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A NOTE ON SIERPIŃSKI'S PROBLEM RELATED TO TRIANGULAR NUMBERS

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Abstract. We show that the system of equations

$$t_x + t_y = t_p, \quad t_y + t_z = t_q, \quad t_x + t_z = t_r,$$

where $t_x = x(x+1)/2$ is a triangular number, has infinitely many solutions in integers. Moreover, we show that this system has a rational three-parameter solution. Using this result we show that the system

$$t_x + t_y = t_p, \quad t_y + t_z = t_q, \quad t_x + t_z = t_r, \quad t_x + t_y + t_z = t_s$$

has infinitely many rational two-parameter solutions.

1. Introduction.

A *triangular number* is a number of the form

$$t_n = 1 + 2 + \cdots + (n-1) + n = \frac{n(n+1)}{2},$$

where n is a natural number. There are a lot of papers related to various types of diophantine equations containing triangular numbers and their various generalizations [3–7]. One of the author's favourites is the little book [6] written by W. Sierpiński.

On page 33 of his book W. Sierpiński asked an interesting question:

QUESTION 1.1. *Is it possible to find three different triangular numbers such that the sum of any pair of them is a triangular number? In other words: is it possible to find solutions of the system of equations*

$$t_x + t_y = t_p, \quad t_y + t_z = t_q, \quad t_z + t_x = t_r, \tag{1}$$

in positive integers x, y, z, p, q, r satisfying the condition $x < y < z$?

In the next section we give all integer solutions of the system (1) satisfying the condition $x < y < z < 1000$ and next we construct two one-parameter polynomial solutions of that system (Theorem 2.1).

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In Section 3 we change the perspective a bit and ask for rational parametric solutions of the problem. Using a very simple reasoning we are able to construct a rational parametric solution with three rational parameters (Theorem 3.2).

Finally, in the last section we consider the system of equations

$$(1') \qquad t_x + t_y = t_p, \quad t_y + t_z = t_q, \quad t_z + t_x = t_r, \quad t_x + t_y + t_z = t_s.$$

We give some integer solutions of this system. Next, we use the parametrization obtained in Theorem 3.2 to obtain infinitely many rational solutions depending on two parameters. In order to get this we show that on a certain elliptic curve defined over the field $\mathbb{Q}(u, v)$ there is a point of infinite order.

2. Integer solutions of (1). In order to find integer solutions of the system (1) we have used the computer. We have looked for solutions satisfying the condition $x < y < z < 1000$. We have found 44 solutions in this range (see Table 1).

Table 1

x	y	z	p	q	r
9	13	44	16	46	45
14	51	104	53	116	105
20	50	209	54	215	210
23	30	90	38	95	93
27	124	377	127	397	378
35	65	86	74	108	93
35	123	629	128	641	630
41	119	285	126	309	288
44	245	989	249	1019	990
51	69	104	86	125	116
54	143	244	153	283	250
62	99	322	117	337	328
65	135	209	150	249	219
66	195	365	206	414	371
74	459	923	465	1031	926
76	90	144	118	170	163
77	125	132	147	182	153
77	125	207	147	242	221
83	284	494	296	570	501
105	170	363	200	401	378
105	363	390	378	533	404
105	551	924	561	1076	930

x	y	z	p	q	r
114	429	650	444	779	660
131	174	714	218	735	726
131	245	714	278	755	726
135	154	531	205	553	548
161	260	924	306	960	938
170	469	755	499	889	774
189	305	406	359	508	448
216	390	854	446	939	881
230	741	870	776	1143	900
237	527	650	578	837	692
245	714	989	755	1220	1019
252	272	702	371	753	746
278	370	594	463	700	656
286	405	494	496	639	571
293	390	854	488	939	903
299	441	560	533	713	635
350	629	781	720	1003	856
476	634	665	793	919	818
581	774	935	968	1214	1101
588	645	689	873	944	906
609	779	923	989	1208	1106
714	798	989	1071	1271	1220

The solutions above show that the answer to Sierpiński's question is easy. However, due to the abundance of solutions it is natural to ask whether we can find infinitely many integer solutions of (1).

