ON AN INTEGRAL OF FRACTIONAL POWER OPERATORS

BY

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Abstract. For a bounded and sectorial linear operator $V$ in a Banach space, with spectrum in the open unit disc, we study the operator $\tilde{V} = \int_0^\infty d\alpha V^\alpha$. We show, for example, that $\tilde{V}$ is sectorial, and asymptotically of type 0. If $V$ has single-point spectrum $\{0\}$, then $\tilde{V}$ is of type 0 with a single-point spectrum, and the operator $I - \tilde{V}$ satisfies the Ritt resolvent condition. These results generalize an example of Lyubich, who studied the case where $V$ is a classical Volterra operator.

1. Introduction. Consider the classical Volterra operator $J$ which acts in the Banach spaces $L^p([0,1])$, $1 \leq p \leq \infty$, by $(Jf)(x) = \int_0^x dy f(y)$. It is well known that $J$ is a bounded operator in $L^p$ with single-point spectrum $\{0\}$, and it can be proved that $J$ is sectorial of type $\pi/2$. See, for example, the arguments of [4, Section 8.5]; more refined estimates for $J$ are given in [8] and in [10, Theorem 1.2]. Here we use a standard definition of sectoriality: a closed linear operator $V$ acting in the complex Banach space $X$ is said to be sectorial, of type $\omega \in [0,\pi)$, if its spectrum $\sigma(V)$ is contained in the closed sector $\Lambda_\omega := \{0\} \cup \{z \in \mathbb{C} : |\arg z| \leq \omega\}$ and if

$$\sup_{\lambda \in \Lambda_\pi - \theta} \|\lambda(\lambda + V)^{-1}\| < \infty$$

for any $\theta \in (\omega,\pi)$ (where $A_\omega$ denotes the open sector $\{z \in \mathbb{C} : z \neq 0, |\arg z| < \omega\}$).

Note that there is a well developed theory for the fractional powers $V^\alpha$, $\alpha > 0$, of any sectorial operator $V$; see, for example, [9] or [4]. For example, a classical result states that if $V$ is of type $\omega$ then $V^\alpha$ is of type $\alpha \omega$ for $\alpha \in (0,1)$.

In [8] Lyubich considered the interesting example of the operator

$$\tilde{J} := \int_0^\infty d\alpha J^\alpha,$$

and showed that it is bounded and sectorial of type 0, with spectrum $\{0\}$.  

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One can wonder if similar results are true if in (1) the Volterra operator \( J \) is replaced by a more general sectorial operator \( V \) in a Banach space. In this note we will show that this is indeed the case under some additional conditions on \( V \), namely, \( V \) should be bounded with spectrum contained in the open unit disc \( \mathbb{D} := \{ z \in \mathbb{C} : |z| < 1 \} \).

For such operators \( V \) we will see that although the operator \( \tilde{V} := \int_0^\infty d\alpha \, V^\alpha \) is not necessarily of type 0, it is of asymptotic type 0. This statement uses the notion of asymptotic type introduced in [3]: a closed linear operator \( V \) is said to be of asymptotic type \( \omega \in [0, \pi) \) if for every \( \theta \in (\omega, \pi) \) there exists an \( \varepsilon > 0 \) such that \( \sigma(V) \cap \mathcal{D}(0; \varepsilon) \subseteq \Lambda_\theta \) and

\[
\sup_{\lambda \in \Lambda_{\pi-\theta} \cap \mathcal{D}(0; \varepsilon)} \| \lambda(\lambda + V)^{-1} \| < \infty
\]

(where \( \mathcal{D}(a; r) := \{ z \in \mathbb{C} : |z-a| \leq r \} \) for \( a \in \mathbb{C}, r \geq 0 \)). Clearly an operator of type \( \omega \) is also of asymptotic type \( \omega \), but the converse is not true.

