COLLOQUIUM MATHEMATICUM

VOL. 130 2013 NO. 2

NON-MEAGER *P*-FILTERS ARE COUNTABLE DENSE HOMOGENEOUS

BY

RODRIGO HERNÁNDEZ-GUTIÉRREZ and MICHAEL HRUŠÁK (Morelia)

Abstract. We prove that if $\mathcal{F}$ is a non-meager P-filter, then both $\mathcal{F}$ and ${}^{\omega}\mathcal{F}$ are countable dense homogeneous spaces.

1. Introduction. *All spaces considered are separable and metrizable.*

A separable space X is *countable dense homogeneous* (*CDH* for short) if whenever D and E are countable dense subsets of X, there exists a homeomorphism $h : X \to X$ such that $h[D] = E$. Using the now well-known back-and-forth argument, Cantor [3] gave the first example of a CDH space: the real line. In fact, many other important spaces are CDH, e.g. the Euclidean spaces, the Hilbert cube and the Cantor set. Results from [2] and [13] provide general classes of CDH spaces that include the examples mentioned. In [6] and [11] the reader can find summaries of past research and bibliography about CDH spaces.

In [5], Fitzpatrick and Zhou posed the following problems.

1.1. Question. Does there exist a CDH metrizable space that is not completely metrizable?

1.2. Question. For which 0-dimensional subsets X of $\mathbb{R}$ is ${}^{\omega}X$ CDH?

Concerning these two problems, the following results have been obtained.

1.3. Theorem ([6]). *Let X be a separable metrizable space.*

- *If X is CDH and Borel, then X is completely metrizable.*
- *If ${}^{\omega}X$ is CDH, then X is a Baire space.*

1.4. Theorem ([4]). *There is a CDH set of reals X of size ω_1 that is a λ-set* (1) *and thus not completely metrizable.*

The techniques used in the proof of Theorem 1.4 produce spaces that are not Baire spaces, so by Theorem 1.3 they cannot answer Question 1.2.

2010 *Mathematics Subject Classification*: 54D80, 54H05, 54B10, 54A35.

Key words and phrases: countable dense homogeneous, non-meager P-filter.

(1) Recall that a set of reals X is a λ-*set* if every countable subset of X is a relative G_δ set.

DOI: 10.4064/cm130-2-9

There is a natural bijection between the Cantor set ${}^{\omega}2$ and $\mathcal{P}(\omega)$ via characteristic functions. In this way we may identify $\mathcal{P}(\omega)$ with the Cantor set. Thus, any subset of $\mathcal{P}(\omega)$ can be thought of as a separable metrizable space.

Recall that a set $\mathcal{F} \subset \mathcal{P}(\omega)$ is a *filter* (on ω) if (a) $X \in \mathcal{F}$ and $\emptyset \notin \mathcal{F}$; (b) if $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$ and (c) if $A \in \mathcal{F}$ and $A \subset B \subset \omega$, then $B \in \mathcal{F}$. For $\mathcal{X} \subset \mathcal{P}(\omega)$, let $\mathcal{X}^* = \{x \subset \omega : \omega \setminus x \in \mathcal{X}\}$. Then a set $\mathcal{I} \subset \mathcal{P}(\omega)$ is an *ideal* if and only if $\mathcal{I}^*$ is a filter. We will assume that all filters contain the Fréchet filter $\{x \subset \omega : \omega \setminus x \text{ is finite}\}$ (dually, all ideals contain the set of finite subsets of ω). An *ultrafilter* is a maximal filter with respect to inclusion.

If A, B are sets, $A \subset^* B$ means that $A \setminus B$ is finite. A filter $\mathcal{F}$ is called a *P-filter* if given $\{X_n : n < \omega\} \subset \mathcal{F}$ there exists $X \in \mathcal{F}$ such that $X \subset^* X_n$ for all $n < \omega$. Such an X is called a *pseudo-intersection* of $\{X_n : n < \omega\}$. Dually, $\mathcal{I}$ is a *P-ideal* if $\mathcal{I}^*$ is a P-filter. An ultrafilter that is a P-filter is called a *P-point*.

Considering ultrafilters as topological spaces, the following results were obtained recently by Medini and Milovich.

