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# ON AN INTEGRAL OF FRACTIONAL POWER OPERATORS

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**Abstract.** For a bounded and sectorial linear operator $V$ in a Banach space, with spectrum in the open unit disc, we study the operator $\widetilde{V} = \int_0^\infty d\alpha\, V^\alpha$. We show, for example, that $\widetilde{V}$ is sectorial, and asymptotically of type 0. If $V$ has single-point spectrum $\{0\}$, then $\widetilde{V}$ is of type 0 with a single-point spectrum, and the operator $I - \widetilde{V}$ satisfies the Ritt resolvent condition. These results generalize an example of Lyubich, who studied the case where $V$ is a classical Volterra operator.

**1. Introduction.** Consider the classical Volterra operator $J$ which acts in the Banach spaces $L^p([0,1])$, $1 \le p \le \infty$, by $(Jf)(x) = \int_0^x dy\, f(y)$. It is well known that $J$ is a bounded operator in $L^p$ with single-point spectrum $\{0\}$, and it can be proved that $J$ is sectorial of type $\pi/2$. See, for example, the arguments of [4, Section 8.5]; more refined estimates for $J$ are given in [8] and in [10, Theorem 1.2]. Here we use a standard definition of sectoriality: a closed linear operator $V$ acting in the complex Banach space $X$ is said to be *sectorial*, of type $\omega \in [0,\pi)$, if its spectrum $\sigma(V)$ is contained in the closed sector $\bar{\Lambda}_\omega := \{0\} \cup \{z \in \mathbb{C} \colon |\arg z| \le \omega\}$ and if

$$\sup_{\lambda \in \Lambda_{\pi-\theta}} \|\lambda(\lambda+V)^{-1}\| < \infty$$

for any $\theta \in (\omega,\pi)$ (where $\Lambda_\omega$ denotes the open sector $\{z \in \mathbb{C} \colon z \neq 0, |\arg z| < \omega\}$).

Note that there is a well developed theory for the fractional powers $V^\alpha$, $\alpha > 0$, of any sectorial operator $V$; see, for example, [9] or [4]. For example, a classical result states that if $V$ is of type $\omega$ then $V^\alpha$ is of type $\alpha\omega$ for $\alpha \in (0,1)$.

In [8] Lyubich considered the interesting example of the operator

$$\widetilde{J} := \int_0^\infty d\alpha\, J^\alpha, \tag{1}$$

and showed that it is bounded and sectorial of type 0, with spectrum $\{0\}$.

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One can wonder if similar results are true if in (1) the Volterra operator $J$ is replaced by a more general sectorial operator $V$ in a Banach space. In this note we will show that this is indeed the case under some additional conditions on $V$, namely, $V$ should be bounded with spectrum contained in the open unit disc $\mathbb{D} := \{z \in \mathbb{C}\colon |z| < 1\}$.

For such operators $V$ we will see that although the operator $\widetilde{V} := \int_0^\infty d\alpha\, V^\alpha$ is not necessarily of type 0, it is of *asymptotic* type 0. This statement uses the notion of asymptotic type introduced in [3]: a closed linear operator $V$ is said to be *of asymptotic type* $\omega \in [0,\pi)$ if for every $\theta \in (\omega,\pi)$ there exists an $\varepsilon > 0$ such that $\sigma(V) \cap \bar{D}(0;\varepsilon) \subseteq \bar{\Lambda}_\theta$ and

$$\sup_{\lambda \in \Lambda_{\pi-\theta} \cap \bar{D}(0;\varepsilon)} \|\lambda(\lambda + V)^{-1}\| < \infty$$

(where $\bar{D}(a;r) := \{z \in \mathbb{C}\colon |z-a| \le r\}$ for $a \in \mathbb{C}$, $r \ge 0$). Clearly an operator of type $\omega$ is also of asymptotic type $\omega$, but the converse is not true.

It was actually shown in [3] that the operator

$$\int_0^1 d\alpha\, V^\alpha$$

is of asymptotic type 0, for a general sectorial operator $V$. (See also [5, p. 466] for a related example when $V$ is a modified Volterra operator.) In the present paper our proof of the asymptotic type property for the operator $\int_0^\infty d\alpha\, V^\alpha$ is rather different from the approaches in [3] and in Lyubich's paper [8]. In fact, our proof depends essentially on the fact that the operator semigroup $\alpha \mapsto V^\alpha \in \mathcal{L}(X)$ extends to a holomorphic semigroup on the half plane $\Lambda_{\pi/2}$, which is exponentially bounded on proper subsectors of $\Lambda_{\pi/2}$. Here, $\mathcal{L}(X)$ denotes the space of all bounded linear operators $T\colon X \to X$.