It was actually shown in [3] that the operator

\[
\int_0^1 d\alpha \, V^\alpha
\]

is of asymptotic type 0, for a general sectorial operator \( V \). (See also [5, p. 466] for a related example when \( V \) is a modified Volterra operator.) In the present paper our proof of the asymptotic type property for the operator \( \int_0^\infty d\alpha \, V^\alpha \) is rather different from the approaches in [3] and in Lyubich’s paper [8]. In fact, our proof depends essentially on the fact that the operator semigroup \( \alpha \mapsto V^\alpha \in \mathcal{L}(X) \) extends to a holomorphic semigroup on the half plane \( \Lambda_{\pi/2} \), which is exponentially bounded on proper subsectors of \( \Lambda_{\pi/2} \). Here, \( \mathcal{L}(X) \) denotes the space of all bounded linear operators \( T : X \to X \).

It is interesting to point out the formal identity

\[
\int_0^\infty d\alpha \, V^\alpha = -1/\log V
\]

obtained by substituting \( V^\alpha = e^{\alpha \log V} \). This identity is actually valid within the usual bounded Dunford functional calculus for the operator \( V \in \mathcal{L}(X) \) if one assumes that \( \sigma(V) \subseteq \mathbb{D} \setminus (-1, 0] \); in that case \( \log V, (\log V)^{-1} \) are elements of \( \mathcal{L}(X) \). However, we wish to allow operators \( V \) with \( 0 \in \sigma(V) \) and which are possibly non-injective, whereas the operator \( \log V \) can generally only be defined for injective sectorial operators (see [4, Section 3.5]). Nevertheless, it might be possible to make sense of (2) even for non-injective \( V \) by considering a multi-valued operator \( \log V \) (compare [4, Remark 3.5.4]). We do not pursue this here.

In [8] Lyubich applied his results on the operator (1) to give a new example of an operator satisfying the well known Ritt condition. Recall
that $T \in \mathcal{L}(X)$ is said to be a Ritt operator if $\sigma(T)$ is contained in the closed disc $\overline{D} = \{z \in \mathbb{C}: |z| \leq 1\}$ and if
\[
\|(\lambda - T)^{-1}\| \leq c|\lambda - 1|^{-1}
\]
for some constant $c > 0$ and all $\lambda \in \mathbb{C}\setminus\overline{D}$. It is a standard theorem that $T \in \mathcal{L}(X)$ is a Ritt operator if and only if $\sigma(T) \subseteq \mathbb{D} \cup \{1\}$ and $I - T$ is of type $\omega$ for some $\omega \in [0, \pi/2)$; or alternatively, if and only if
\[
\sup_{n \in \mathbb{N}} (\|T^n\| + n\|T^n - T^{n+1}\|) < \infty
\]
where $\mathbb{N} := \{1, 2, 3, \ldots\}$ (see [1, 2, 7, 11, 12]). In particular, the properties of $\tilde{J}$ mentioned above imply that the operator
\[
T := I - \tilde{J},
\]
acting in $L^p([0, 1])$, is a Ritt operator with spectrum equal to $\{1\}$. Thus Lyubich answered affirmatively a question of J. Zemánek as to whether there exist Ritt operators $T$ with single-point spectrum $\{1\}$.

We will obtain a similar conclusion for the operator $T := I - \int_0^\infty d\alpha V^\alpha$, for any bounded sectorial operator $V$ such that $\sigma(V) = \{0\}$.

Finally, let us speculate on possible generalizations. For a positive measure $\mu$ on $(0, \infty)$ and a suitable sectorial operator $V$ one could consider an integral
\[
\tilde{V}_\mu := \int_0^\infty \mu(\alpha) V^\alpha.
\]
It seems reasonable to conjecture that $\tilde{V}_\mu$ is of asymptotic type 0 when the measure $\mu$ is non-vanishing near 0 in the sense that $\mu((0, \varepsilon)) > 0$ for all $\varepsilon > 0$. Note that measures of the form $\mu = \sum_{k=1}^\infty a_k \delta_{\alpha_k}$ with $a_k, \alpha_k > 0$, $\sum_k a_k < \infty$ and $\lim_{k \to \infty} \alpha_k = 0$ satisfy this hypothesis. We shall not, however, develop these ideas here.

