 2013 (5598)