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RECENT DEVELOPMENTS IN THE THEORY OF SEPARATELY HOLOMORPHIC MAPPINGS

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Abstract. We describe a part of the recent developments in the theory of separately holomorphic mappings between complex analytic spaces. Our description focuses on works using the technique of holomorphic discs.

1. Introduction. In this exposition all complex manifolds are assumed to be of finite dimension and countable at infinity, and all complex analytic spaces are assumed to be reduced, irreducible and countable at infinity. For a subset S of a topological space $\mathcal{M}$, $\overline{S}$ denotes the closure of S in $\mathcal{M}$. For two complex analytic spaces (resp. topological spaces) D and Z, $\mathcal{O}(D,Z)$ (resp. $\mathcal{C}(D,Z)$) denotes the set of all holomorphic (resp. continuous) mappings from D to Z.

The main purpose of this work is to describe the recent developments around the following two problems.

PROBLEM 1. *Let X, Y be two complex manifolds, let D (resp. G) be an open subset of X (resp. Y), let A (resp. B) be a subset of $\overline{D}$ (resp. $\overline{G}$) and let Z be a complex analytic space. Define the* cross

$$W := ((D \cup A) \times B) \cup (A \times (G \cup B)).$$

We want to determine the envelope of holomorphy *of the cross W, that is, an "optimal" open subset of $X \times Y$, denoted by $\widehat{\widetilde{W}}$, which is characterized by the following properties:*

For every mapping $f : W \to Z$ that satisfies, in essence, the following condition:

$$f(a,\cdot) \in \mathcal{C}(G \cup B, Z) \cap \mathcal{O}(G,Z), \quad a \in A,$$
$$f(\cdot,b) \in \mathcal{C}(D \cup A, Z) \cap \mathcal{O}(D,Z), \quad b \in B,$$

there exists an $\hat{f} \in \mathcal{O}(\widehat{\widetilde{W}}, Z)$ such that for every $(\zeta,\eta) \in W$, $\hat{f}(z,w)$ tends to $f(\zeta,\eta)$ as $(z,w) \in \widehat{\widetilde{W}}$ tends, in some sense, to (ζ,η).

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The second problem generalizes the first one to the case where we add a set M of singularities to the cross. In order to understand this problem we need to introduce some more notation and terminology. Let X, Y, D, G, A, B and Z and W be as in Problem 1 and let $M \subset W$. Then the set $M_a := \{w \in G : (a, w) \in M\}$, $a \in A$, is called the *vertical fiber* of M over a (resp. the set $M^b := \{z \in D : (z, b) \in M\}$, $b \in B$, is called the *horizontal fiber* of M over b). We say that M has a certain property *in fibers over* A (*resp.* B) if all vertical fibers M_a, $a \in A$, (resp. all horizontal fibers M^b, $b \in B$) have this property.

Problem 2. *Under the above hypotheses and notation, let* $\widehat{\widetilde{W}}$ *be the envelope of holomorphy of* W *given by Problem* 1. *For every subset* $M \subset W$ *which is relatively closed and locally pluripolar* (*resp. thin*) (1) *in fibers over* A *and* B ($M = \emptyset$ *is allowed*) *we want to know if there exists an "optimal" set of singularities* $\widehat{M} \subset \widehat{\widetilde{W}}$, *which is relatively closed locally pluripolar* (*resp. relatively closed analytic*) *and which is characterized by the following property*:

For every mapping $f : W\backslash M \to Z$ *that satisfies*, *in essence*, *the following condition*:

$$f(a, \cdot) \in \mathcal{C}((G \cup B) \setminus M_a, Z) \cap \mathcal{O}(G \setminus M_a, Z), \quad a \in A,$$
$$f(\cdot, b) \in \mathcal{C}((D \cup A) \setminus M^b, Z) \cap \mathcal{O}(D \setminus M^b, Z), \quad b \in B,$$

there exists an $\hat{f} \in \mathcal{O}(\widehat{\widetilde{W}} \setminus \widehat{M}, Z)$ *such that for all* $(\zeta, \eta) \in W \setminus M$, $\hat{f}(z, w)$ *tends to* $f(\zeta, \eta)$ *as* $(z, w) \in \widehat{\widetilde{W}} \setminus \widehat{M}$ *tends*, *in some sense*, *to* (ζ, η).

The motivation for Problem 2 will be explained in Sections 2 and 8 below. These problems play a fundamental role in the theory of separately holomorphic (resp. meromorphic) mappings, and they have been intensively studied during the last decades. There are two recent surveys by Nguyên Thanh Vân (see [34]) and by Peter Pflug (see [47]) which summarize the historical developments up to 2001 on Problems 1 and 2 under the hypotheses that $A \subset D$ *and* $B \subset G$ *and* X, Y *are Stein manifolds and* Z *is a complex analytic space which has the Hartogs extension property* (2).

Both survey articles give interesting insights and suggest new research trends in this subject. Our exposition may be considered as a continuation of the above works. Namely, we describe a part of the recent developments using the technique of holomorphic discs. This will permit us to obtain partial (but reasonable) solutions to Problems 1 and 2 in the

(1) The notion of local pluripolarity (resp. thinness) will be defined in Subsection 3.1 (resp. Section 8) below.

(2) This notion will be defined in Subsection 3.4 below.

case where Z is a complex analytic space with the Hartogs extension property.

We close the introduction with a brief outline of the paper.

In Section 2 we describe briefly the historical developments on Problem 1 and 2.

In Section 3 we provide the framework for an exact formulation of both problems and for their solution.

The technique of holomorphic discs and related results are described in Section 4.

In Section 5 we present some ideas of our new approach to the theory of separate holomorphy. More precisely, we apply the results of Section 4 in order to completely solve Problem 1 in a special case.

Section 6 is devoted to various partial results on Problem 1.

Some approaches to Problems 1 and 2 are presented in Section 7 and 8 respectively. In fact, Sections 6 and 8 are obtained in collaboration with Pflug (see [48, 49, 50, 51, 43, 44]).

Various applications of our solutions are given in Section 9.

Section 10 concludes the article with some remarks and open questions.

2. History. Now we briefly recall the main developments around Problems 1 and 2. All the results obtained so far may be divided into two directions. The first direction investigates the results in the "interior" context: $A \subset D$ and $B \subset G$, while the second one explores the "boundary" context: $A \subset \partial D$ and $B \subset \partial G$.

The first fundamental result in the field of separate holomorphy is the well-known Hartogs extension theorem for separately holomorphic functions (see [15]). In the language of Problem 1 the case: $X = \mathbb{C}^n,\ Y = \mathbb{C}^m,\ A = D,\ B = G,\ Z = \mathbb{C}$ is solved, and the result is $\widehat{\widetilde{W}} = D \times G$. In particular, this theorem may be considered as the first main result in the first direction. In 1912 Bernstein obtained, in his famous article [8], a positive solution to Problem 1 for certain cases where $A \subset D$, $B \subset G$, $X = Y = \mathbb{C}$ and $Z = \mathbb{C}$.

The next important development came about very much later. In 1969–1970 Siciak established some significant generalizations of the Hartogs extension theorem (see [59, 60]). In fact, Siciak's formulation of these generalizations gives rise to Problem 1: to determine the envelope of holomorphy for separately holomorphic functions defined on some *cross sets* W. The theorems obtained under this formulation are often called *cross theorems.* Using the so-called *relative extremal function* (see Section 3 below), Siciak completely solved Problem 1 for the case where $A \subset D$, $B \subset G$, $X = Y = \mathbb{C}$ and $Z = \mathbb{C}$.

The next deep steps were initiated by Zahariuta in 1976 (see [62]) when he started to use the method of common bases of Hilbert spaces. This original approach permitted him to obtain new cross theorems for some cases where $A \subset D$, $B \subset G$ and $D = X$, $G = Y$ are Stein manifolds. As a consequence, he was able to generalize the result of Siciak to higher dimensions.

Later, Nguyên Thanh Vân and Zeriahi (see [36, 37, 38]) developed the method of doubly orthogonal bases of Bergman type in order to generalize the result of Zahariuta. This is a significantly simpler and more constructive version of Zahariuta's original method. Nguyên Thanh Vân and Zeriahi have recently achieved an elegant improvement of their method (see [35], [63]).

Using Siciak's method, Shiffman [56] was the first to generalize some results of Siciak to separately holomorphic mappings with values in a complex analytic space Z. Shiffman's result of [57] shows that the natural "target spaces" for obtaining satisfactory generalizations of cross theorems are the ones which have the *Hartogs extension property* (see Subsection 3.4 below for more explanations).

In 2001 Alehyane and Zeriahi solved Problem 1 for the case where $A \subset D$, $B \subset G$ and X, Y are Stein manifolds and Z is a complex analytic space with the Hartogs extension property. The envelope of holomorphy $\widehat{\widetilde{W}}$ is then given by

$$\widehat{\widetilde{W}} := \{(z, w) \in D \times G : \widetilde{\omega}(z, A, D) + \widetilde{\omega}(w, B, G) < 1\},$$

where $\widetilde{\omega}(\cdot, A, D)$ and $\widetilde{\omega}(\cdot, B, G)$ are the plurisubharmonic measures, which are generalizations of Siciak's relative extremal function (see Section 3 below for this notion). This is the most general result on Problem 1 under the hypothesis $A \subset D$, $B \subset G$. More precisely,

THEOREM 1 (Alehyane–Zeriahi [5]). *Let X, Y be Stein manifolds, let $D \subset X$, $G \subset Y$ be domains, and let $A \subset D$, $B \subset G$ be nonpluripolar subsets. Let Z be a complex analytic space with the Hartogs extension property. Then for every mapping f as in the hypotheses of Problem 1, there is a unique mapping $\hat{f} \in \mathcal{O}(\widehat{\widetilde{W}}, Z)$ such that $\hat{f} = f$ on $W \cap \widehat{\widetilde{W}}$.*

In fact, Theorem 1 is still valid for N-fold crosses W ($N \geq 2$). For the notion of an N-fold cross see, for example, [47] or [39].