In what follows we always use the principal branch of the logarithm $z \mapsto \log z$ and of the power function $z \mapsto z^\alpha = e^{\alpha \log z}$ ($\alpha \in \mathbb{C}$), so that these functions are holomorphic on the domain $\mathbb{C} \setminus (-\infty, 0]$.

2. Proof of the main result. Before stating and proving our main result, let us recall some essential facts about fractional powers of operators (see [4] or [9]).

For a sectorial operator $V$ in the complex Banach space $X$, one can define the fractional power $V^\alpha$ for every $\alpha \in \Lambda_{\pi/2} \subseteq \mathbb{C}$. If $V$ is also injective one can define $V^\alpha$ for all $\alpha \in \mathbb{C}$, but we will avoid any injectivity assumption in what follows. Here are a few standard properties, in which we assume that $V \in \mathcal{L}(X)$ is a bounded sectorial operator. (For further details and complete proofs see [9] or [4].)
(i) \( V^\alpha \in \mathcal{L}(X) \), and \( V^\alpha V^\beta = V^{\alpha+\beta} \) for all \( \alpha, \beta \in \Lambda_{\pi/2} \).

(ii) For \( 0 < \Re \alpha < 1 \) one has the Balakrishnan formula

\[
V^\alpha = \frac{\sin \alpha \pi}{\pi} \int_0^\infty dt \, t^{\alpha-1} (t + V)^{-1} V.
\]

(iii) The mapping \( \alpha \in \Lambda_{\pi/2} \mapsto V^\alpha \in \mathcal{L}(X) \) is holomorphic.

(It is not difficult to derive (iii) from (i) and (ii)).

Observe that \( V^\alpha \) is uniquely determined for all \( \alpha \in \Lambda_{\pi/2} \) by properties (i) and (ii). We mention that \( V^\alpha \) is also given by a Dunford integral

\[
(2\pi i)^{-1} \int_\gamma dz \, z^{\alpha}(z - V)^{-1}
\]

where \( \gamma \) is the positively oriented boundary of a truncated sector \( \overline{\Lambda}_\theta \cap \overline{D}(0; R) \), for large enough \( \theta \in (0, \pi) \) and \( R > \|V\| \).

Here is our main result.

**Theorem 2.1.** Let \( V \in \mathcal{L}(X) \) be a bounded sectorial operator such that \( \sigma(V) \subseteq \mathbb{D} \). Define the operator

\[
\tilde{V} := \int_0^\infty d\alpha \, V^\alpha.
\]

Then \( \tilde{V} \in \mathcal{L}(X) \), and

\[
\sigma(\tilde{V}) = \left\{-1/\log \lambda : \lambda \in \sigma(V)\right\}
\]

with the convention that \( 1/\log 0 := 0 \). Moreover, \( \tilde{V} \) is of asymptotic type 0: more precisely, if \( M_0 > 0, M_1 \geq 1 \) are such that

\[
\|V\| \leq M_0, \quad \sup_{\lambda > 0} \|\lambda(\lambda + V)^{-1}\| \leq M_1,
\]

then for each \( \theta \in (0, \pi) \) there exist \( c, \delta > 0 \) depending only on \( \theta, M_0, M_1 \) such that

\[
\|\lambda(\lambda + \tilde{V})^{-1}\| \leq c
\]

for all \( \lambda \in \Lambda_{\pi-\theta} \cap \overline{D}(0; \delta) \).