1.5. Theorem ([10, Theorems 15, 21, 24, 41, 43 and 44]). *Assume MA(countable). Then there are ultrafilters $\mathcal{U} \subset \mathcal{P}(\omega)$ with any of the following properties:*

(a) *$\mathcal{U}$ is CDH and a P-point,*
(b) *$\mathcal{U}$ is CDH and not a P-point,*
(c) *$\mathcal{U}$ is not CDH and not a P-point, and*
(d) *${}^{\omega}\mathcal{U}$ is CDH.*

Since ultrafilters do not even have the Baire property ([1, 4.1.1]), Theorem 1.5 gives a consistent answer to Question 1.1 and a consistent example for Question 1.2.

The purpose of this note is to extend these results on ultrafilters to a wider class of filters on ω. In particular, we prove the following result, which answers Questions 3, 5 and 11 of [10].

1.6. Theorem. *Let $\mathcal{F}$ be a non-meager P-filter on $\mathcal{P}(\omega)$ extending the Fréchet filter. Then both $\mathcal{F}$ and ${}^{\omega}\mathcal{F}$ are CDH.*

It is known that non-meager filters do not have the Baire property ([1, 4.1.1]). However, the existence of non-meager P-filters is an open question (in ZFC). It is known that the existence of non-meager P-filters follows from $\mathbf{cof}([\mathfrak{d}]^{\omega}) = \mathfrak{d}$ (where $\mathfrak{d}$ is the dominating number, see [1, 1.3.A]). Hence, if all P-filters are meager then there is an inner model with large cardinals. See [1, 4.4.C] or [7] for a detailed description of this problem.

Note that every CDH filter has to be non-definable in the following sense.

1.7. Proposition. *Let $\mathcal{F}$ be a filter on $\mathcal{P}(\omega)$ extending the Fréchet filter. If one of $\mathcal{F}$ or ${}^{\omega}\mathcal{F}$ is CDH, then $\mathcal{F}$ is non-meager.*

Proof. If ${}^{\omega}\mathcal{F}$ is CDH, then $\mathcal{F}$ is non-meager by [6, Theorem 3.1]. Assume that $\mathcal{F}$ is CDH. If $\mathcal{F}$ is the Fréchet filter, then $\mathcal{F}$ is countable, hence not CDH. If $\mathcal{F}$ is not the Fréchet filter, there exists $x \in \mathcal{F}$ such that $\omega \setminus x$ is infinite. Thus, $C = \{y : x \subset y \subset \omega\}$ is a copy of the Cantor set contained in $\mathcal{F}$.

If $\mathcal{F}$ were meager, we arrive at a contradiction as follows: Let $D \subset \mathcal{F}$ be a countable dense subset of $\mathcal{F}$ such that $D \cap C$ is dense in C. Since $\mathcal{F}$ is meager in itself, by [6, Lemma 2.1], there is a countable dense subset E of $\mathcal{F}$ that is a G_δ set relative to $\mathcal{F}$. Let $h : \mathcal{F} \to \mathcal{F}$ be a homeomorphism such that $h[D] = E$. Then $h[D \cap C]$ is a countable dense subset of the Cantor set $h[C]$ that is a relative G_δ subset of $h[C]$, which is impossible. So $\mathcal{F}$ is non-meager and the proof is complete. ∎

Notice that $(\mathcal{P}(\omega), \triangle, \emptyset)$ is a topological group (where $A \triangle B$ denotes the symmetric difference of A and B) as it corresponds to addition modulo 2 in ${}^{\omega}2$. Given a filter $\mathcal{F} \subset \mathcal{P}(\omega)$, the dual ideal $\mathcal{F}^*$ is homeomorphic to $\mathcal{F}$ by means of the map that sends each subset of ω to its complement. Notice that $\emptyset \in \mathcal{F}^*$ and $\mathcal{F}^*$ is closed under $\triangle$. Moreover, for each $x \in \mathcal{P}(\omega)$, the function $y \mapsto y \triangle x$ is an autohomeomorphism of $\mathcal{P}(\omega)$. From this it is easy to see that $\mathcal{F}$ is homogeneous. Thus, by [10, Proposition 3], "non-meager" in Proposition 1.7 can be replaced by "a Baire space".