Problem 2 originated with a paper by Öktem in 1998 (see [45, 46]) investigating the range problem in mathematical tomography. The reader will find in Section 8 below a concise description of the range problem and its relations to the theory of separate holomorphy. On the other hand, Henkin and Shananin gave, in an earlier work [16], some applications of Bernstein's result [8] to mathematical tomography. Here is the most general result in this direction. In fact, we state it in a somewhat simplified from.

THEOREM 2 (Jarnicki–Pflug [25, 28]). *Let X and Y be Riemann–Stein domains, let $D \subset X$, $G \subset Y$ be two subdomains, and let $A \subset D$ and $B \subset G$ be nonpluripolar subsets. Suppose in addition that $\mathcal{O}(D, \mathbb{C})$ (resp. $\mathcal{O}(G, \mathbb{C})$) separates points in D (resp. in G)* ([3]). *Let $M \subset W$ be a relatively closed subset which is pluripolar (resp. thin) in fibers over A and B.*

Then there exists a relatively closed pluripolar set (resp. relatively closed analytic set) $\widehat{M} \subset \widehat{\widetilde{W}}$ such that:

- $\widehat{M} \cap W \cap \widetilde{W} \subset M$ ([4]);
- *for every function f as in the hypothesis of Problem 2 with $Z = \mathbb{C}$, there exists a unique function $\hat{f} \in \mathcal{O}(\widehat{\widetilde{W}} \setminus \widehat{M}, \mathbb{C})$ such that $\hat{f} = f$ on $(W \cap \widetilde{W}) \setminus M$.*

We refer the reader to [25, 28] for complete versions of this theorem.

The first result in the second direction (i.e. "boundary context") is contained in the work of Malgrange–Zerner [64] in the 1960s. Further results in this direction were obtained by Komatsu [32] and Drużkowski [11], but only for some special cases. Recently, Gonchar [13, 14] has proved a more general result where the following case of Problem 1 has been solved: D and G are Jordan domains in $\mathbb{C}$, A (resp. B) is an open boundary subset of ∂D (resp. ∂G), and $Z = \mathbb{C}$. Namely, we have

THEOREM 3 (Gonchar [13, 14]). *Let $X = Y = \mathbb{C}$, let $D \subset X$, $G \subset Y$ be Jordan domains and A (resp. B) a nonempty open subset of the boundary ∂D (resp. ∂G). Then, for every function $f \in \mathcal{C}(W, \mathbb{C})$ which satisfies the hypotheses of Problem 1 with $Z = \mathbb{C}$, there exists a unique function $\hat{f} \in \mathcal{C}(\widehat{W} \cup W, \mathbb{C}) \cap \mathcal{O}(\widehat{W}, \mathbb{C})$ such that $\hat{f} = f$ on W. Here*

$$\widehat{W} := \{(z, w) \in D \times G : \omega(z, A, D) + \omega(w, B, G) < 1\},$$

where $\omega(\cdot, A, D)$ and $\omega(\cdot, B, G)$ are the harmonic measures (see Subsection 3.1 below for this notion).

Theorem 3 may be rephrased as follows: $\widehat{\widetilde{W}} = \widehat{W}$ (see also [51]). It should be observed that before Gonchar's works, Airapetyan and Henkin published a version of the edge-of-the-wedge theorem for CR manifolds (see [2] for a brief version and [3] for a complete proof). Gonchar's theorem could be deduced from the latter result.

3. New formulations. Our purpose is to develop a theory which unifies all results obtained so far. First we develop some new notions such as a

([3]) We say that $\mathcal{O}(D, \mathbb{C})$ *separates points in D* if for all points z_1, z_2 with $z_1 \neq z_2$, there exists $f \in \mathcal{O}(D, \mathbb{C})$ such that $f(z_1) \neq f(z_2)$.

([4]) The set $\widetilde{W}$ is defined in Subsection 3.3 below.

system of approach regions for an open set in a complex manifold, and the corresponding plurisubharmonic measure. These will provide the framework for an exact formulation of Problems 1 and 2, and for our solution.

3.1. *Approach regions, local pluripolarity and plurisubharmonic measure*

DEFINITION 3.1. Let X be a complex manifold and let $D \subset X$ be an open subset. A *system of approach regions* for D is a collection $\mathcal{A} = (\mathcal{A}_\alpha(\zeta))_{\zeta\in\overline{D},\,\alpha\in I_\zeta}$ ($I_\zeta \neq \emptyset$ for all $\zeta \in \partial D$) (5) of open subsets of D with the following properties:

(i) For all $\zeta \in D$, the system $(\mathcal{A}_\alpha(\zeta))_{\alpha\in I_\zeta}$ forms a basis of open neighborhoods of ζ (i.e., for any open neighborhood U of a point $\zeta \in D$, there is $\alpha \in I_\zeta$ such that $\zeta \in \mathcal{A}_\alpha(\zeta) \subset U$).
(ii) For all $\zeta \in \partial D$ and $\alpha \in I_\zeta$, $\zeta \in \overline{\mathcal{A}_\alpha(\zeta)}$.

$\mathcal{A}_\alpha(\zeta)$ is often called an *approach region* at ζ.

Moreover, $\mathcal{A}$ is said to be *canonical* if it satisfies (i) and the following property (which is stronger than (ii)):

(ii′) For every point $\zeta \in \partial D$, there is a basis $(U_\alpha)_{\alpha\in I_\zeta}$ of open neighborhoods of ζ in X such that $\mathcal{A}_\alpha(\zeta) = U_\alpha \cap D$ for all $\alpha \in I_\zeta$.

Various systems of approach regions which one often encounters in complex analysis will be described in the next subsection. Systems of approach regions for D are used to deal with the limit at points in $\overline{D}$ of mappings defined on some open subsets of D. Consequently, we deduce from Definition 3.1 that the subfamily $(\mathcal{A}_\alpha(\zeta))_{\zeta\in D,\,\alpha\in I_\zeta}$ is, in a certain sense, independent of the choice of a system $\mathcal{A}$ of approach regions. In addition, any two canonical systems of approach regions are, in some sense, equivalent. These observations lead us to use, throughout the paper, the following *convention*:

We fix, *for every open set* $D \subset X$, *a* canonical system of approach regions. *When we want to define a system* $\mathcal{A}$ *of approach regions for an open set* $D \subset X$, *we only need to specify the subfamily* $(\mathcal{A}_\alpha(\zeta))_{\zeta\in\partial D,\,\alpha\in I_\zeta}$.

In what follows we fix an open subset $D \subset X$ and a system of approach regions $\mathcal{A} = (\mathcal{A}_\alpha(\zeta))_{\zeta\in\overline{D},\,\alpha\in I_\zeta}$ for D.

For every function $u : D \to [-\infty, \infty)$, let

$$(\mathcal{A}\text{-}\limsup u)(z) := \sup_{\alpha\in I_z}\ \limsup_{w\in\mathcal{A}_\alpha(z),\, w\to z} u(w), \qquad z \in \overline{D}.$$

By Definition 3.1(i), $(\mathcal{A}\text{-}\limsup u)|_D$ coincides with the usual *upper semicontinuous regularization* of u.

(5) Note that this definition is slightly different from Definition 2.1 in [41].

For a set $A \subset \overline{D}$ put

$$h_{A,D} := \sup\{u : u \in \mathcal{PSH}(D),\ u \le 1 \text{ on } D,\ \mathcal{A}\text{-}\limsup u \le 0 \text{ on } A\},$$

where $\mathcal{PSH}(D)$ denotes the cone of all functions plurisubharmonic on D.

A set $A \subset D$ is said to be *thin* in D if for every point $a \in D$ there is a connected neighborhood $U = U_a \subset D$ and a holomorphic function f on U, not identically zero, such that $U \cap A \subset f^{-1}(0)$. The set A is said to be *pluripolar* in D if there is $u \in \mathcal{PSH}(D)$ such that u is not identically $-\infty$ on every connected component of D and $A \subset \{z \in D : u(z) = -\infty\}$. The set A is said to be *locally pluripolar* in D if for any $z \in A$, there is an open neighborhood $V \subset D$ of z such that $A \cap V$ is pluripolar in V. Finally, A is said to be *nonpluripolar* (resp. *non-locally pluripolar*) if it is not pluripolar (resp. not locally pluripolar). According to a classical result of Josefson and Bedford (see [30], [6]), if D is a Riemann–Stein domain then $A \subset D$ is locally pluripolar if and only if it is pluripolar.

Definition 3.2. For $A \subset \overline{D}$, the *relative extremal function of A relative to D* is the function $\omega(\cdot, A, D)$ defined by

$$\omega(z, A, D) = \omega_{\mathcal{A}}(z, A, D) := (\mathcal{A}\text{-}\limsup h_{A,D})(z), \qquad z \in \overline{D}\ (^6).$$

Note that when $A \subset D$, Definition 3.2 coincides with the classical definition of Siciak's relative extremal function. When D is a complex manifold of dimension 1 and $\mathcal{A}$ is the canonical system, the function $\omega(\cdot, A, D)$ is often called the *harmonic measure of A relative to D* (see Theorem 3 above).