It is interesting to point out the formal identity

$$\int_0^\infty d\alpha\, V^\alpha = -1/\log V \tag{2}$$

obtained by substituting $V^\alpha = e^{\alpha \log V}$. This identity is actually valid within the usual bounded Dunford functional calculus for the operator $V \in \mathcal{L}(X)$ if one assumes that $\sigma(V) \subseteq \mathbb{D} \setminus (-1,0]$; in that case $\log V$, $(\log V)^{-1}$ are elements of $\mathcal{L}(X)$. However, we wish to allow operators $V$ with $0 \in \sigma(V)$ and which are possibly non-injective, whereas the operator $\log V$ can generally only be defined for injective sectorial operators (see [4, Section 3.5]). Nevertheless, it might be possible to make sense of (2) even for non-injective $V$ by considering a *multi-valued* operator $\log V$ (compare [4, Remark 3.5.4]). We do not pursue this here.

In [8] Lyubich applied his results on the operator (1) to give a new example of an operator satisfying the well known Ritt condition. Recall that $T \in \mathcal{L}(X)$ is said to be a *Ritt operator* if $\sigma(T)$ is contained in the closed disc $\overline{\mathbb{D}} = \{z \in \mathbb{C}\colon |z| \le 1\}$ and if

$$\|(\lambda - T)^{-1}\| \le c|\lambda - 1|^{-1}$$

for some constant $c > 0$ and all $\lambda \in \mathbb{C}\backslash\overline{\mathbb{D}}$. It is a standard theorem that $T \in \mathcal{L}(X)$ is a Ritt operator if and only if $\sigma(T) \subseteq \mathbb{D} \cup \{1\}$ and $I - T$ is of type $\omega$ for some $\omega \in [0, \pi/2)$; or alternatively, if and only if

$$\sup_{n\in\mathbb{N}}(\|T^n\| + n\|T^n - T^{n+1}\|) < \infty$$

where $\mathbb{N} := \{1, 2, 3, \ldots\}$ (see [1, 2, 7, 11, 12]). In particular, the properties of $\widetilde{J}$ mentioned above imply that the operator

$$T := I - \widetilde{J},$$

acting in $L^p([0, 1])$, is a Ritt operator with spectrum equal to $\{1\}$. Thus Lyubich answered affirmatively a question of J. Zemánek as to whether there exist Ritt operators $T$ with single-point spectrum $\{1\}$.

We will obtain a similar conclusion for the operator $T := I - \int_0^\infty d\alpha\, V^\alpha$, for any bounded sectorial operator $V$ such that $\sigma(V) = \{0\}$.

Finally, let us speculate on possible generalizations. For a positive measure $\mu$ on $(0, \infty)$ and a suitable sectorial operator $V$ one could consider an integral

$$\widetilde{V}_\mu := \int_0^\infty d\mu(\alpha)\, V^\alpha.$$

It seems reasonable to conjecture that $\widetilde{V}_\mu$ is of asymptotic type 0 when the measure $\mu$ is non-vanishing near 0 in the sense that $\mu((0, \varepsilon)) > 0$ for all $\varepsilon > 0$. Note that measures of the form $\mu = \sum_{k=1}^\infty a_k \delta_{\alpha_k}$ with $a_k, \alpha_k > 0$, $\sum_k a_k < \infty$ and $\lim_{k\to\infty} \alpha_k = 0$ satisfy this hypothesis. We shall not, however, develop these ideas here.

In what follows we always use the principal branch of the logarithm $z \mapsto \log z$ and of the power function $z \mapsto z^\alpha = e^{\alpha \log z}$ ($\alpha \in \mathbb{C}$), so that these functions are holomorphic on the domain $\mathbb{C} \setminus (-\infty, 0]$.

**2. Proof of the main result.** Before stating and proving our main result, let us recall some essential facts about fractional powers of operators (see [4] or [9]).