THEOREM 2.1. *The system* (1) *has infinitely many solutions in integers.*

Proof. Examining Table 1, we find that for any $u \in \mathbb{N}$ the values of the polynomials given by

$$\begin{aligned} x &= (u+1)(2u+5), \\ y &= (u+2)(2u^2+8u+7), \\ z &= (2u^2+7u+4)(2u^2+7u+7)/2, \\ p &= 2u^3+12u^2+24u+15, \\ q &= (4u^4+28u^3+73u^2+87u+40)/2, \\ r &= (4u^4+28u^3+71u^2+77u+30)/2, \end{aligned}$$

or

$$\begin{aligned} x &= (u+3)(2u+3), \\ y &= (u+1)(2u^2+10u+13), \\ z &= (2u^2+9u+8)(2u^2+9u+11)/2, \\ p &= 2u^3+12u^2+24u+16, \\ q &= (4u^4+36u^3+121u^2+177u+92)/2, \\ r &= (u+2)(u+3)(2u+3)(2u+5)/2, \end{aligned}$$

are integers and are solutions of the system (1). ∎

Using the parametric solutions we have obtained above we get

COROLLARY 2.2. *Let* $f(X) = X(X+a)$, *where* $a \in \mathbb{Z} \setminus \{0\}$. *Then there are infinitely many positive integer solutions of the system*

$$(\star) \qquad f(x)+f(y)=f(p), \qquad f(y)+f(z)=f(q), \qquad f(z)+f(x)=f(r).$$

Proof. It is clear that we can assume $a > 0$. Now, note that if x, y, z, p, q, r is a solution of (1), then the sextuple ax, ay, az, ap, aq, ar is a solution of $(\star)$. ∎

The above corollary leads to the following question:

QUESTION 2.3. *Let* $f \in \mathbb{Z}[x]$ *be a polynomial of degree two with two distinct roots in* $\mathbb{C}$. *Is the system of equations*

$$f(x)+f(y)=f(p), \qquad f(y)+f(z)=f(q), \qquad f(z)+f(x)=f(r),$$

solvable in different positive integers?

3. Rational solutions of (1). In view of Theorem 2.1 it is natural to state the following

QUESTION 3.1. *Is the set of one-parameter polynomial solutions of the system* (1) *infinite?*

We suppose that the answer to this question is YES. Unfortunately, we are unable to prove this. So, it is natural to ask if instead of polynomials we can find rational parametric solutions.

THEOREM 3.2. *There is a three-parameter rational solution of the system* (1).

Proof. Let u, v, w be parameters. Note that (1) is equivalent to the system

$$\begin{cases} y = u(p-x), & u(y+1) = p+x+1, \\ z = v(q-y), & v(z+1) = q+y+1, \\ x = w(r-z), & w(x+1) = r+z+1. \end{cases} \tag{2}$$

Because we are interested in rational solutions, we can look at (2) as a system of linear equations with unknowns x, y, z, p, q, r. This system has a solution given by

$$x(u,v,w) = \frac{(u-1)w(1+u-2v+2uv+v^2+uv^2+(-1-u+v^2+uv^2)w)}{(1-u^2)(v^2-1)(w^2-1)+8uvw},$$
$$y(u,v,w) = \frac{u(v-1)(1-u+v-uv+(-2+2v)w+(1+u+v+uv)w^2)}{(1-u^2)(v^2-1)(w^2-1)+8uvw},$$
$$z(u,v,w) = \frac{v(w-1)(1-2u+u^2-v+u^2v+(1+2u+u^2-v+u^2v)w)}{(1-u^2)(v^2-1)(w^2-1)+8uvw},$$

and the quantities p, q, r can by calculated from the identities

$$p(u,v,w) = \frac{ux(u,v,w)+y(u,v,w)}{u},$$
$$q(u,v,w) = \frac{vy(u,v,w)+z(u,v,w)}{v},$$
$$r(u,v,w) = \frac{wz(u,v,w)+x(u,v,w)}{w}. \ \blacksquare$$