Next, we say that a set $A \subset \overline{D}$ is *locally pluriregular at a point $a \in \overline{A}$* if $\omega(a, A \cap U, D \cap U) = 0$ for all open neighborhoods U of a. Moreover, A is said to be *locally pluriregular* if it is locally pluriregular at all points $a \in A$. It should be noted from Definition 3.1 that if $a \in \overline{A} \cap D$ then the property of local pluriregularity of A at a does not depend on any particular choices of a system $\mathcal{A}$ of approach regions, while the situation is different when $a \in \overline{A} \cap \partial D$: the property does depend on $\mathcal{A}$.

We denote by A^* the set

$$(A \cap \partial D) \cup \{a \in \overline{A} \cap D : A \text{ is locally pluriregular at } a\}.$$

If $A \subset D$ is not locally pluripolar, then a classical result of Bedford and Taylor (see [6, 7]) says that A^* is locally pluriregular and $A \setminus A^*$ is locally pluripolar. Moreover, A^* is locally of type $\mathcal{G}_\delta$, that is, for every $a \in A^*$ there is an open neighborhood $U \subset D$ of a such that $A^* \cap U$ is a countable intersection of open sets.

Now we are in a position to formulate the following version of the plurisubharmonic measure.

(^{6}) Observe that this function depends on the system of approach regions.

DEFINITION 3.3. For a set $A \subset \overline{D}$, let $\widetilde{A} = \widetilde{A}(\mathcal{A}) := \bigcup_{P \in \mathcal{E}(A)} P$, where

$$\mathcal{E}(A) = \mathcal{E}(A, \mathcal{A}) := \{P \subset \overline{D} : P \text{ is locally pluriregular, } \overline{P} \subset A^*\},$$

The *plurisubharmonic measure of A relative to D* is the function $\widetilde{\omega}(\cdot, A, D)$ defined by

$$\widetilde{\omega}(z, A, D) := \omega(z, \widetilde{A}, D), \qquad z \in D.$$

It is worth remarking that $\widetilde{\omega}(\cdot, A, D) \in \mathcal{PSH}(D)$ and $0 \le \widetilde{\omega}(z, A, D) \le 1$ for $z \in D$. Moreover,

$$(\mathcal{A}\text{-}\limsup \widetilde{\omega}(\cdot, A, D))(z) = 0, \qquad z \in \widetilde{A}. \tag{3.1}$$

An example in [4] shows that, in general, $\omega(\cdot, A, D) \ne \widetilde{\omega}(\cdot, A, D)$ on D. Sections 6 and 9 below are devoted to the study of $\widetilde{\omega}(\cdot, A, D)$ in some important cases. As we will see later, in most applications one can obtain good and simple characterizations of $\widetilde{\omega}(\cdot, A, D)$ (see Theorems 5, 6, 7, 9 and Corollaries 2, 3 below).

Now we compare the plurisubharmonic measure $\widetilde{\omega}(\cdot, A, D)$ with Siciak's relative extremal function $\omega(\cdot, A, D)$. We only consider two important special cases: $A \subset D$ and $A \subset \partial D$. For the moment, we only focus on the former; the latter will be discussed in Sections 6 and 9.

If A is an open subset of an arbitrary complex manifold D, then it can be shown that

$$\widetilde{\omega}(z, A, D) = \omega(z, A, D), \qquad z \in D.$$

If A is a (not necessarily open) subset of an arbitrary complex manifold D, then we have, by Proposition 7.1 in [41],

$$\widetilde{\omega}(z, A, D) = \omega(z, A^*, D), \qquad z \in D.$$

On the other hand, if, morever, D is a bounded open subset of $\mathbb{C}^n$ then (see, for example, Lemma 3.5.3 in [23]) $\omega(z, A, D) = \omega(z, A^*, D)$ for $z \in D$. Consequently, under the last assumption,

$$\widetilde{\omega}(z, A, D) = \omega(z, A, D), \qquad z \in D.$$

Our discussion shows that at least in the case where $A \subset D$, the notion of the plurisubharmonic measure is a good candidate for generalizing Siciak's relative extremal function to the manifold context in the theory of separate holomorphy.

For a good background of pluripotential theory, see the books [23] or [31].

3.2. *Examples of systems of approach regions.* There are many systems of approach regions which are useful in complex analysis. In this subsection we present some of them.

1. *Canonical system of approach regions.* It has been given by Definition 3.1(i)–(ii′). It is the most natural one.

2. *System of angular* (*or Stolz*) *approach regions for the open unit disc.* Let E be the open unit disc of $\mathbb{C}$. Put

$$\mathcal{A}_\alpha(\zeta) := \left\{ t \in E : \left| \arg\left(\frac{\zeta - t}{\zeta}\right) \right| < \alpha \right\}, \qquad \zeta \in \partial E,\, 0 < \alpha < \pi/2,$$

where $\arg : \mathbb{C} \to (-\pi, \pi]$ is as usual the argument function. The system $\mathcal{A} = (\mathcal{A}_\alpha(\zeta))_{\zeta\in\partial E,\, 0<\alpha<\pi/2}$ is referred to as the *system of angular* (*or Stolz*) *approach regions for E*. In this context $\mathcal{A}$-lim is also called *angular limit*.

3. *System of angular approach regions for certain "good" open subsets of Riemann surfaces.* Now we generalize the previous construction (for the open unit disc) to a global situation. More precisely, we will use as the local model the system of angular approach regions for E. Let X be a complex manifold of dimension 1 (in other words, X is a Riemann surface), and let $D \subset X$ be an open set. Then D is said to be *good at a point $\zeta \in \partial D$* ([7]) if there is a Jordan domain $U \subset X$ such that $\zeta \in U$ and $U \cap \partial D$ is the interior of a Jordan curve.

Suppose that D is good at ζ. This point is said to be *of type 1* if there is a neighborhood V of ζ such that $V_0 = V \cap D$ is a Jordan domain. Otherwise, ζ is said to be *of type 2*. We see easily that if ζ is of type 2, then there are an open neighborhood V of ζ and two disjoint Jordan domains V_1, V_2 such that $V \cap D = V_1 \cup V_2$. Moreover, D is said to be *good on a subset A of ∂D* if D is good at all points of A.

Here is a simple example which may clarify the above definitions. Let G be the open square in $\mathbb{C}$ with vertices $1+i$, $-1+i$, $-1-i$, and $1-i$. Define $D := G \setminus [-1/2, 1/2]$. Then D is good on $\partial G \cup (-1/2, 1/2)$. All points of ∂G are of type 1 and all points of $(-1/2, 1/2)$ are of type 2.

Suppose now that D is good on a nonempty subset A of ∂D. We define the *system of angular approach regions supported on A*, $\mathcal{A} = (\mathcal{A}_\alpha(\zeta))_{\zeta\in\overline{D},\, \alpha\in I_\zeta}$, as follows:

- If $\zeta \in \overline{D} \setminus A$, then $(\mathcal{A}_\alpha(\zeta))_{\alpha\in I_\zeta}$ coincide with the canonical approach regions.
- If $\zeta \in A$, then by using a conformal mapping Φ from V_0 (resp. V_1 and V_2) onto E when ζ is of type 1 (resp. 2), we can "transfer" the angular approach regions at the point $\Phi(\zeta) \in \partial E$, $(\mathcal{A}_\alpha(\Phi(\zeta)))_{0<\alpha<\pi/2}$, to those at the point $\zeta \in \partial D$ (see [49] for more detailed explanations).

Making use of conformal mappings in a local way, we can transfer, in the same way, many notions which exist on E (resp. ∂E) to those on D (resp. ∂D).

([7]) In [49] we use the more appealing word *Jordan-curve-like* for this notion.

4. *System of conical approach regions.* Let $D \subset \mathbb{C}^n$ be a domain and $A \subset \partial D$. Suppose in addition that for every point $\zeta \in A$ the (real) tangent space T_ζ to ∂D at ζ exists. We define the *system of conical approach regions supported on* A, $\mathcal{A} = (\mathcal{A}_\alpha(\zeta))_{\zeta \in \overline{D},\, \alpha \in I_\zeta}$ as follows:

- If $\zeta \in \overline{D} \setminus A$, then $(\mathcal{A}_\alpha(\zeta))_{\alpha \in I_\zeta}$ coincide with the canonical approach regions.
- If $\zeta \in A$, then
$$\mathcal{A}_\alpha(\zeta) := \{z \in D : |z - \zeta| < \alpha \cdot \operatorname{dist}(z, T_\zeta)\},$$
where $I_\zeta := (1, \infty)$ and $\operatorname{dist}(z, T_\zeta)$ denotes the Euclidean distance from the point z to T_ζ.

We can also generalize the previous construction to a global situation:

X is an arbitrary complex manifold, $D \subset X$ is an open set and $A \subset \partial D$ is a subset with the property that at every point $\zeta \in A$ the (real) tangent space T_ζ to ∂D exists.

We can also formulate the notion of points of type 1 or 2 in this general context in the same way as in item 3 above.