For a sectorial operator $V$ in the complex Banach space $X$, one can define the fractional power $V^\alpha$ for every $\alpha \in \Lambda_{\pi/2} \subseteq \mathbb{C}$. If $V$ is also injective one can define $V^\alpha$ for all $\alpha \in \mathbb{C}$, but we will avoid any injectivity assumption in what follows. Here are a few standard properties, in which we assume that $V \in \mathcal{L}(X)$ is a *bounded* sectorial operator. (For further details and complete proofs see [9] or [4].)

(i) $V^\alpha \in \mathcal{L}(X)$, and $V^\alpha V^\beta = V^{\alpha+\beta}$ for all $\alpha, \beta \in \Lambda_{\pi/2}$.

(ii) For $0 < \operatorname{Re}\alpha < 1$ one has the Balakrishnan formula

$$V^\alpha = \frac{\sin \alpha\pi}{\pi} \int_0^\infty dt\, t^{\alpha-1}(t+V)^{-1}V. \tag{3}$$

(iii) The mapping $\alpha \in \Lambda_{\pi/2} \mapsto V^\alpha \in \mathcal{L}(X)$ is holomorphic.

(It is not difficult to derive (iii) from (i) and (ii)).

Observe that $V^\alpha$ is uniquely determined for all $\alpha \in \Lambda_{\pi/2}$ by properties (i) and (ii). We mention that $V^\alpha$ is also given by a Dunford integral

$$(2\pi i)^{-1} \int_\gamma dz\, z^\alpha (z-V)^{-1}$$

where $\gamma$ is the positively oriented boundary of a truncated sector $\bar{\Lambda}_\theta \cap \bar{D}(0; R)$, for large enough $\theta \in (0, \pi)$ and $R > \|V\|$.

Here is our main result.

THEOREM 2.1. *Let $V \in \mathcal{L}(X)$ be a bounded sectorial operator such that $\sigma(V) \subseteq \mathbb{D}$. Define the operator*

$$\widetilde{V} := \int_0^\infty d\alpha\, V^\alpha. \tag{4}$$

*Then $\widetilde{V} \in \mathcal{L}(X)$, and*

$$\sigma(\widetilde{V}) = \{-1/\log\lambda \colon \lambda \in \sigma(V)\} \tag{5}$$

*with the convention that $1/\log 0 := 0$. Moreover, $\widetilde{V}$ is of asymptotic type $0$: more precisely, if $M_0 > 0$, $M_1 \geq 1$ are such that*

$$\|V\| \leq M_0, \qquad \sup_{\lambda>0} \|\lambda(\lambda+V)^{-1}\| \leq M_1, \tag{6}$$

*then for each $\theta \in (0, \pi)$ there exist $c, \delta > 0$ depending only on $\theta, M_0, M_1$ such that*

$$\|\lambda(\lambda+\widetilde{V})^{-1}\| \leq c$$

*for all $\lambda \in \Lambda_{\pi-\theta} \cap \bar{D}(0; \delta)$.*

*The operator $\widetilde{V}$ is sectorial. More precisely, if $r_0 \in (0,1)$ and $\omega_0 \in [0, \pi)$ are chosen with*

$$\sigma(V) \subseteq \bar{D}(0; r_0) \cap \bar{\Lambda}_{\omega_0}, \tag{7}$$

*then $\widetilde{V}$ is of type $\widetilde{\omega}$, where*

$$\widetilde{\omega} := \arg(-\log r_0 + i\omega_0) \in [0, \pi/2). \tag{8}$$

*In particular, if $\sigma(V) \subseteq [0, 1)$ then $\widetilde{V}$ is of type $0$.*

In the special case where $\sigma(V) = \{0\}$, then (5) gives $\sigma(\widetilde{V}) = \{0\}$, and $\widetilde{V}$ is of type 0. Thus one obtains the following corollary, which generalizes Lyubich's example of a single-point spectrum Ritt operator discussed in Section 1.

COROLLARY 2.2. *Let $V \in \mathcal{L}(X)$ be a bounded sectorial operator with $\sigma(V) = \{0\}$, and define $\widetilde{V}$ as in Theorem 2.1. Then the operator $T := I - \widetilde{V} = I - \int_0^\infty d\alpha\, V^\alpha$ is a Ritt operator with spectrum $\sigma(T) = \{1\}$, and the operator $I - T = \widetilde{V}$ is of type 0.*

In the rest of this section we prove Theorem 2.1. Let $V$ satisfy the hypotheses of the theorem.