REMARK 3.3. It is clear that the same reasoning can be used to prove that the system

$$f(x)+f(y)=f(p), \quad f(y)+f(z)=f(q), \quad f(z)+f(x)=f(r),$$

where $f \in \mathbb{Z}[x]$ is a polynomial of degree two with two different rational roots, has a rational parametric solution depending on three parameters.

4. Rational solutions of (1′). We start this section with a quite natural

QUESTION 4.1. *Is the system of equations*

$$(1') \qquad t_x + t_y = t_p, \qquad t_y + t_z = t_q, \qquad t_z + t_x = t_r, \qquad t_x + t_y + t_z = t_s$$

solvable in integers?

This question is mentioned in the very interesting book [1, p. 292] and it is attributed to K. R. S. Sastry. In this book we can find the triple $x = 11$, $y = z = 14$ (found by Ch. Ashbacher) which satisfies (1′) with $p = r = 18$, $q = 20$, $s = 23$. It is clear that the solutions with x, y, z, p, q, r, s different are more interesting. Using a computer search we have found some 7-tuples of different integers satisfying (1′) (see Table 2).

Table 2

x	y	z	p	q	r	s
230	741	870	776	1143	900	1166
609	779	923	989	1208	1106	1353
714	798	989	1071	1271	1220	1458
1224	1716	3219	2108	3648	3444	3848

It is quite possible that there is a polynomial solution of (1′). However, we have been unable to find one.

Now we use the parametric solutions obtained in Theorem 3.2 to deduce the following

THEOREM 4.2. *The system of diophantine equations* (1′) *has infinitely many rational solutions depending on two parameters.*

Proof. We know that the functions $x, y, z, p, q, r \in \mathbb{Q}(u, v, w)$ we have obtained in the proof of Theorem 3.2 satisfy (1). Thus, in order to find solutions of (1′) it is enough to consider the last equation $t_x + t_y + t_z = t_s$. If we put the calculated quantities x, y, z into the equation $t_x + t_y + t_z = t_s$, then use the identity $8t_s + 1 = (2s + 1)^2$ and clear the denominators, we obtain the equation of a quartic curve C defined over the field $\mathbb{Q}(u, v)$:

$$C\colon h^2 = a_4(u, v)w^4 + a_3(u, v)w^3 + a_2(u, v)w^2 + a_1(u, v)w + a_0(u, v) =: h(w),$$

where

$$a_0(u, v) = a_4(-u, v) = (u - 1)^2(-1 + u + 2v + 2uv - v^2 + uv^2)^2,$$

$$\begin{aligned} a_1(u,v) &= a_3(-u,v) \\ &= 4(u-1)(v^2-1)\left(\frac{u^4-1}{u-1}(v^2+1)+2(u-1)(u^2+4u+1)v\right), \\ a_2(u,v) &= 4(1-10u^2+u^4)v^2+8(u^4-1)v(1+v^2) \\ &\quad +2(3+2u^2+3u^4)(1+v^4). \end{aligned}$$

Because the polynomial $h \in \mathbb{Q}(u,v)[w]$ has no multiple roots, the curve C is smooth. Moreover, we have a $\mathbb{Q}(u,v)$-rational point on C given by

$$Q = (0, (u-1)(-1+u+2v+2uv-v^2+uv^2)).$$

If we treat Q as a point at infinity of C and use the method of [2, p. 77], we conclude that C is birationally equivalent over $\mathbb{Q}(u,v)$ to the elliptic curve with the Weierstrass equation