3.3. *Cross and separate holomorphy and $\mathcal{A}$-limit.* Let X, Y be two complex manifolds, let $D \subset X$, $G \subset Y$ be two nonempty open sets, and let $A \subset \overline{D}$ and $B \subset \overline{G}$. Moreover, suppose D (resp. G) is equipped with a system of approach regions $\mathcal{A}(D) = (\mathcal{A}_\alpha(\zeta))_{\zeta \in \overline{D},\, \alpha \in I_\zeta}$ (resp. $\mathcal{A}(G) = (\mathcal{A}_\alpha(\eta))_{\eta \in \overline{G},\, \alpha \in I_\eta}$) ([8]). We define a *2-fold cross* W, its *interior* W^{o} and its *regular part* $\widetilde{W}$ (with respect to $\mathcal{A}(D)$ and $\mathcal{A}(G)$) as

$$\begin{aligned}
W &= \mathbb{X}(A, B; D, G) := ((D \cup A) \times B) \cup (A \times (B \cup G)),\\
W^{\mathrm{o}} &= \mathbb{X}^{\mathrm{o}}(A, B; D, G) := (A \times G) \cup (D \times B),\\
\widetilde{W} &= \widetilde{\mathbb{X}}(A, B; D, G) := ((D \cup \widetilde{A}) \times \widetilde{B}) \cup (\widetilde{A} \times (G \cup \widetilde{B})),
\end{aligned}$$

where $\widetilde{A}$ and $\widetilde{B}$ are as in Definition 3.3. Moreover, put

$$\begin{aligned}
\omega(z, w) &:= \omega(z, A, D) + \omega(w, B, G), \qquad (z, w) \in D \times G,\\
\widetilde{\omega}(z, w) &:= \widetilde{\omega}(z, A, D) + \widetilde{\omega}(w, B, G), \qquad (z, w) \in D \times G.
\end{aligned}$$

For a 2-fold cross $W := \mathbb{X}(A, B; D, G)$ let

$$\begin{aligned}
\widehat{W} &:= \widehat{\mathbb{X}}(A, B; D, G) = \{(z, w) \in D \times G : \omega(z, w) < 1\},\\
\widehat{\widetilde{W}} &:= \widehat{\mathbb{X}}(\widetilde{A}, \widetilde{B}; D, G) = \{(z, w) \in D \times G : \widetilde{\omega}(z, w) < 1\}.
\end{aligned}$$

Let Z be a complex analytic space and $M \subset W$ a subset which is relatively closed in fibers over A and B. We say that a mapping $f : W^{\mathrm{o}} \setminus M \to Z$

([8]) In fact, we should have written $I_\zeta(D)$, resp. $I_\eta(G)$; but we skip D and G here to make the notions as simple as possible.

is *separately holomorphic* and write $f \in \mathcal{O}_{\mathrm{s}}(W^{\mathrm{o}} \setminus M, Z)$ if, for any $a \in A$ (resp. $b \in B$) the mapping $f(a,\cdot)|_{G\setminus M_a}$ (resp. $f(\cdot,b)|_{D\setminus M^b}$) is holomorphic.

We say that a mapping $f : W\setminus M \to Z$ is *separately continuous* and write $f \in \mathcal{C}_{\mathrm{s}}(W\setminus M, Z)$ if, for any $a \in A$ (resp. $b \in B$) the mapping $f(a,\cdot)|_{(G\cup B)\setminus M_a}$ (resp. $f(\cdot,b)|_{(D\cup A)\setminus M^b}$) is continuous.

Let Ω be an open subset of $D\times G$. A point $(\zeta,\eta) \in \overline{D}\times\overline{G}$ is said to be an *end-point* of Ω with respect to $\mathcal{A} = \mathcal{A}(D)\times\mathcal{A}(G)$ if for any $(\alpha,\beta) \in I_\zeta \times I_\eta$ there exist open neighborhoods U of ζ in X and V of η in Y such that

$$(U \cap \mathcal{A}_\alpha(\zeta)) \times (V \cap \mathcal{A}_\beta(\eta)) \subset \Omega.$$

The set of all end-points of Ω is denoted by $\mathrm{End}(\Omega)$.

It follows from (3.1) that if $\widetilde{A}, \widetilde{B} \neq \emptyset$, then $\widetilde{W} \subset \mathrm{End}(\widehat{\widetilde{W}})$.

Let S be a relatively closed subset of $\widehat{\widetilde{W}}$ and let $(\zeta,\eta) \in \mathrm{End}(\widehat{\widetilde{W}} \setminus S)$. Then a mapping $f : \widehat{\widetilde{W}} \setminus S \to Z$ is said to *admit the $\mathcal{A}$-limit λ at* (ζ,η), and one writes

$$(\mathcal{A}\text{-}\lim f)(\zeta,\eta) = \lambda \ (^9),$$

if, for all $\alpha \in I_\zeta$ and $\beta \in I_\eta$,

$$\lim_{\widehat{\widetilde{W}}\setminus S \ni (z,w)\to(\zeta,\eta),\, z\in\mathcal{A}_\alpha(\zeta),\, w\in\mathcal{A}_\beta(\eta)} f(z,w) = \lambda.$$

We conclude this introduction with a notion we need later. Let $\mathcal{M}$ be a topological space. A mapping $f : \mathcal{M} \to Z$ is said to be *bounded* if there exists an open neighborhood U of $f(\mathcal{M})$ in Z and a holomorphic embedding ϕ of U into the unit polydisc of $\mathbb{C}^k$ such that $\phi(U)$ is an analytic set in this polydisc. f is said to be *locally bounded along* $\mathcal{N} \subset \mathcal{M}$ if for every point $z \in \mathcal{N}$, there is an open neighborhood U of z (in $\mathcal{M}$) such that $f|_U : U \to Z$ is bounded. Finally, f is said to be *locally bounded* if it is so for $\mathcal{N} = \mathcal{M}$. It is clear that if $Z = \mathbb{C}$, then the above notions of boundedness coincide with the usual ones.

3.4. *Hartogs extension property.* The following example (see Shiffman [57]) shows that an additional hypothesis on the "target space" Z is necessary in order that Problems 1 and 2 make sense. Consider the mapping $f : \mathbb{C}^2 \to \mathbb{P}^1$ given by

$$f(z,w) := \begin{cases} [(z+w)^2 : (z-w)^2], & (z,w) \neq (0,0), \\ [1:1], & (z,w) = (0,0). \end{cases}$$

Then $f \in \mathcal{O}_{\mathrm{s}}(\mathbb{X}^{\mathrm{o}}(\mathbb{C},\mathbb{C};\mathbb{C},\mathbb{C}),\mathbb{P}^1)$, but f is not continuous at $(0,0)$.

We recall here the following notion (see, for example, Shiffman [56]). Let $p \geq 2$ be an integer. For $0 < r < 1$, the *Hartogs figure* in dimension p,

(^{9}) Note that here $\mathcal{A} = \mathcal{A}(D) \times \mathcal{A}(G)$.

denoted by $H_p(r)$, is given by

$$H_p(r) := \{(z', z_p) \in E^p : \|z'\| < r \text{ or } |z_p| > 1 - r\},$$

where E is the open unit disc of $\mathbb{C}$ and $z' = (z_1, \dots, z_{p-1})$, $\|z'\| := \max_{1\leq j\leq p-1} |z_j|$.

DEFINITION 3.4. A complex analytic space Z is said to have the *Hartogs extension property in dimension* p if every mapping $f \in \mathcal{O}(H_p(r), Z)$ extends to a mapping $\hat{f} \in \mathcal{O}(E^p, Z)$. Moreover, Z is said to have the *Hartogs extension property* if it has this property in all dimensions $p \geq 2$.

It is a classical result of Ivashkovich (see [19]) that if Z has the Hartogs extension property in dimension 2, then it has this property in all dimensions $p \geq 2$. Some typical examples of complex analytic spaces with the Hartogs extension property are the complex Lie groups (see [1]), the taut spaces (see [61]), the Hermitian manifolds with negative holomorphic sectional curvature (see [56]), and the holomorphically convex Kähler manifolds without rational curves (see [19]).

Here we mention an important characterization.

THEOREM 4 (Shiffman [56]). *A complex analytic space Z has the Hartogs extension property if and only if for every domain D of any Stein manifold $\mathcal{M}$, every mapping $f \in \mathcal{O}(D, Z)$ extends to a mapping $\hat{f} \in \mathcal{O}(\widehat{D}, Z)$, where $\widehat{D}$ is the envelope of holomorphy* ([10]) *of D.*

In the light of Definition 3.4 and Shiffman's theorem, the natural "target spaces" Z for obtaining satisfactory answers to Problem 1 are the complex analytic spaces with the Hartogs extension property.

4. A new approach: Poletsky's theory of discs and Rosay theorem. Poletsky's theory of discs was invented by Poletsky (see [52, 53]) in the late 1980s. A new approach to the theory of separate holomorphy based on Poletsky's theory of discs was developed in our work [39]. Let us recall some elements of this theory.

Let E denote as usual the open unit disc in $\mathbb{C}$. For a complex manifold $\mathcal{M}$, let $\mathcal{O}(\overline{E}, \mathcal{M})$ denote the set of all holomorphic mappings $\phi : E \to \mathcal{M}$ which extend holomorphically to a neighborhood of $\overline{E}$. Such a mapping ϕ is called a *holomorphic disc* on $\mathcal{M}$. Moreover, for a subset A of $\mathcal{M}$, let

$$1_{A,\mathcal{M}}(z) := \begin{cases} 1, & z \in A, \\ 0, & z \in \mathcal{M} \setminus A. \end{cases}$$

In 2003 Rosay proved the following remarkable result.

([10]) For the notion of the envelope of holomorphy, see, for example, [23].

Rosay Theorem ([54]). *Let u be an upper semicontinuous function on a complex manifold $\mathcal{M}$. Then the Poisson functional of u defined by*

$$\mathcal{P}[u](z) := \inf\left\{\frac{1}{2\pi}\int_0^{2\pi} u(\phi(e^{i\theta}))\,d\theta : \phi \in \mathcal{O}(\overline{E},\mathcal{M}),\ \phi(0)=z\right\}$$

is plurisubharmonic on $\mathcal{M}$.

The Rosay theorem may be viewed as an important development in Poletsky's theory of discs. Observe that special cases of this theorem have been considered by Poletsky (see [52, 53]), Lárusson–Sigurdsson (see [33]) and Edigarian (see [12]).

The next result describes the situation in dimension 1.