The operator \( \tilde{V} \) is sectorial. More precisely, if \( r_0 \in (0, 1) \) and \( \omega_0 \in [0, \pi) \) are chosen with

\[
\sigma(V) \subseteq \overline{D}(0; r_0) \cap \overline{\Lambda}_{\omega_0},
\]

then \( \tilde{V} \) is of type \( \tilde{\omega} \), where

\[
\tilde{\omega} := \arg(-\log r_0 + i\omega_0) \in [0, \pi/2).
\]

In particular, if \( \sigma(V) \subseteq [0, 1) \) then \( \tilde{V} \) is of type 0.
In the special case where \( \sigma(V) = \{0\} \), then (5) gives \( \sigma(\tilde{V}) = \{0\} \), and \( \tilde{V} \) is of type 0. Thus one obtains the following corollary, which generalizes Lyubich’s example of a single-point spectrum Ritt operator discussed in Section 1.

**Corollary 2.2.** Let \( V \in \mathcal{L}(X) \) be a bounded sectorial operator with \( \sigma(V) = \{0\} \), and define \( \tilde{V} \) as in Theorem 2.1. Then the operator \( T := I - \tilde{V} = I - \int_{0}^{\infty} \alpha V^{\alpha} \) is a Ritt operator with spectrum \( \sigma(T) = \{1\} \), and the operator \( I - T = \tilde{V} \) is of type 0.

In the rest of this section we prove Theorem 2.1. Let \( V \) satisfy the hypotheses of the theorem.

**Lemma 2.3.** Given any \( \varphi \in (0, \pi/2) \), there exist \( c, \rho > 0 \) depending only on \( \varphi \) and on \( M_0, M_1 \) in (6) such that
\[
\|V^{\alpha}\| \leq ce^{\rho|\alpha|}, \quad \alpha \in \Lambda_{\varphi}.
\]
Moreover, there exist \( C, \sigma > 0 \) such that
\[
\|V^{\alpha}\| \leq Ce^{-\sigma \alpha}, \quad \alpha > 0.
\]

**Proof.** Given \( \varphi \in (0, \pi/2) \), we first claim that there is a \( c_0 \geq 1 \) depending only on \( \varphi, M_0, M_1 \) such that
\[
\sup\{\|V^{\alpha}\| : \alpha \in \Lambda_{\varphi} \cap \overline{D}(0; 1/2)\} \leq c_0.
\]
This can be seen from (3): apply the bounds
\[
\|t^{\alpha-1}(t + V)^{-1}V\| = t^{\Re(\alpha)-1}\|I - t(t + V)^{-1}\| \leq t^{\Re(\alpha)-1}(1 + M_1)
\]
for \( t \in (0, 1] \) and
\[
\|t^{\alpha-1}(t + V)^{-1}V\| \leq t^{\Re(\alpha)-2}M_1M_0
\]
for \( t \geq 1 \), noting also that \( |(\Re \alpha)^{-1} \sin(\alpha \pi)| \) is uniformly bounded for \( \alpha \in \Lambda_{\varphi} \cap \overline{D}(0; 1/2) \). We leave the reader to check the details.

Next, for any \( \alpha \in \Lambda_{\varphi} \), take an integer \( n \in (|\alpha|, |\alpha| + 1] \) and use (11) to write \( \|V^{\alpha}\| \leq (\|V^{\alpha/2n}\|)^{2n} \leq c^n_0 \leq c_0c_{|\alpha|} \). Then (9) follows.

Finally, the hypothesis \( \sigma(T) \subseteq \mathbb{D} \) means that \( \lim_{n \in \mathbb{N}, n \to \infty} \|V^n\|^{1/n} < 1 \), hence there exists a \( \sigma > 0 \) with \( \sup\{e^{\sigma n}\|V^n\| : n \in \mathbb{N}\} < \infty \). Because \( \sup\{\|V^{\alpha}\| : \alpha \in (0, 1]\} < \infty \), it is easy to deduce (10).

By (10), the integral (4) converges and defines an element \( \tilde{V} \in \mathcal{L}(X) \).

To study the resolvent of \( \tilde{V} \) we require the following lemma.