LEMMA 2.3. *Given any $\varphi \in (0, \pi/2)$, there exist $c, \rho > 0$ depending only on $\varphi$ and on $M_0, M_1$ in (6) such that*

$$\|V^\alpha\| \le ce^{\rho|\alpha|}, \qquad \alpha \in \Lambda_\varphi. \tag{9}$$

*Moreover, there exist $C, \sigma > 0$ such that*

$$\|V^\alpha\| \le Ce^{-\sigma\alpha}, \qquad \alpha > 0. \tag{10}$$

*Proof.* Given $\varphi \in (0, \pi/2)$, we first claim that there is a $c_0 \ge 1$ depending only on $\varphi, M_0, M_1$ such that

$$\sup\{\|V^\alpha\| \colon \alpha \in \Lambda_\varphi \cap \overline{D}(0; 1/2)\} \le c_0. \tag{11}$$

This can be seen from (3): apply the bounds

$$\|t^{\alpha-1}(t+V)^{-1}V\| = t^{\mathrm{Re}(\alpha)-1}\|I - t(t+V)^{-1}\| \le t^{\mathrm{Re}(\alpha)-1}(1 + M_1)$$

for $t \in (0, 1]$ and

$$\|t^{\alpha-1}(t+V)^{-1}V\| \le t^{\mathrm{Re}(\alpha)-2}M_1M_0$$

for $t \ge 1$, noting also that $|(\mathrm{Re}\,\alpha)^{-1}\sin(\alpha\pi)|$ is uniformly bounded for $\alpha \in \Lambda_\varphi \cap \overline{D}(0; 1/2)$. We leave the reader to check the details.

Next, for any $\alpha \in \Lambda_\varphi$, take an integer $n \in (|\alpha|, |\alpha| + 1]$ and use (11) to write $\|V^\alpha\| \le (\|V^{\alpha/2n}\|)^{2n} \le c_0^n \le c_0 c_0^{|\alpha|}$. Then (9) follows.

Finally, the hypothesis $\sigma(T) \subseteq \mathbb{D}$ means that $\lim_{n\in\mathbb{N},\, n\to\infty} \|V^n\|^{1/n} < 1$, hence there exists a $\sigma > 0$ with $\sup\{e^{\sigma n}\|V^n\| \colon n \in \mathbb{N}\} < \infty$. Because $\sup\{\|V^\alpha\| \colon \alpha \in (0, 1]\} < \infty$, it is easy to deduce (10). ■

By (10), the integral (4) converges and defines an element $\widetilde{V} \in \mathcal{L}(X)$. To study the resolvent of $\widetilde{V}$ we require the following lemma.

LEMMA 2.4. *One has*

$$(\lambda + \widetilde{V})^{-1} = \lambda^{-1} - \lambda^{-2}\int_0^\infty d\alpha\, e^{-\lambda^{-1}\alpha}V^\alpha \tag{12}$$

*for all $\lambda \in \Lambda_{\pi/2}$.*

Heuristically, one derives (12) by writing $\widetilde{V} = -(\log V)^{-1}$ (recall (2)) so that

$$(\lambda + \widetilde{V})^{-1} = \lambda^{-1} - \lambda^{-2}(\lambda^{-1} - \log V)^{-1},$$

which equals the right side of (12) by writing $V^\alpha = e^{\alpha \log V}$.

*Proof of Lemma 2.4.* Let $R(\lambda)$ denote the operator on the right hand side of (12). It is clear from (10) that $R(\lambda) \in \mathcal{L}(X)$ and that $R(\lambda)\widetilde{V} = \widetilde{V}R(\lambda)$, so the lemma will follow if we show that $(\lambda + \widetilde{V})R(\lambda) = I$. Now

$$\begin{aligned}(\lambda + \widetilde{V})R(\lambda) &= \Big(\lambda + \int_0^\infty d\beta\, V^\beta\Big)\Big(\lambda^{-1} - \lambda^{-2}\int_0^\infty d\alpha\, e^{-\lambda^{-1}\alpha}V^\alpha\Big) \\ &= I + \lambda^{-1}\int_0^\infty d\beta\, V^\beta - \lambda^{-1}\int_0^\infty d\alpha\, e^{-\lambda^{-1}\alpha}V^\alpha \\ &\quad - \lambda^{-2}\int_0^\infty d\beta \int_0^\infty d\alpha\, e^{-\lambda^{-1}\alpha}V^{\alpha+\beta}.\end{aligned}$$