Lemma 1 ([39, Lemma 3.3]). *Let T be an open subset of $\overline{E}$. Then*

$$\omega(0, T\cap E, E) \le \frac{1}{2\pi}\int_0^{2\pi} 1_{\partial E\setminus T, T}(e^{i\theta})\,d\theta.$$

The last result, which is an important consequence of the Rosay theorem, gives the connection between the Poisson functional and the plurisubharmonic measure.

Lemma 2 ([39, Proposition 3.4]). *Let $\mathcal{M}$ be a complex manifold and A a nonempty open subset of $\mathcal{M}$. Then $\omega(z, A, \mathcal{M}) = \mathcal{P}[1_{\mathcal{M}\setminus A,\mathcal{M}}](z)$ for $z \in \mathcal{M}$.*

5. Problem 1 for $A \subset D$, $B \subset G$. We will give the first application of the previous section. Observe that under the hypothesis $A \subset D$, $B \subset G$ and the notation of Subsection 3.3, we have $W = W^{\circ}$ and $W \cap \widetilde{W} \subset W \cap \widehat{\widetilde{W}}$. Since $\widetilde{W} \subset D \times G$, the notion of $\mathcal{A}$-lim at a point of $\widetilde{W}$ coincides with the ordinary notion of a limit, that is, $\mathcal{A}$ can be taken as the canonical system. Moreover, it can be shown that $W \setminus \widetilde{W}$ is a locally pluripolar subset of $D \times G$. Therefore, from the viewpoint of pluripotential theory, $W \cap \widetilde{W}$ is "almost" equal to W. Now we are able to state the following generalization of Theorem 1.

Theorem 5 ([39, Theorem A]). *Let X, Y be arbitrary complex manifolds, let $D \subset X$ and $G \subset Y$ be open sets and $A \subset D$, $B \subset G$ non-locally pluripolar subsets. Let Z be a complex analytic space with the Hartogs extension property. Then for every mapping $f \in \mathcal{O}_{\mathrm{s}}(W^{\circ}, Z)$, there is a unique mapping $\hat{f} \in \mathcal{O}(\widehat{\widetilde{W}}, Z)$ such that $\hat{f} = f$ on $W \cap \widehat{\widetilde{W}}$.*

A remark is in order. Theorem 5 removes all the assumptions of pseudoconvexity of the "source spaces" X, Y stated in Theorem 1. Namely, now X and Y can be arbitrary complex manifolds. The sketchy proof given below explains our approach: how Poletsky's theory of discs and the Rosay theorem may apply to the theory of separate holomorphy. The proof is divided into four steps. In Steps 3 and 4 below we use some ideas of our previous joint work with Pflug [48].

STEP 1: *The case where D is an arbitrary complex manifold, A is an open subset of D, and G is a bounded open subset of $\mathbb{C}^n$.*

Sketchy proof of Step 1. We define $\hat{f}$ as follows: Let $\mathcal{W}$ be the set of all pairs $(z,w)\in D\times G$ with the property that there are a holomorphic disc $\phi\in\mathcal{O}(\overline{E},D)$ and $t\in E$ such that $\phi(t)=z$ and $(t,w)\in\widehat{\widetilde{\mathbb{X}}}(\phi^{-1}(A)\cap E,B;E,G)$. In virtue of Theorem 1 and the observation made at the beginning of the section, let $\hat{f}_\phi$ be the unique mapping in $\mathcal{O}(\widehat{\widetilde{\mathbb{X}}}(\phi^{-1}(A)\cap E,B;E,G),Z)$ such that

$$\begin{aligned}&\hat{f}_\phi(t,w)=f(\phi(t),w),\\ &(t,w)\in\mathbb{X}(\phi^{-1}(A)\cap E,B;E,G)\cap\widetilde{\mathbb{X}}(\phi^{-1}(A)\cap E,B;E,G).\end{aligned}\tag{5.1}$$

Then we may define the desired extension mapping $\hat{f}$ as follows:

$$\hat{f}(z,w):=\hat{f}_\phi(t,w).\tag{5.2}$$

Using the uniqueness of Theorem 1, we can prove that $\hat{f}$ is well-defined on $\mathcal{W}$. Using Lemmas 1 and 2, one can show that

$$\mathcal{W}=\widehat{\widetilde{W}}.$$

Moreover, it follows from the above construction that for every fixed $z\in D$, the restricted mapping $\hat{f}(z,\cdot)$ is holomorphic on the open set $\{w\in G:(z,w)\in\widehat{\widetilde{W}}\}$. However, it is quite difficult to see that $\hat{f}$ is holomorphic in both variables (z,w). A complete proof of this fact is given in Theorem 4.1 of [39]. Now we only explain briefly why $\hat{f}$ is holomorphic in a neighborhood of an arbitrary point $(z_0,w_0)\in\widehat{\widetilde{W}}$. For this purpose we "add" one complex dimension more to a suitable neighborhood of (z_0,w_0), and this makes our initial 2-fold cross W a 3-fold one. Finally, we apply the version of Theorem 1 for 3-fold cross to finish the proof. ∎

STEP 2: *The case where D, G are arbitrary complex manifolds, but $A\subset D$, $B\subset G$ are open subsets.*

Sketchy proof of Step 2. It follows from the discussion at the end of Subsection 3.1 that under the hypothesis of Step 2, $\widehat{\widetilde{W}}=\widehat{W}$ and $W=W\cap\widetilde{W}\subset\widehat{W}$.

We will determine the value of $\hat{f}$ at an arbitrary fixed point $(z_0, w_0) \in \widehat{W}$. To this end fix any $\epsilon > 0$ such that

$$2\epsilon < 1 - \omega(z_0, A, D) - \omega(w_0, B, G). \tag{5.3}$$

By the Rosay theorem and Lemma 2, there is a holomorphic disc $\phi \in \mathcal{O}(\overline{E}, D)$ (resp. $\psi \in \mathcal{O}(\overline{E}, G)$) such that $\phi(0) = z_0$ (resp. $\psi(0) = w_0$) and

$$\frac{1}{2\pi}\int_0^{2\pi} 1_{D\setminus A}(\phi(e^{i\theta}))\, d\theta < \omega(z_0, A, D) + \epsilon,$$
$$\frac{1}{2\pi}\int_0^{2\pi} 1_{G\setminus B}(\psi(e^{i\theta}))\, d\theta < \omega(w_0, B, G) + \epsilon.$$

Using this with estimate (5.3) and Lemma 1, we see that

$$(0,0) \in \widehat{\mathbb{X}}(\phi^{-1}(A) \cap E, \psi^{-1}(B) \cap E; E, E).$$

Moreover, since $f \in \mathcal{O}_{\mathrm{s}}(W^{\mathrm{o}}, Z)$, the mapping h given by

$$h(t,\tau) := f(\phi(t), \psi(\tau)), \quad (t,\tau) \in \mathbb{X}(\phi^{-1}(A) \cap E, \psi^{-1}(B) \cap E; E, E),$$

belongs to $\mathcal{O}_{\mathrm{s}}(\mathbb{X}(\phi^{-1}(A) \cap E, \psi^{-1}(B) \cap E; E, E), Z)$. By Theorem 1, let $\hat{h} \in \mathcal{O}(\widehat{\mathbb{X}}(\phi^{-1}(A) \cap E, \psi^{-1}(B) \cap E; E, E), Z)$ be the unique mapping such that

$$\hat{h}(t,\tau) = h(t,\tau) = f(\phi(t), \psi(\tau)),$$
$$(t,\tau) \in \mathbb{X}(\phi^{-1}(A) \cap E, \psi^{-1}(B) \cap E; E, E).$$

Then we can define

$$\hat{f}(z_0, w_0) = \hat{h}(0,0), \quad (z_0, w_0) \in \widehat{W}.$$

We leave to the interested reader the verification that $\hat{f}$ is well-defined on $\widehat{W}$. Now we explain why $\hat{f} \in \mathcal{O}(\widehat{W}, Z)$.

If we fix ϕ and let ψ be free (or conversely, fix ψ and let ϕ be free) in the above construction, then this procedure is very similar to the one carried out in (5.1)–(5.2). Consequently, we may apply the result of Step 1 twice to conclude that for all $(z_0, w_0) \in \widehat{W}$, $\hat{f}(z_0, \cdot)$ is holomorphic in $\{w \in G : (z_0, w) \in \widehat{W}\}$ (resp. $\hat{f}(\cdot, w_0)$ is holomorphic in $\{z \in D : (z, w_0) \in \widehat{W}\}$). The classical Hartogs extension theorem yields $\hat{f} \in \mathcal{O}(\widehat{W}, Z)$. ∎

To continue the proof we need to introduce some more notation.