**Lemma 2.4.** One has
\[
(\lambda + \tilde{V})^{-1} = \lambda^{-1} - \lambda^{-2} \int_{0}^{\infty} d\alpha e^{-\lambda^{-1}\alpha}V^{\alpha}
\]
for all \( \lambda \in \Lambda_{\pi/2} \).
Heuristically, one derives (12) by writing \( \tilde{V} = -(\log V)^{-1} \) (recall (2)) so that
\[
(\lambda + \tilde{V})^{-1} = \lambda^{-1} - \lambda^{-2}(\lambda^{-1} - \log V)^{-1},
\]
which equals the right side of (12) by writing \( V^\alpha = e^{\alpha \log V} \).

Proof of Lemma 2.4. Let \( R(\lambda) \) denote the operator on the right hand side of (12). It is clear from (10) that \( R(\lambda) \in \mathcal{L}(X) \) and that \( R(\lambda)\tilde{V} = \tilde{V} R(\lambda) \), so the lemma will follow if we show that \((\lambda + \tilde{V})R(\lambda) = I\). Now
\[
(\lambda + \tilde{V})R(\lambda) = \left( \lambda + \int_0^\infty d\beta V^\beta \right) \left( \lambda^{-1} - \lambda^{-2} \int_0^\infty d\alpha e^{-\lambda^{-1}\alpha V^\alpha} \right)
\]
\[= I + \lambda^{-1} \int d\beta V^\beta - \lambda^{-1} \int d\alpha e^{-\lambda^{-1}\alpha V^\alpha} \]
\[= -\lambda^{-2} \int d\beta \int d\alpha e^{-\lambda^{-1}\alpha V^\alpha + \beta}.
\]
In the last line, make a change of variable \( u = \alpha + \beta \) to see that
\[
\int_0^\infty d\beta \int_0^\infty d\alpha e^{-\lambda^{-1}\alpha V^{\alpha + \beta}} = \int_0^\infty d\beta \int_0^\infty du e^{-\lambda^{-1}u} e^{\lambda^{-1}\beta V^u}
\]
\[= \int_0^\infty du e^{-\lambda^{-1}u} V^u \left[ \int_0^\infty d\beta e^{\lambda^{-1}\beta} \right]
\]
\[= \lambda \int_0^\infty du V^u - \lambda \int_0^\infty du e^{-\lambda^{-1}u} V^u.
\]
Thus after cancellation we obtain \((\lambda + \tilde{V})R(\lambda) = I\).

We remark that (12) and the bound \( ||V^\alpha|| \leq C \) from (10) yield
\[
||(\lambda + \tilde{V})^{-1}|| \leq |\lambda|^{-1} + C|\lambda|^{-2}(\text{Re}(\lambda^{-1}))^{-1}
\]
for all \( \lambda \in \Lambda_{\pi/2} \). It follows easily that \( \tilde{V} \) is sectorial of type \( \pi/2 \); however, the value \( \pi/2 \) will later be improved.

We will establish (5) by an approximation argument. Because \( \sigma(V) \subseteq \mathbb{D} \) we may choose an \( \varepsilon_0 > 0 \) such that \( \sigma(\varepsilon + V) \subseteq \mathbb{D} \) for all \( \varepsilon \in (0, \varepsilon_0) \). For such \( \varepsilon \) the operators \( \log(\varepsilon + V) \in \mathcal{L}(X) \) and \( \tilde{V}_\varepsilon := -(\log(\varepsilon + V))^{-1} \in \mathcal{L}(X) \) are defined by the Dunford functional calculus for \( V \), and the usual spectral mapping theorem for that calculus yields
\[
\sigma(\tilde{V}_\varepsilon) = \{ -1/\log(\varepsilon + \lambda) : \lambda \in \sigma(V) \}.
\]
Note that
\[ \widetilde{V}_\varepsilon = \int_0^\infty d\alpha\ e^{\alpha \log(\varepsilon + V)} = \int_0^\infty d\alpha\ (\varepsilon + V)^\alpha. \]