$$E:\ Y^2 = X^3 - 27f(u,v)X - 27g(u,v),$$

where

$$\begin{aligned} f(u,v) &= u^4(v^8+1)+4u^2(u^4-1)v(v^6+1) \\ &\quad +(1+8u^2-22u^4+8u^6+u^8)v^2(1+v^4) \\ &\quad +4(u^4-1)(u^4-3u^2-1)v^3(1+v^2) \\ &\quad +2(3-16u^2+29u^4-16u^6+3u^8)v^4, \\ g(u,v) &= (u^2(v^4+1)+2(u^4-1)v(v^2+1)+2(2-5u^2+2u^4)v^2) \\ &\quad \times(-2f(u,v)+3(u^2-1)^2v^2(1+u^2-2v+2u^2v+v^2+u^2v^2)^2). \end{aligned}$$

The mapping $\varphi:\ C \ni (w,h) \mapsto (X,Y) \in E$ is given by

$$\begin{aligned} w &= a_4(u,v)^{-1}\left(\left(\frac{16a_4(u,v)^{3/2}Y-27d(u,v)}{24a_4(u,v)X-54c(u,v)}\right)-\frac{a_3(u,v)}{4}\right), \\ h &= a_4(u,v)^{-3/2}\left(-\left(\frac{16a_4(u,v)^{3/2}Y-27d(u,v)}{24a_4(u,v)X-54c(u,v)}\right)^2+\frac{8a_4(u,v)X}{9}+c(u,v)\right). \end{aligned}$$

Note that the quantity $a_4(u,v)^{3/2} = ((u+1)(1+u+\cdots))^3$ is a polynomial in $\mathbb{Z}[u,v]$, whence our mapping is clearly rational. The quantities $c, d \in \mathbb{Z}[u,v]$ are given below:

$$\begin{aligned} c(u,v) = &-4(u+1)^2\big(-u^2(u+1)^2(v^8+1) \\ &+(u^2-1)(1-10u-2u^2-10u^3+u^4)v(v^6+1) \\ &+2(u-1)^2(3-4u-6u^2-4u^3+3u^4)v^2(v^4+1) \\ &+(u-1)^2(15+10u+2u^2+10u^3+15u^4)v^3(v^2+1) \\ &+2(10+20u-5u^2-46u^3-5u^4+20u^5+10u^6)v^4\big)/3, \end{aligned}$$

$$d(u,v)=16(u-1)u(u+1)^4v(v^2-1)$$
$$\times(-1+u+2v+2uv-v^2+uv^2)(1+u^2-2v+2u^2v+v^2+u^2v^2)$$
$$\times(-1+u^2-4v-4u^2v+10v^2-10u^2v^2-4v^3-4u^2v^3-v^4+u^2v^4).$$

To finish the proof we must show that the set of $\mathbb{Q}(u,v)$-rational points on the elliptic curve E is infinite. This will be proved if we find a point of infinite order in the group $E(\mathbb{Q}(u,v))$ of all $\mathbb{Q}(u,v)$-rational points on E. In general this is not an easy task. First of all note that there is a torsion point T of order 2 on E given by

$$T=(3u^2(v^4+1)+6(u^4-1)v(v^2+1)+6(u^2-2)(2u^2-1)v^2,0).$$

It is clear that this point is not suitable for our purposes. Fortunately in our case we can find another point

$$P=\Big(\frac{3}{4}\,((3-2u^2+3u^4)(v^4+1)$$
$$+8(u^4-1)v(v^2+1)+2(5-14u^2+5u^4)v^2),$$
$$\frac{27}{8}\,(u^2-1)^2(v^2-1)((u^2+1)(v^4+1)$$
$$+4(u^2-1)v(v^2+1)+6(u^2+1)v^2\Big).$$