Suppose without loss of generality that D and G are domains and let m (resp. n) be the dimension of D (resp. of G). For every $a \in A^*$ (resp. $b \in B^*$), fix an open neighborhood U_a of a (resp. V_b of b) such that U_a (resp. V_b) is biholomorphic to a bounded domain in $\mathbb{C}^m$ (resp. in $\mathbb{C}^n$). For any

$0 < \delta \le 1/2$, define

$$
(5.4)\qquad
\begin{aligned}
U_{a,\delta} &:= \{z \in U_a : \widetilde{\omega}(z, A \cap U_a, U_a) < \delta\}, \qquad a \in A \cap A^*,\\
V_{b,\delta} &:= \{w \in V_b : \widetilde{\omega}(w, B \cap V_b, V_b) < \delta\}, \qquad b \in B \cap B^*,\\
A_\delta &:= \bigcup_{a \in A \cap A^*} U_{a,\delta},\\
B_\delta &:= \bigcup_{b \in B \cap B^*} V_{b,\delta},\\
D_\delta &:= \{z \in D : \widetilde{\omega}(z, A, D) < 1 - \delta\},\\
G_\delta &:= \{w \in G : \widetilde{\omega}(w, B, G) < 1 - \delta\}.
\end{aligned}
$$

Observe that $U_{a,\delta}$ (resp. $V_{b,\delta}$) is an open neighborhood of a (resp. b). Moreover, one has the following inclusion (which will be implicitly used below):

$$\mathbb{X}(A \cap A^*, B \cap B^*; D, G) \subset W \cap \widehat{\widetilde{W}}.$$

Step 3: *The case where G is a bounded open subset in $\mathbb{C}^n$.*

Sketchy proof of Step 3. We only describe the construction of $\hat{f}$. For each $a \in A \cap A^*$, let $f_a := f|_{\mathbb{X}(A \cap U_a, B; U_a, G)}$. Since $f \in \mathcal{O}_s(W^o, Z)$, we deduce that $f_a \in \mathcal{O}_s(\mathbb{X}(A \cap U_a, B; U_a, G), Z)$. Recall that U_a (resp. G) is biholomorphic to a bounded open set in $\mathbb{C}^m$ (resp. in $\mathbb{C}^n$). Therefore, applying Theorem 1 to f_a shows that there is a unique mapping $\hat{f}_a \in \mathcal{O}(\widehat{\widetilde{\mathbb{X}}}(A \cap U_a, B; U_a, G), Z)$ such that

$$
(5.5)\qquad \hat{f}_a(z, w) = f_a(z, w) = f(z, w), \qquad (z, w) \in \mathbb{X}(A \cap A^* \cap U_a, B \cap B^*; U_a, G).
$$

Let $0 < \delta \le 1/2$. In virtue of (5.4)–(5.5), we are able to "glue" the family $(\hat{f}_a|_{U_{a,\delta} \times G_\delta})_{a \in A \cap A^*}$. Let

$$
(5.6)\qquad \tilde{\tilde{f}}_\delta \in \mathcal{O}(A_\delta \times G_\delta, Z)
$$

denote the resulting mapping after the gluing process. By (5.5)–(5.6), we can define a new mapping $\tilde{f}_\delta$ on $\mathbb{X}(A_\delta, B \cap B^*; D, G_\delta)$ as follows:

$$
\tilde{f}_\delta := \begin{cases} \tilde{\tilde{f}}_\delta & \text{on } A_\delta \times G_\delta,\\ f & \text{on } D \times (B \cap B^*). \end{cases}
$$

Using (5.5)–(5.6) again, we see that $\tilde{f}_\delta \in \mathcal{O}_s(\mathbb{X}(A_\delta, B \cap B^*; D, G_\delta), Z)$, and

$$\tilde{f}_\delta = f \quad \text{on } \mathbb{X}(A \cap A^*, B \cap B^*; D, G_\delta).$$

Since A_δ is an open subset of the complex manifold D, and G_δ is biholomorphic to a bounded open set in $\mathbb{C}^n$, we can apply Step 1 to $\tilde{f}_\delta$ in order to obtain a mapping $\hat{f}_\delta \in \mathcal{O}(\widehat{\widehat{\mathbb{X}}}(A_\delta, B\cap B^*; D, G_\delta), Z)$ such that

$$\hat{f}_\delta = \tilde{f}_\delta \quad \text{on } \mathbb{X}(A_\delta, B\cap B^*; D, G_\delta).$$

We are now in a position to define the desired extension mapping $\hat{f}$. Indeed, one glues $(\hat{f}_\delta)_{0<\delta\le 1/2}$ together to obtain $\hat{f}$ in the following way:

$$\hat{f} := \lim_{\delta\to 0} \hat{f}_\delta \quad \text{on } \widehat{\widetilde{W}}.$$

In fact, the equality $\widehat{\widetilde{W}} = \bigcup_{0<\delta<1/2} \widehat{\widehat{\mathbb{X}}}(A_\delta, B\cap B^*; D, G_\delta)$ follows essentially from (5.4). ■

Step 4: *Completion of the proof of Theorem 5.*

Sketchy proof of Step 4. For each $a \in A\cap A^*$, let $f_a := f|_{\mathbb{X}(A\cap U_a, B; U_a, G)}$. Since $f \in \mathcal{O}_{\mathrm{s}}(W^{\mathrm{o}}, Z)$, we deduce that $f_a \in \mathcal{O}_{\mathrm{s}}(\mathbb{X}(A\cap U_a, B; U_a, G), Z)$. Since U_a is biholomorphic to a bounded domain in $\mathbb{C}^m$, we can apply Step 3 to f_a. Consequently, there is a mapping $\hat{f}_a \in \mathcal{O}(\widehat{\widehat{\mathbb{X}}}(A\cap U_a, B; U_a, G), Z)$ such that

$$\hat{f}_a(z,w) = f(z,w), \quad (z,w) \in \mathbb{X}(A\cap A^*\cap U_a, B\cap B^*; U_a, G). \tag{5.7}$$

Let $0<\delta\le 1/2$. In virtue of (5.7), we can "glue" the family $(\hat{f}_a|_{U_{a,\delta}\times G_\delta})_{a\in A\cap A^*}$ to obtain a mapping $\tilde{f}'_\delta \in \mathcal{O}(A_\delta \times G_\delta, Z)$.

Similarly, for each $b \in B\cap B^*$, one obtains a mapping $\hat{f}_b \in \mathcal{O}(\widehat{\widehat{\mathbb{X}}}(A, B\cap V_b; D, V_b), Z)$ such that

$$\hat{f}_b(z,w) = f(z,w), \quad (z,w) \in \mathbb{X}(A\cap A^*, B\cap B^*\cap V_b; D, V_b). \tag{5.8}$$

Moreover, one can "glue" the family $(\hat{f}_b|_{D_\delta\times V_{b,\delta}})_{b\in B\cap B^*}$ to obtain a mapping $\tilde{f}''_\delta \in \mathcal{O}(D_\delta \times B_\delta, Z)$.

Next, using (5.7)–(5.8) and (5.4) we can prove that

$$\tilde{f}'_\delta = \tilde{f}''_\delta \quad \text{on } A_\delta \times B_\delta.$$

Using this we define $\tilde{f}_\delta : \mathbb{X}(A_\delta, B_\delta; D_\delta, G_\delta) \to Z$ by

$$\tilde{f}_\delta := \begin{cases} \tilde{f}'_\delta & \text{on } A_\delta \times G_\delta, \\ \tilde{f}''_\delta & \text{on } D_\delta \times B_\delta. \end{cases}$$

It can be readily checked that $\tilde{f}_\delta \in \mathcal{O}_{\mathrm{s}}(\mathbb{X}(A_\delta, B_\delta; D_\delta, G_\delta), Z)$. Since we know from (5.4) that A_δ (resp. B_δ) is an open subset of D_δ (resp. G_δ), we can apply Step 2 to $\tilde{f}_\delta$ for every $0<\delta\le 1/2$. Consequently, one obtains a mapping $\hat{f}_\delta \in \mathcal{O}(\widehat{\mathbb{X}}(A_\delta, B_\delta; D_\delta, G_\delta), Z)$ such that

$$\hat{f}_\delta = \tilde{f}_\delta \quad \text{on } \mathbb{X}(A_\delta, B_\delta; D_\delta, G_\delta).$$

We are now in a position to define the desired extension mapping $\hat{f}$:

$$\hat{f} := \lim_{\delta\to 0} \hat{f}_\delta \quad \text{on } \widehat{\widetilde{W}}.$$

In fact, the equality $\widehat{\widetilde{W}} = \bigcup_{0<\delta<1/2} \widehat{\mathbb{X}}(A_\delta, B_\delta; D_\delta, G_\delta)$ follows essentially from (5.4). ∎

6. Problem 1 for $A \subset \partial D$, $B \subset \partial G$**.** In this section we present two particular cases of Problem 1 using two different systems of approach regions defined in Subsection 3.2. These results are obtained in collaboration with Pflug (see [48, 49, 50]). Firstly, we start with the case of dimension 1.

6.1. *System of angular approach regions.* Our main purpose is to establish a boundary cross theorem which is an optimal version of Theorem 3. This constitutes the first step of our strategy to extend the theory of separately holomorphic mappings. We will use the terminology and notation of item 3 of Subsection 3.2. More precisely, if D is an open set of a Riemann surface such that D is good on a nonempty part of ∂D, we equip D with the system of angular approach regions supported on this part. Moreover, the notions such as set of positive length, set of zero length, locally pluriregular point, which exist on ∂E, can be transferred to ∂D using conformal mappings in a local way (see [49] for more details).

THEOREM 6 ([49]). *Let X, Y be Riemann surfaces and $D \subset X$, $G \subset Y$ open subsets, and let A (resp. B) be a subset of ∂D (resp. ∂G) such that D (resp. G) is good on A (resp. B) and both A and B are of positive length. Define*

$$\begin{aligned} W &:= \mathbb{X}(A, B; D, G), \qquad W' := \mathbb{X}(A', B'; D, G),\\ \widehat{W} &:= \{(z, w) \in D \times G : \omega(z, A, D) + \omega(w, B, G) < 1\},\\ \widehat{W'} &:= \{(z, w) \in D \times G : \omega(z, A', D) + \omega(w, B', G) < 1\}, \end{aligned}$$

where A' (resp. B') is the set of points at which A (resp. B) is locally pluriregular with respect to the system of angular approach regions supported on A (resp. B), and $\omega(\cdot, A, D)$, $\omega(\cdot, A', D)$ (resp. $\omega(\cdot, B, G)$, $\omega(\cdot, B', G)$) are calculated using the canonical system of approach regions.