In the last line, make a change of variable $u = \alpha + \beta$ to see that

$$\begin{aligned}\int_0^\infty d\beta \int_0^\infty d\alpha\, e^{-\lambda^{-1}\alpha}V^{\alpha+\beta} &= \int_0^\infty d\beta \int_\beta^\infty du\, e^{-\lambda^{-1}u}e^{\lambda^{-1}\beta}V^u \\ &= \int_0^\infty du\, e^{-\lambda^{-1}u}V^u\Big[\int_0^u d\beta\, e^{\lambda^{-1}\beta}\Big] \\ &= \lambda\int_0^\infty du\, V^u - \lambda\int_0^\infty du\, e^{-\lambda^{-1}u}V^u.\end{aligned}$$

Thus after cancellation we obtain $(\lambda + \widetilde{V})R(\lambda) = I$. ∎

We remark that (12) and the bound $\|V^\alpha\| \le C$ from (10) yield

$$\|(\lambda + \widetilde{V})^{-1}\| \le |\lambda|^{-1} + C|\lambda|^{-2}(\mathrm{Re}(\lambda^{-1}))^{-1}$$

for all $\lambda \in \Lambda_{\pi/2}$. It follows easily that $\widetilde{V}$ is sectorial of type $\pi/2$; however, the value $\pi/2$ will later be improved.

We will establish (5) by an approximation argument. Because $\sigma(V) \subseteq \mathbb{D}$ we may choose an $\varepsilon_0 > 0$ such that $\sigma(\varepsilon + V) \subseteq \mathbb{D}$ for all $\varepsilon \in (0, \varepsilon_0)$. For such $\varepsilon$ the operators $\log(\varepsilon + V) \in \mathcal{L}(X)$ and $\widetilde{V}_\varepsilon := -(\log(\varepsilon + V))^{-1} \in \mathcal{L}(X)$ are defined by the Dunford functional calculus for $V$, and the usual spectral mapping theorem for that calculus yields

$$\sigma(\widetilde{V}_\varepsilon) = \{-1/\log(\varepsilon + \lambda) \colon \lambda \in \sigma(V)\}. \tag{13}$$

Note that

$$\widetilde{V}_\varepsilon = \int_0^\infty d\alpha\, e^{\alpha \log(\varepsilon+V)} = \int_0^\infty d\alpha\, (\varepsilon + V)^\alpha.$$

It is a standard fact, derivable from the above properties (i) and (ii) of fractional powers, that $\lim_{\varepsilon\downarrow 0} \|(\varepsilon+V)^\alpha - V^\alpha\| = 0$ for each $\alpha > 0$. Using the Lebesgue dominated convergence theorem one finds that

$$\lim_{\varepsilon\downarrow 0} \|\widetilde{V}_\varepsilon - \widetilde{V}\| \le \lim_{\varepsilon\downarrow 0} \int_0^\infty d\alpha\, \|(\varepsilon+V)^\alpha - V^\alpha\| = 0.$$

By standard results in spectral theory it follows that $\sigma(\widetilde{V})$ is the limit of the sets $\sigma(\widetilde{V}_\varepsilon)$ as $\varepsilon \downarrow 0$, in the Hausdorff metric for compact subsets of $\mathbb{C}$; see for example [6, Theorem IV.3.6]. But (13) shows that the sets $\sigma(\widetilde{V}_\varepsilon)$ converge to $\{-1/\log\lambda \colon \lambda \in \sigma(V)\}$. Thus (5) follows.

That $\widetilde{V}$ is of asymptotic type 0 is really a consequence of the resolvent identity (12) and the fact that the semigroup $\alpha \mapsto V^\alpha$ is exponentially bounded on any proper subsector of the half plane $\Lambda_{\pi/2}$. The details are as follows.