In [9, Theorem 1.2] it is proved that a filter $\mathcal{F}$ is hereditarily Baire if and only if $\mathcal{F}$ is a non-meager P-filter. By Theorem 1.5, it is consistent that not all CDH ultrafilters are P-points so it is consistent that there are CDH filters that are not hereditarily Baire. These observations answer Question 4 in [10].

Recall that Theorem 1.5 also shows that it is consistent that there exist non-CDH non-meager filters.

1.8. Question. Is there a combinatorial characterization of CDH filters?

1.9. Question. Is there a CDH filter (ultrafilter) in ZFC? Is there a non-CDH and non-meager filter (ultrafilter) in ZFC?

2. Proof of Theorem 1.6. For any set X, let $[X]^{<\omega}$ and $[X]^{\omega}$ denote the sets of its finite and countably infinite subsets, respectively. Also ${}^{<\omega}X = \bigcup\{{}^{n}X : n < \omega\}$.

Since any filter is homeomorphic to its dual ideal, we may alternatively speak about a filter or its dual ideal. In particular, the following result is better expressed in the language of ideals. Its proof follows from [10, Lemma 20], we include it for the sake of completeness.

2.1. Lemma. *Let $\mathcal{I} \subset \mathcal{P}(\omega)$ be an ideal, $f : \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ a continuous function and D a countable dense subset of $\mathcal{I}$. If there exists $x \in \mathcal{I}$ such that $\{d \triangle f(d) : d \in D\} \subset \mathcal{P}(x)$, then $f[\mathcal{I}] = \mathcal{I}$.*

Proof. Since D is dense in $\mathcal{P}(\omega)$ and $d \triangle f(d) \subset x$ for all $d \in D$, by continuity it follows that $y \triangle f(y) \subset x$ for all $y \in \mathcal{P}(\omega)$. Then $y \triangle f(y) \in \mathcal{I}$ for all $y \in \mathcal{P}(\omega)$. Since $\mathcal{I}$ is closed under $\triangle$ and $a \triangle a = \emptyset$ for all $a \in \mathcal{P}(\omega)$, it is easy to see that $y \in \mathcal{I}$ if and only if $f(y) \in \mathcal{I}$ for all $y \in \mathcal{P}(\omega)$. ∎

Let $\mathcal{X} \subset [\omega]^\omega$. A tree $T \subset {}^{<\omega}([\omega]^{<\omega})$ is called an *$\mathcal{X}$-tree of finite sets* if for each $s \in T$ there is $X_s \in \mathcal{X}$ such that for every $a \in [X_s]^{<\omega}$ we have $s^\frown a \in T$. It turns out that non-meager P-filters have a very useful combinatorial characterization as follows.

2.2. Lemma ([8, Lemma 1.3]). *Let $\mathcal{F}$ be a filter on $\mathcal{P}(\omega)$ that extends the Fréchet filter. Then $\mathcal{F}$ is a non-meager P-filter if and only if every $\mathcal{F}$-tree of finite sets has a branch whose union is in $\mathcal{F}$.*

Next we prove a combinatorial property that will allow us to construct autohomeomorphisms of the Cantor set that restrict to a given ideal. For $x \in \mathcal{P}(\omega)$, let $\chi(x) \in {}^\omega 2$ be its characteristic function.

2.3. Lemma. *Let $\mathcal{I}$ be a non-meager P-ideal and D_0, D_1 be two countable dense subsets of $\mathcal{I}$. Then there exists $x \in \mathcal{I}$ such that*

(i) *for each $d \in D_0 \cup D_1$, $d \subset^* x$ and*
(ii) *for each $i \in 2$, $d \in D_i$, $n < \omega$ and $t \in {}^{n \cap x}2$, there exists $e \in D_{1-i}$ such that $d \setminus x = e \setminus x$ and $\chi(e)\restriction_{n \cap x} = t$.*

Proof. Let $\mathcal{F} = \mathcal{I}^*$. We will construct an $\mathcal{F}$-tree of finite sets T and use Lemma 2.2 to find $x \in \mathcal{I}$ with the properties listed. Let us give an enumeration $(D_0 \cup D_1) \times {}^{<\omega}2 = \{(d_n, t_n) : n < \omega\}$ such that $\{d_n : n \equiv i \pmod 2\} = D_i$ for $i \in 2$.