Then for every function $f : W \to \mathbb{C}$ which satisfies the following conditions:

(i) *for every $a \in A$ the function $f(a, \cdot)|_G$ is holomorphic and has the angular limit $f(a, b)$ at all points $b \in B$, and for every $b \in B$ the function $f(\cdot, b)|_D$ is holomorphic and has the angular limit $f(a, b)$ at all points $a \in A$;*

(ii) *f is locally bounded;*

(iii) *$f|_{A\times B}$ is continuous,*

there exists a unique function $\hat{f} \in \mathcal{O}(\widehat{W'}, \mathbb{C})$ *which admits the angular limit* f *at all points of* $W \cap W'$.

If A *and* B *are Borel sets or if* $X = Y = \mathbb{C}$ *then* $\widehat{W} = \widehat{W'}$.

Theorem 6 is the "measurable" version of Theorem 3. Indeed, the hypotheses of the latter such as open boundary sets A and B etc. are now replaced by measurable boundary sets A and B etc. The question of optimality of Theorem 6 has been settled in [51].

Our method consists of two steps. In the first step we suppose that D and G are Jordan domains in $\mathbb{C}$. In the second step we treat the general case. Now we give a brief outline of the proof.

For every $0 < \delta < 1$ the set $D_\delta := \{z \in D : \omega(z, A, D) < 1 - \delta\}$ (resp. $G_\delta := \{w \in G : \omega(w, B, G) < 1 - \delta\}$) is called a *level set* (of the harmonic measure $\omega(\cdot, A, D)$ (resp. $\omega(\cdot, B, G)$). In the first step, we improve Gonchar's method [13, 14] by making intensive use of Carleman's formula (see [5]) and of geometric properties of level sets of harmonic measures. More precisely, when adapting Gonchar's method to our "measurable" situation, we meet some difficulties as the geometry of D_δ and G_δ is complicated. To overcome this, we construct Jordan domains with rectifiable boundary which are contained in D_δ and G_δ and which touch the boundary of these level sets along a set of positive length. Consequently, the analysis on the complicated open sets D_δ and G_δ can be reduced to that on certain Jordan domains.

The main ingredient for the second step is a mixed cross type theorem. The idea is to adapt Theorem 1 to the following "mixed" situation:

D (*resp.* G) *is an open set of a Riemann surface*, A *is an open subset of* D, *but* B *is a subset of* ∂G *such that* G *is good on* B. *This explains the terminology "mixed cross".*

Our key observation is that the classical method of doubly orthogonal bases of Bergman type, discussed in Section 2, still applies in the present mixed context. We also use a recent work of Zeriahi (see [63]).

In the second step we apply this mixed cross type theorem to prove Theorem 6 with D (resp. G) replaced by D_δ (resp. G_δ). Then we construct the solution for the original open sets D and G by means of a gluing procedure. The method for the second step (called "the method of level sets") has appeared for the first time in [48]. We will discuss it in the next subsection.

6.2. *Canonical system of approach regions.* For every open subset $U \subset \mathbb{R}^{2n-1}$ and every continuous function $h : U \to \mathbb{R}$, the graph

$$\{z = (z', z_n) = (z', x_n + iy_n) \in \mathbb{C}^n : (z', x_n) \in U \text{ and } y_n = h(z', x_n)\}$$

is called a *topological hypersurface in* $\mathbb{C}^n$.

Let X be a complex manifold of dimension n. A subset $A \subset X$ is said to be a *topological hypersurface* if, for every point $a \in A$, there is a local chart $\phi : U \to \mathbb{C}^n$ around a such that $\phi(A \cap U)$ is a topological hypersurface in $\mathbb{C}^n$.

Now let $D \subset X$ be an open subset and let $A \subset \partial D$ be an open subset (in the topology induced on ∂D). Suppose in addition that A is a topological hypersurface. A point $a \in A$ is said to be *of type 1* (*with respect to* D) if for every neighborhood U of a there is an open neighborhood V of a such that $V \subset U$ and $V \cap D$ is a domain. Otherwise, a is said to be *of type 2*. We see easily that if a is of type 2, then for every neighborhood U of a, there are an open neighborhood V of a and two domains V_1, V_2 such that $V \subset U$, $V \cap D = V_1 \cup V_2$ and all points in $A \cap V$ are of type 1 with respect to V_1 and V_2.

In virtue of Proposition 3.7 in [50] we have the following

PROPOSITION 6.1. *Let X be a complex manifold and D an open subset of X, equipped with the canonical system of approach regions. Suppose that $A \subset \partial D$ is an open boundary subset which is also a topological hypersurface. Then A is locally pluriregular and $\widetilde{A} = A$.*

The main result of this subsection is

THEOREM 7 ([50]). *Let X, Y be two complex manifolds, and $D \subset X$, $G \subset Y$ two nonempty open sets. Suppose that D (resp. G) is equipped with the canonical system of approach regions. Let A (resp. B) be a nonempty open subset of ∂D (resp. ∂G) which is also a topological hypersurface. Define*

$$W := \mathbb{X}(A, B; D, G),$$
$$\widehat{W} := \{(z, w) \in D \times G : \omega(z, A, D) + \omega(w, B, G) < 1\}.$$

Let $f : W \to \mathbb{C}$ be such that:

(i) *$f \in \mathcal{C}_{\mathrm{s}}(W, \mathbb{C}) \cap \mathcal{O}_{\mathrm{s}}(W^{\mathrm{o}}, \mathbb{C})$;*
(ii) *f is locally bounded on W;*
(iii) *$f|_{A\times B}$ is continuous.*

Then there exists a unique function $\hat{f} \in \mathcal{O}(\widehat{W}, \mathbb{C})$ such that

$$\lim_{\widehat{W} \ni (z,w) \to (\zeta,\eta)} \hat{f}(z, w) = f(\zeta, \eta), \qquad (\zeta, \eta) \in W.$$

A weaker version of Theorem 7 where D (resp. G) is a pseudoconvex open subset of $\mathbb{C}^m$ (resp. $\mathbb{C}^n$) was previously proved in [48]. In order to tackle "arbitrary" complex manifolds we follow our new approach introduced in Sections 4 and 5. The next key technique is to apply a mixed cross type theorem in the following context.

D is an open subset of $\mathbb{C}^m$ and G is the open unit disc in $\mathbb{C}$, A is an open subset of D but B is an open connected subset (an arc) of ∂G.

The last key technique is to use *level sets* of the plurisubharmonic measure (see [48, 49]). More precisely, we exhaust D (resp. G) by the level sets of the plurisubharmonic measure $\omega(\cdot, A, D)$ (resp. $\omega(\cdot, B, G)$), that is, by $D_\delta := \{z \in D : \omega(z, A, D) < 1 - \delta\}$ (resp. $G_\delta := \{w \in G : \omega(w, B, G) < 1 - \delta\}$) for $0 < \delta < 1$.

Our method consists of three steps. In the first step we suppose that G is a domain in $\mathbb{C}^m$ and A is an open subset of D. In the second step we treat the case where the pairs (D, A) and (G, B) are "good" enough in the sense of the slicing method. In the last one we consider the general case. For the first step we combine the above mentioned mixed cross theorem with the technique of holomorphic discs. For the second step we apply the slicing method and Theorem 3 ([11]). The general philosophy is to prove Theorem 7 with D (resp. G) replaced by D_δ (resp. G_δ). Then we construct the solution for the original open sets D and G by means of a gluing procedure (that is, the method of level sets). In the last step we transfer the holomorphy from local situations to the global context using Poletsky's theory of discs and the Rosay theorem.

7. Problem 1 in the general case. In Sections 5 and 6 we have solved Problem 1 in some particular but important cases. These results make us hope that a reasonable solution to Problem 1 in the general case may exist. The main purpose of this section is to confirm this speculation. In our work [41] we have introduced the formulations given in Section 3 above and developed a unified approach which improves the one given in Section 4. We keep the notation introduced in Section 3, and state the main results.

THEOREM 8 ([41]). *Let X, Y be two complex manifolds, let $D \subset X$, $G \subset Y$ be two open sets, let A (resp. B) be a subset of $\overline{D}$ (resp. $\overline{G}$). Suppose that D (resp. G) is equipped with a system of approach regions $(\mathcal{A}_\alpha(\zeta))_{\zeta\in\overline{D},\,\alpha\in I_\zeta}$ (resp. $(\mathcal{A}_\beta(\eta))_{\eta\in\overline{G},\,\beta\in I_\eta}$). Suppose in addition that $\widetilde{\omega}(\cdot, A, D) < 1$ on D and $\widetilde{\omega}(\cdot, B, G) < 1$ on G. Let Z be a complex analytic space with the Hartogs extension property. Then, for every mapping $f : W \to Z$ which satisfies the following conditions:*

- $f \in \mathcal{C}_{\mathrm{s}}(W, Z) \cap \mathcal{O}_{\mathrm{s}}(W^{\mathrm{o}}, Z)$;

([11]) It is worth remarking here that a weaker version of Theorem 3 will suffice for this argument. Namely, we only need Theorem 3 for the case where A and B are arcs. This weaker version of Theorem 3 is also known under the name of Drużkowski's theorem (see [11]). In fact, we also obtain, in this way, a new proof of Theorem 3 starting from Drużkowski's theorem.

- *f is locally bounded along* $\mathbb{X}(A\cap\partial D, B\cap\partial G; D, G)$ ([12]);
- $f|_{A\times B}$ *is continuous at all points of* $(A\cap\partial D)\times(B\cap\partial G)$,

there exists a unique mapping $\hat{f}\in\mathcal{O}(\widehat{\widetilde{W}}, Z)$ *which admits the* $\mathcal{A}$*-limit* $f(\zeta,\eta)$ *at every point* $(\zeta,\eta)\in W\cap\widetilde{W}$.

Theorem 8 has an important corollary. Before stating it, we need to introduce some terminology. A complex manifold $\mathcal{M}$ is said to be a *Liouville manifold* if $\mathcal{PSH}(\mathcal{M})$ does not contain any nonconstant functions bounded above. We see clearly that the class of Liouville manifolds contains the class of connected compact manifolds.