Now, if we specialize the curve E for $u=2, v=3$, we obtain the elliptic curve

$$E_{2,3}:\ Y^2=X^3-28802736X+40355763840$$

with the point $P_{2,3}=(5736,252720)$, which is the specialization of P. As points of finite order on the elliptic curve $y^2=x^3+ax+b$, $a,b\in\mathbb{Z}$, have integer coordinates [8, p. 177], while

$$2P_{2,3}=(765489/100,\ -518102487/1000),$$

we see that $P_{2,3}$ is not a point of finite order on $E_{2,3}$, which means that P cannot be a point of finite order on E. Therefore, E is a curve of positive rank. Hence, its set of $\mathbb{Q}(u,v)$-rational points is infinite and our theorem is proved. ■

Let us note the obvious

COROLLARY 4.3. *Let $f\in\mathbb{Z}[x]$ be a polynomial of degree two with two distinct rational roots. Then the system of equations*

$$f(x)+f(y)=f(p),$$
$$f(z)+f(x)=f(r),$$

$$f(y)+f(z)=f(q),$$
$$f(x)+f(y)+f(z)=f(s),$$

has infinitely many rational parametric solutions depending on two parameters.

EXAMPLE 4.4. Using the method of the above proof we now produce an example of rational functions $x, y, z \in \mathbb{Q}(u)$ that satisfy $(1')$ for some $p, q, r, s \in \mathbb{Q}(u)$ which can be easily found (by computer of course). Because the quantities considered are rather large we put here $v = 2$. Then

$$P_2 + T_2 = (1404u^4 + 219u^2 + 4,\ 8(9u^2+1)^2(81u^2+1)),$$

where P_2, T_2 are the specializations of P, T respectively at $v = 2$. Now,

$$(w,h) = \varphi^{-1}(P_2+T_2) = \left(\frac{(9u-1)^2(9u+1)(63u^2+17)}{3(u+1)(6561u^4+1134u^3+306u-1)},\right.$$
$$\left.\frac{8(9u-1)(9u+1)F(u)}{9(u+1)^2(6561u^4+1134u^3+306u-1)^2}\right)$$

where

$$F(u) = 43046721u^8 + 11573604u^7 + 6388956u^5$$
$$+ 1285956u^6 + 680886u^4 + 919836u^3 + 93636u^2 + 10404u + 1.$$

Using the calculated values and the definition of x, y, z given in the proof of Theorem 3.2 we find that the functions

$$x(u) = \frac{2(u-1)(9u-1)^2(9u+1)^2(63u^2+17)(81u^2+1)}{G(u)},$$
$$y(u) = \frac{3u(11+42u^2+2187u^4)(1+2754u^2+3645u^4)}{G(u)},$$
$$z(u) =$$
$$\frac{2(27u^2-18u-5)(81u^2-48u-1)(135u^2+18u+7)(243u^3-99u^2+57u-1)}{3G(u)},$$

where

$$G(u) = -23914845u^9 + 110008287u^8 - 18528264u^7 + 15956352u^6$$
$$- 473850u^5 - 940410u^4 - 91008u^3 - 96264u^2 - 33u + 35,$$

satisfy $(1')$.

REFERENCES

[1] R. K. Guy, *Unsolved Problems in Number Theory.* 3rd ed., Springer, 2004.
[2] L. J. Mordell, *Diophantine Equations*, Academic Press, London, 1969.

[3] W. Sierpiński, *Sur les triangles rectangulaires dont les deux côtés sont des nombres triangulaires,* Bull. Soc. Math. Phys. Serbie 13 (1961), 145-147.
[4] —, *Triangular numbers*, Biblioteczka Matematyczna 12, PZWS, Warszawa, 1962 (in Polish).
[5] —, *Trois nombres tétraédraux en progression arithmétique*, Elem. Math. 18 (1962), 54–55.
[6] —, *Sur un problème de A. Mąkowski concernant les nombres tétraédraux*, Publ. Inst. Math. 2 (16) (1963), 115–119.
[7] —, *Un théorème sur les nombres triangulaires*, Elem. Math. 23 (1968), 31–32.
[8] J. Silverman, *The Arithmetic of Elliptic Curves*, Springer, New York, 1986.

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