The definition of T will be by recursion. For each $s \in T$ we also define $n(s) < \omega$, $F_s \in \mathcal{F}$ and $\phi_s : \mathrm{dom}(s) \to D_0 \cup D_1$ so that the following properties hold:

(1) $\forall s, t \in T\ (s \subsetneq t \Rightarrow n(s) < n(t))$,
(2) $\forall s \in T\ \forall k < \mathrm{dom}(s)\ (s(k) \subset n(s\restriction_{k+1}) \setminus n(s\restriction_k))$,
(3) $\forall s, t \in T\ (s \subset t \Rightarrow F_t \subset F_s)$,
(4) $\forall s \in T\ (F_s \subset \omega \setminus n(s))$,
(5) $\forall s, t \in T\ (s \subset t \Rightarrow \phi_s \subset \phi_t)$,
(6) $\forall s \in T$, if $k = \mathrm{dom}(s)$ $((d_{k-1} \cup \phi_s(k-1)) \setminus n(s) \subset \omega \setminus F_s)$.

Since $\emptyset \in T$, let $n(\emptyset) = 0$ and $F_\emptyset = \omega$. Assume we have $s \in T$ and $a \in F_s$, we have to define everything for $s^\frown a$. Let $k = \mathrm{dom}(s)$. We start by defining $n(s^\frown a) = \max\{k, \max(a), \mathrm{dom}(t_k)\} + 1$. Next we define $\phi_{s^\frown a}$. We only have to do it at k because of (5). We have two cases.

CASE 1: There exists $m < \mathrm{dom}(t_k)$ with $t_k(m) = 1$ and $m \in s(0) \cup \cdots \cup s(k-1)$. We simply declare $\phi_{s^\frown a}(k) = d_k$.

CASE 2: Not Case 1. We define $r_{s^\frown a} \in {}^{n(s^\frown a)}2$ in the following way.

$$r_{s^\frown a}(m) = \begin{cases} d_k(m) & \text{if } m \in s(0) \cup \cdots \cup s(k-1) \cup a, \\ t_k(m) & \text{if } m \in \mathrm{dom}(t_k) \setminus (s(0) \cup \cdots \cup s(k-1) \cup a), \\ 1 & \text{in any other case.} \end{cases}$$

Let $i \in 2$ be such that $i \equiv k \pmod 2$. So $d_k \in D_i$, let $\phi_{s^\frown a}(k) \in D_{1-i}$ be such that $\phi_{s^\frown a}(k) \cap n(s^\frown a) = (r_{s^\frown a})^{-1}(1)$, this is possible because D_{1-i} is dense in $\mathcal{P}(\omega)$. Finally, define

$$F_{s^\frown a} = (F_s \cap (\omega \setminus d_{k-1}) \cap (\omega \setminus \phi_{s^\frown a}(k-1))) \setminus n(s^\frown a).$$

Clearly, $F_{s^\frown a} \in \mathcal{F}$ and it is easy to see that conditions (1)–(6) hold.

By Lemma 2.2, there exists a branch $\{(y_0, \ldots, y_n) : n < \omega\}$ of T whose union $y = \bigcup\{y_n : n < \omega\}$ is in $\mathcal{F}$. Let $x = \omega \setminus y \in \mathcal{I}$. We prove that x is the element we were looking for. It is easy to prove that (6) implies (i).

We next prove that (ii) holds. Let $i \in 2$, $n < \omega$, $t \in {}^{n\cap x}2$ and $d \in D_i$. Let $k < \omega$ be such that $(d_k, t_k) = (d, t')$, where $t' \in {}^n2$ is such that $t'\restriction_{n\cap x} = t$ and $t'\restriction_{n-x} = \underline{0}$. Consider step $k+1$ in the construction of y, that is, the step when $y(k+1)$ was defined. Notice that we are in Case 2 of the construction and $r_{y\restriction_{k+1}}$ is defined. Then $\phi_{y\restriction_{k+1}}(k) = e$ is an element of D_{1-i}. It is not hard to see that $d \setminus x = e \setminus x$ and $\chi(e)\restriction_{n\cap x} = t$. This completes the proof of the lemma. ∎

We will now show that it is enough to prove Theorem 1.6 for $\mathcal{F}$. Recall the following characterization of non-meager filters.