It is a standard fact, derivable from the above properties (i) and (ii) of fractional powers, that \( \lim_{\varepsilon \downarrow 0} \| (\varepsilon + V)^\alpha - V^\alpha \| = 0 \) for each \( \alpha > 0 \). Using the Lebesgue dominated convergence theorem one finds that
\[ \lim_{\varepsilon \downarrow 0} \| \widetilde{V}_\varepsilon - \widetilde{V} \| \leq \lim_{\varepsilon \downarrow 0} \int_0^\infty d\alpha\ (\varepsilon + V)^\alpha - V^\alpha = 0. \]

By standard results in spectral theory it follows that \( \sigma(\widetilde{V}) \) is the limit of the sets \( \sigma(\widetilde{V}_\varepsilon) \) as \( \varepsilon \downarrow 0 \), in the Hausdorff metric for compact subsets of \( \mathbb{C} \); see for example [6, Theorem IV.3.6]. But (13) shows that the sets \( \sigma(\widetilde{V}_\varepsilon) \) converge to \( \{ -1/\log \lambda : \lambda \in \sigma(V) \} \). Thus (5) follows.

That \( \widetilde{V} \) is of asymptotic type 0 is really a consequence of the resolvent identity (12) and the fact that the semigroup \( \alpha \mapsto V^\alpha \) is exponentially bounded on any proper subsector of the half plane \( \Lambda_{\pi/2} \). The details are as follows.

Given any \( \varphi \in (0, \pi/2) \), choose \( c, \rho \) as in (9). In (12), we may shift the integration to a complex contour \( \{ r e^{i\theta} : r \geq 0 \} \), where \( \theta \in (-\varphi, \varphi) \), and then analytically continue in the variable \( \lambda \). In this way one sees that
\[ (\lambda + \widetilde{V})^{-1} = \lambda^{-1} - \lambda^{-2} e^{i\theta} \int_0^\infty dr\ e^{-\lambda^{-1} r e^{i\theta}} V r e^{i\theta} \]
whenever \( \lambda \in \mathbb{C} \) with \( \lambda = |\lambda| e^{i(\theta + \tau)} \) where \( \theta, \tau \in (-\varphi, \varphi) \) and \( 0 < |\lambda| < \rho^{-1} \cos \varphi \). These conditions on \( \lambda \) ensure that
\[ \Re(\lambda^{-1} r e^{i\theta}) \geq |\lambda|^{-1} r \cos \varphi > \rho r \]
so that the integral in (14) converges, thanks to (9). Choosing \( \tau = \theta \) we obtain
\[ \| \lambda(\lambda + \widetilde{V})^{-1} \| \leq 1 + c|\lambda|^{-1} \int_0^\infty dr\ e^{-|\lambda|^{-1} r \cos \varphi + \rho r} \]
\[ \leq 1 + c|\lambda|^{-1} (|\lambda|^{-1} \cos \varphi - \rho)^{-1} \]
\[ \leq 1 + 2c(\cos \varphi)^{-1} \]
valid for all \( \lambda \in \Lambda_{2\varphi} \) such that \( |\lambda| < 2^{-1} \rho^{-1} \cos \varphi \). This proves that \( \widetilde{V} \) is of asymptotic type 0 with resolvent estimates of the required form.

Let us prove the final statement of the theorem. It follows straightforwardly from (7) and (5) that \( \sigma(\widetilde{V}) \subseteq \overline{\Lambda} \) where \( \overline{\Lambda} \) is defined by (8). Then, since \( \widetilde{V} \in \mathcal{L}(X) \), one must have
\[ \sup\{ \| \lambda(\lambda + \widetilde{V})^{-1} \| : \lambda \in \Lambda_{\pi-\theta}, |\lambda| \geq \varepsilon \} < \infty \]
for every $\theta \in (\tilde{\omega}, \pi)$ and $\varepsilon > 0$. Because $\tilde{V}$ is of asymptotic type 0 it follows that $\tilde{V}$ is actually of type $\tilde{\omega}$. The proof of Theorem 2.1 is complete.

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