Given any $\varphi \in (0, \pi/2)$, choose $c, \rho$ as in (9). In (12), we may shift the integration to a complex contour $\{re^{i\theta} \colon r \ge 0\}$, where $\theta \in (-\varphi, \varphi)$, and then analytically continue in the variable $\lambda$. In this way one sees that

$$(\lambda + \widetilde{V})^{-1} = \lambda^{-1} - \lambda^{-2} e^{i\theta} \int_0^\infty dr\, e^{-\lambda^{-1} r e^{i\theta}} V^{re^{i\theta}} \tag{14}$$

whenever $\lambda \in \mathbb{C}$ with $\lambda = |\lambda| e^{i(\theta+\tau)}$ where $\theta, \tau \in (-\varphi, \varphi)$ and $0 < |\lambda| < \rho^{-1}\cos\varphi$. These conditions on $\lambda$ ensure that

$$\mathrm{Re}(\lambda^{-1} r e^{i\theta}) \ge |\lambda|^{-1} r \cos\varphi > \rho r$$

so that the integral in (14) converges, thanks to (9). Choosing $\tau = \theta$ we obtain

$$\begin{aligned}\|\lambda(\lambda + \widetilde{V})^{-1}\| &\le 1 + c|\lambda|^{-1} \int_0^\infty dr\, e^{-|\lambda|^{-1} r \cos\varphi + \rho r} \\ &\le 1 + c|\lambda|^{-1}(|\lambda|^{-1}\cos\varphi - \rho)^{-1} \\ &\le 1 + 2c(\cos\varphi)^{-1}\end{aligned}$$

valid for all $\lambda \in \Lambda_{2\varphi}$ such that $|\lambda| < 2^{-1}\rho^{-1}\cos\varphi$. This proves that $\widetilde{V}$ is of asymptotic type 0 with resolvent estimates of the required form.

Let us prove the final statement of the theorem. It follows straightforwardly from (7) and (5) that $\sigma(\widetilde{V}) \subseteq \overline{\Lambda}_{\widetilde{\omega}}$ where $\widetilde{\omega}$ is defined by (8). Then, since $\widetilde{V} \in \mathcal{L}(X)$, one must have

$$\sup\{\|\lambda(\lambda + \widetilde{V})^{-1}\| \colon \lambda \in \Lambda_{\pi-\theta},\, |\lambda| \ge \varepsilon\} < \infty$$

for every $\theta \in (\widetilde{\omega}, \pi)$ and $\varepsilon > 0$. Because $\widetilde{V}$ is of asymptotic type 0 it follows that $\widetilde{V}$ is actually of type $\widetilde{\omega}$. The proof of Theorem 2.1 is complete.

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*REFERENCES*

[1] S. Blunck, *Maximal regularity of discrete and continuous time evolution equations*, Studia Math. 146 (2001), 157–176.

[2] —, *Analyticity and discrete maximal regularity on $L_p$-spaces*, J. Funct. Anal. 183 (2001), 211–230.

[3] N. Dungey, *Asymptotic type for sectorial operators and an integral of fractional powers*, preprint, Macquarie Univ., 2008.

[4] M. Haase, *The Functional Calculus for Sectorial Operators*, Birkhäuser, Basel, 2006.

[5] N. Kalton, S. Montgomery-Smith, K. Oleszkiewicz and Y. Tomilov, *Power-bounded operators and related norm estimates*, J. London Math. Soc. 70 (2004), 463–478.

[6] T. Kato, *Perturbation Theory for Linear Operators*, Grundlehren Math. Wiss. 132, Springer, 1980.

[7] Yu. Lyubich, *Spectral localization, power boundedness and invariant subspaces under Ritt's type condition*, Studia Math. 134 (1999), 153–167.

[8] —, *The single-point spectrum operators satisfying Ritt's resolvent condition*, ibid. 145 (2001), 135–142.

[9] C. Martínez Carracedo and M. Sanz Alix, *The Theory of Fractional Powers of Operators*, North-Holland Math. Stud. 187, North-Holland, Amsterdam, 2001.

[10] A. Montes-Rodríguez, J. Sánchez-Álvarez and J. Zemánek, *Uniform Abel–Kreiss boundedness and the extremal behaviour of the Volterra operator*, Proc. London Math. Soc. 91 (2005), 761–788.

[11] B. Nagy and J. Zemánek, *A resolvent condition implying power boundedness*, Studia Math. 134 (1999), 143–151.

[12] O. Nevanlinna, *On the growth of the resolvent operators for power bounded operators*, in: Linear Operators, Banach Center Publ. 38, Inst. Math., Polish Acad. Sci., 1997, 247–264.

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