2.4. LEMMA ([1, Theorem 4.1.2]). *Let $\mathcal{F}$ be a filter. Then $\mathcal{F}$ is non-meager if and only if for every partition of ω into finite sets $\{J_n : n < \omega\}$, there is $X \in \mathcal{F}$ such that $\{n < \omega : X \cap J_n = \emptyset\}$ is infinite.*

The following was originally proved by Shelah (see [12, Fact 4.3, p. 327]). We include a proof for the convenience of the reader.

2.5. LEMMA. *If $\mathcal{F}$ is a non-meager P-filter, then ${}^\omega\mathcal{F}$ is homeomorphic to a non-meager P-filter.*

Proof. Let

$$\mathcal{G} = \{A \subset \omega \times \omega : \forall n < \omega\ (A \cap (\{n\} \times \omega) \in \mathcal{F})\}.$$

Notice that $\mathcal{G}$ is homeomorphic to ${}^\omega\mathcal{F}$. It is easy to see that G is a filter on $\omega \times \omega$. We next prove that $\mathcal{G}$ is a non-meager P-filter.

Let $\{A_k : k < \omega\} \subset \mathcal{G}$. For each $\{k, n\} \subset \omega$, we define $A_k^n = \{x \in \omega : (n, x) \in A_k\} \in \mathcal{F}$. Since $\mathcal{F}$ is a P-filter, there is $A \in \mathcal{F}$ such that $A \subset^* A_k^n$

for all $\{k,n\} \subset \omega$. Let $f : \omega \to \omega$ be such that $A \setminus f(n) \subset A_k^n$ for all $k \leq n$. Let

$$B = \bigcup \{\{n\} \times (A \setminus f(n)) : n < \omega\}.$$

Then it is easy to see that $B \in \mathcal{G}$ and B is a pseudointersection of $\{A_n : n < \omega\}$. So $\mathcal{G}$ is a P-filter.

Let $\{J_k : k < \omega\}$ be a partition of $\omega \times \omega$ into finite subsets. Recursively, we define a sequence $\{F_n : n < \omega\} \subset \mathcal{F}$ and a sequence $\{A_n : n < \omega\} \subset [\omega]^\omega$ such that $A_{n+1} \subset A_n$ and $A_n \subset \{k < \omega : J_k \cap (\{n\} \times F_n) = \emptyset\}$ for all $n < \omega$.

For $n = 0$, since $\mathcal{F}$ is non-meager, by Lemma 2.4 there is $F_0 \in \mathcal{F}$ such that $\{k < \omega : J_k \cap (\{0\} \times F_0) = \emptyset\}$ is infinite; call this last set A_0. Assume that we have the construction up to $m < \omega$, then $\mathcal{B} = \{J_k \cap (\{m+1\} \times \omega) : k \in A_m\}$ is a family of pairwise disjoint finite subsets of $\{m+1\} \times \omega$. If $\bigcup \mathcal{B}$ is finite, let $F_{m+1} \in \mathcal{F}$ be such that $F_{m+1} \cap \bigcup \mathcal{B} = \emptyset$ and let $A_{m+1} = A_m$. If $\bigcup \mathcal{B}$ is infinite, let $\{B_k : k \in A_m\}$ be any partition of $(\{m+1\} \times \omega) \setminus \bigcup \mathcal{B}$ into finite subsets (some possibly empty). For each $k \in A_m$, let $C_k = (J_k \cap (\{m+1\} \times \omega)) \cup B_k$. Then $\{C_k : k \in A_m\}$ is a partition of $\{m+1\} \times \omega$ into finite sets, so by Lemma 2.4 there is $F_{m+1} \in \mathcal{F}$ such that $\{k \in A_m : C_k \cap (\{m+1\} \times F_{m+1}) = \emptyset\}$ is infinite; call this set A_{m+1}. This completes the recursion.

Define an increasing function $s : \omega \to \omega$ such that $s(0) = \min A_0$ and $s(k+1) = \min(A_{k+1} \setminus \{s(0), \ldots, s(k)\})$ for $k < \omega$. Also, define $t : \omega \to \omega$ such that $t(0) = 0$ and

$$t(n+1) = \min\{m < \omega : (J_{s(0)} \cup \cdots \cup J_{s(n)}) \cap (\{n+1\} \times \omega) \subset \{n+1\} \times m\}.$$

Finally, let

$$G = \bigcup \{\{n\} \times (F_n \setminus t(n)) : n < \omega\}.$$

Then $G \in \mathcal{G}$ and for all $k < \omega$, $G \cap J_{s(k)} = \emptyset$. Thus, $\mathcal{G}$ is non-meager by Lemma 2.4. ∎

We now have everything ready to prove our result.

Proof of Theorem 1.6. By Lemma 2.5, it is enough to prove that $\mathcal{F}$ is CDH, equivalently that $\mathcal{I} = \mathcal{F}^*$ is CDH. Let D_0 and D_1 be two countable dense subsets of $\mathcal{I}$ and let $x \in \mathcal{I}$ be given by Lemma 2.3.

We will construct a homeomorphism $h : \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ such that $h[D_0] = D_1$ and

$$(\star) \qquad \forall d \in D \ (d \triangle h(d) \subset x).$$

By Lemma 2.1, $h[\mathcal{I}] = \mathcal{I}$ and we will have finished.

We shall define h by approximations. By this we mean the following. We will give a strictly increasing sequence $\{n(k) : k < \omega\} \subset \omega$ and in step $k < \omega$ a homeomorphism (permutation) $h_k : \mathcal{P}(n(k)) \to \mathcal{P}(n(k))$ such that

$$(*) \qquad \forall j < k < \omega \ \forall a \in \mathcal{P}(n(k)) \ (h_k(a) \cap n(j) = h_j(a \cap n(k))).$$

By $(*)$, we can define $h : \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ to be the inverse limit of $\{h_k : k < \omega\}$, which is a homeomorphism.

Let $D_0 \cup D_1 = \{d_n : n < \omega\}$ in such a way that $\{d_n : n \equiv i \pmod 2\} = D_i$ for $i \in 2$. To make sure that $h[D_0] = D_1$, in step k we have to decide the value of $h(d_k)$ when k is even and the value of $h^{-1}(d_k)$ when k is odd. We do this by approximating a bijection $\pi : D_0 \to D_1$ in ω steps by a chain of finite bijections $\{\pi_k : k < \omega\}$ and letting $\pi = \bigcup\{\pi_k : k < \omega\}$. In step $k < \omega$, we would like to have π_k defined on some finite set so that the following conditions hold whenever $\pi_k \subset \pi$:

(a)$_k$ if $j < k$ is even, then $h_k(d_j \cap n(k)) = \pi(d_j) \cap n(k)$, and
(b)$_k$ if $j < k$ is odd, then $h_k(d_j \cap n(k)) = \pi^{-1}(d_j) \cap n(k)$.

Notice that once π is completely defined, if (a)$_k$ and (b)$_k$ hold for all $k < \omega$, then $h[D] = E$. During the construction, we need to make sure that the following two conditions hold:

(c)$_k$ $\forall i \in n(k) \setminus x\ \forall a \in \mathcal{P}(n(k))\ (i \in a \Leftrightarrow i \in h_k(a))$, and
(d)$_k$ $\forall d \in \mathrm{dom}(\pi_k)\ (d \setminus x = \pi_k(d) \setminus x)$.

Condition (c)$_k$ is a technical condition that will help us carry out the recursion. Notice that if we have condition (d)$_k$ for all $k < \omega$, then $(\star)$ will hold.

Assume that we have defined $n(0) < \cdots < n(s-1)$, $h_0, \ldots, h_{s-1}$ and a finite bijection $\pi_s \subset D_0 \times D_1$ with $\{d_r : r < s\} \subset \mathrm{dom}(\pi_s) \cup \mathrm{dom}(\pi_s^{-1})$ in such a way that if $\pi \supset \pi_s$, then (a)$_{s-1}$–(d)$_{s-1}$ hold. Let us consider the case when s is even; the other case can be treated in a similar fashion.

If $d_s = \pi_s^{-1}(d_r)$ for some odd $r < s$, let $n(s) = n(s-1)+1$. If we let $\pi_{s+1} = \pi_s$, it is easy to define h_s so that it is compatible with h_{s-1} in the sense of $(*)$, in such a way that (a)$_s$–(d)$_s$ hold for any $\pi \supset \pi_{s+1}$. So we may assume this is not the case.

Notice that the set $S = \{d_r : r < s+1\} \cup \{\pi_s(d_r) : r < s, r \equiv 0 \pmod 2\} \cup \{\pi_s^{-1}(d_r) : r < s, r \equiv 1 \pmod 2\}$ is finite. Choose $p < \omega$ so that $d_s \setminus p \subset x$. Let $r_0 = h_{s-1}(d_s \cap n(s-1)) \in \mathcal{P}(n(s-1))$. Choose $n(s-1) < m < \omega$ and $t \in {}^{m \cap x}2$ in such a way that $t^{-1}(1) \cap n(s-1) = r_0 \cap n(s-1) \cap x$ and t is not extended by any element of $\{\chi(a) : a \in S\}$. By Lemma 2.3, there exists $e \in E$ such that $d_s \setminus x = e \setminus x$ and $\chi(e){\restriction}_{m \cap x} = t$. Notice that $e \notin S$ and $\chi(e){\restriction}_{n(s-1)} = r_0$. We define $\pi_{s+1} = \pi_s \cup \{(d_s, e)\}$. Notice that (d)$_s$ holds in this way.

Now that we have decided where π will send d_s, let $n(s) > \max\{p, m\}$ be such that there are no two distinct $a, b \in S \cup \{\pi_{s+1}(d_s)\}$ with $a \cap n(s) = b \cap n(s)$. Topologically, all elements of $S \cup \{\pi_{s+1}(d_s)\}$ are contained in distinct basic open sets of measure $1/(n(s)+1)$.

Finally, we define the bijection $h_s : \mathcal{P}(n(s)) \to \mathcal{P}(n(s))$. For this part of the proof we will use characteristic functions instead of subsets of ω (otherwise the notation would become cumbersome). Therefore, we may say $h_r : {}^{n(r)}2 \to {}^{n(r)}2$ is a homeomorphism for $r < s$.

Let $(q, q') \in {}^{n(s-1)}2 \times {}^{n(s)\setminus x}2$ be a pair of compatible functions. Notice that $(h_{s-1}(q), q')$ are also compatible by $(\mathrm{c})_{s-1}$. Consider the following condition:

$$\nabla(q, q') : \quad \forall a \in {}^{n(s)}2 \ (q \cup q' \subset a \Leftrightarrow h_{s-1}(q) \cup q' \subset h_s(a)).$$

Notice that if we define h_s so that $\nabla(q, q')$ holds for each pair $(q, q') \in {}^{n(s-1)}2 \times {}^{n(s)\setminus x}2$ of compatible functions, then $(*)$ and $(\mathrm{c})_s$ hold as well.

Then for each pair $(q, q') \in {}^{n(s-1)}2 \times {}^{n(s)\setminus x}2$ of compatible functions we only have to find a bijection $g : {}^T2 \to {}^T2$, where $T = (n(s) \cap x) \setminus n(s-1)$ (this bijection will depend on the pair) and define $h_s : {}^{n(s)}2 \to {}^{n(s)}2$ as

$$h_s(a) = h_{s-1}(q) \cup q' \cup g(f{\restriction}_T)$$

whenever $a \in {}^{n(s)}2$ and $q \cup q' \subset a$. There is only one restriction in the definition of g and it is imposed by conditions $(\mathrm{a})_s$ and $(\mathrm{b})_s$; namely that g is compatible with the bijection π_{s+1} already defined. However by the choice of $n(s)$ this is not hard to achieve. This finishes the inductive step and the proof. ∎

Acknowledgements. This paper is part of the first-named author's doctoral dissertation. Research was supported by CONACyT scholarship for Doctoral Students. The second-named author acknowledges support from PAPIIT grant IN102311 and CONACyT grant 177758.

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Rodrigo Hernández-Gutiérrez, Michael Hrušák
Centro de Ciencias Matemáticas
UNAM, A.P. 61-3
Xangari, Morelia, Michoacán, 58089, México
E-mail: rod@matmor.unam.mx
michael@matmor.unam.mx

Received 27 April 2012;
revised 28 February 2013 (5676)