COROLLARY 1. *We keep the hypotheses and notation in Theorem* 8. *Suppose in addition that G is a Liouville manifold. Then, for every mapping* $f: W\to Z$ *which satisfies the following conditions*:

- $f\in\mathcal{C}_{\mathrm{s}}(W,Z)\cap\mathcal{O}_{\mathrm{s}}(W^{\mathrm{o}},Z)$;
- *f is locally bounded along* $\mathbb{X}(A\cap\partial D, B\cap\partial G; D, G)$;
- $f|_{A\times B}$ *is continuous at all points of* $(A\cap\partial D)\times(B\cap\partial G)$,

there is a unique mapping $\hat{f}\in\mathcal{O}(D\times G, Z)$ *which admits the* $\mathcal{A}$*-limit* $f(\zeta,\eta)$ *at every point* $(\zeta,\eta)\in W\cap\widetilde{W}$.

Corollary 1 follows immediately from Theorem 8 since $\widetilde{\omega}(\cdot, B, G)\equiv 0$. This theorem generalizes, in some sense, all results obtained in Sections 5 and 6. On the other hand, we will see many other applications of Theorem 8 in Section 9. We will explain our unified approach and techniques for the proof of Theorem 8 in the following special "local" case.

PROPOSITION 7.1. *Let* $D\subset\mathbb{C}^n$, $G\subset\mathbb{C}^m$ *be bounded connected open sets. Suppose D (resp. G) is equipped with a system of approach regions* $(\mathcal{A}_\alpha(\zeta))_{\zeta\in\overline{D},\,\alpha\in I_\zeta}$ *(resp.* $(\mathcal{A}_\alpha(\eta))_{\eta\in\overline{G},\,\alpha\in I_\eta}$*). Let A (resp. B) be a nonempty subset of $\overline{D}$ (resp. $\overline{G}$) such that A and B are locally pluriregular. Put*

$$W:=\mathbb{X}(A,B;D,G),\qquad \overline{W}:=\mathbb{X}(\overline{A},\overline{B};D,G),$$
$$\overline{W}^{\mathrm{o}}:=\mathbb{X}^{\mathrm{o}}(\overline{A},\overline{B};D,G),\qquad \widehat{W}:=\widehat{\mathbb{X}}(A,B;D,G).$$

Then, for every bounded function $f:\overline{W}\to\mathbb{C}$ *such that* $f\in\mathcal{C}_{\mathrm{s}}(\overline{W},\mathbb{C})\cap\mathcal{O}_{\mathrm{s}}(\overline{W}^{\mathrm{o}},\mathbb{C})$ *and* $f|_{\overline{A}\times\overline{B}}$ *is continuous at all points of* $(\overline{A}\cap\partial D)\times(\overline{B}\cap\partial G)$, *there exists a unique bounded function* $\hat{f}\in\mathcal{O}(\widehat{W},\mathbb{C})$ *which admits the* $\mathcal{A}$*-limit* $f(\zeta,\eta)$ *at all points* $(\zeta,\eta)\in W$.

This result constitutes the core of the proof of Theorem 8. Indeed, the latter is, in some sense, the "global " version of Proposition 7.1. By using

([12]) It follows from Subsection 3.3 that

$$\mathbb{X}(A\cap\partial D, B\cap\partial G; D, G) = ((D\cup A)\times(B\cap\partial G))\cup((A\cap\partial D)\times(G\cup B)).$$

the approach developed in Section 4. we can go from local extensions to global ones. In addition, the formulation of Proposition 7.1 gives rise to Definition 3.3 of the plurisubharmonic measure $\widetilde{\omega}(\cdot, A, D)$. The core of our unified approach will be presented below. Our idea is to use an adapted version of Poletsky's theory of discs in order to reduce Proposition 7.1 to the case where D and G are simply the unit discs and $A \subset \partial D$, $B \subset \partial G$ are measurable sets (that is, a special case of Theorem 6).

Let us comment on the needed version of Poletsky's theory of discs. Let mes denote the Lebesgue measure on the unit circle ∂E. For a bounded mapping $\phi \in \mathcal{O}(E, \mathbb{C}^n)$ and $\zeta \in \partial E$, $f(\zeta)$ denotes the angular limit value of f at ζ if it exists. A classical theorem of Fatou says that $\text{mes}(\{\zeta \in \partial E : f(\zeta \text{ exists})\}) = 2\pi$.

Proposition 7.2. *Let D be a bounded open set in $\mathbb{C}^n$, $\emptyset \neq A \subset \overline{D}$, $z_0 \in D$ and $\epsilon > 0$. Let $\mathcal{A}$ be a system of approach regions for D. Suppose in addition that A is locally pluriregular (relative to $\mathcal{A}$) and that $\omega(\cdot, A, D) < 1$ on D. Then there exist a bounded mapping $\phi \in \mathcal{O}(E, \mathbb{C}^n)$ and a measurable subset $\Gamma_0 \subset \partial E$ with the following properties:*

(1) *Every point of Γ_0 is a density point of Γ_0, $\phi(0) = z_0$, $\phi(E) \subset \overline{D}$, $\Gamma_0 \subset \{\zeta \in \partial E : \phi(\zeta) \in \overline{A}\}$, and*
$$1 - \frac{1}{2\pi}\,\text{mes}(\Gamma_0) < \omega(z_0, A, D) + \epsilon.$$
(2) *Let $f \in \mathcal{C}(D \cup \overline{A}, \mathbb{C}) \cap \mathcal{O}(D, \mathbb{C})$ with $f(D)$ bounded. Then there exists a bounded function $g \in \mathcal{O}(E, \mathbb{C})$ such that $g = f \circ \phi$ in a neighborhood of $0 \in E$ and* [13] *$g(\zeta) = (f \circ \phi)(\zeta)$ for all $\zeta \in \Gamma_0$. Moreover, $g|_{\Gamma_0} \in \mathcal{C}(\Gamma_0, \mathbb{C})$.*

This result is proved by adapting the original disc construction of Poletsky in [52, 53]. Recall here that Poletsky considered the case where $A \subset D$ and $\mathcal{A}$ is the canonical system of approach regions. But his method still works in our context by using the Montel theorem on normal families. It is worth remarking that $\phi(E) \subset \overline{D}$, but in general $\phi(E) \not\subset D$.

Proposition 7.2 motivates the following

Definition 7.3. We keep the hypotheses and notation of Proposition 7.2. Then every pair (ϕ, Γ_0) satisfying the conclusions (1)–(2) of this proposition is said to be an *ϵ-candidate for the triplet* (z_0, A, D).

Proposition 7.2 says that there always exist ϵ-candidates for all triplets (z, A, D). Now we arrive at

Sketchy proof of Proposition 7.1. Firstly, we give the construction of $\hat{f}$. Fix a point $(z, w) \in \widehat{W}$; we want to determine the value $\hat{f}(z, w)$. To do this

([13]) Note here that by part (1), $(f \circ \phi)(\zeta)$ exists for all $\zeta \in \Gamma_0$.

let $\epsilon > 0$ be such that

$$\omega(z, A, D) + \omega(w, B, G) + 2\epsilon < 1. \tag{7.1}$$

By Proposition 7.2 and Definition 7.3, there is an ϵ-candidate (ϕ, Γ) (resp. (ψ, Δ)) for (z, A, D) (resp. (w, B, G)). Moreover, using the hypotheses, we see that the function $f_{\phi,\psi}$ defined by

$$f_{\phi,\psi}(t, \tau) := f(\phi(t), \psi(\tau)), \qquad (t, \tau) \in \mathbb{X}(\Gamma, \Delta; E, E),$$

satisfies the hypotheses of Theorem 6. By this theorem, let $\hat{f}_{\phi,\psi}$ be the unique function in $\widehat{\mathbb{X}}(\Gamma, \Delta; E, E)$ such that

$$(\mathcal{A}\text{-lim}\, \hat{f}_{\phi,\psi})(t, \tau) = f_{\phi,\psi}(t, \tau), \qquad (t, \tau) \in \mathbb{X}^{\mathrm{o}}(\Gamma, \Delta; E, E),$$

where $\mathcal{A}$-lim is the angular limit. In virtue of (7.1) and Proposition 7.2, $(0, 0) \in \widehat{\mathbb{X}}(\Gamma, \Delta; E, E)$. Then we can define the value of the desired extension function $\hat{f}$ at (z, w) as follows:

$$\hat{f}(z, w) := \hat{f}_{\phi,\psi}(0, 0).$$

It remains to prove that the $\hat{f}$ so defined has the required properties of Proposition 7.1: namely, $\hat{f}$ is holomorphic and admits the $\mathcal{A}$-limit f at all points of W.

In fact, using the technique of level sets, the holomorphy of $\hat{f}$ is reduced to proving the following mixed cross version of Proposition 7.1.

Assertion. *A is a measurable subset of ∂E with* $\mathrm{mes}(A) > 0$,

$$D := \{w \in E : \omega(w, A, E) < 1 - \delta\} \quad \text{for some } \delta \text{ with } 0 \le \delta < 1,$$

and B is an open subset of an arbitrary complex manifold G.

Using the Rosay theorem, the case $\delta = 0$ of the assertion can be reduced to the special case of Theorem 6 where D and G are merely the unit discs and $A \subset \partial D$, $B \subset \partial G$ are measurable sets.

The case where $0 < \delta < 1$ can be reduced to the previous case by using conformal mappings from every connected component of D onto E. In fact, all connected components of D are simply connected. This idea has been developed in [41], and it is called the *technique of conformal mappings*. The interesting point of this proof of the assertion is that we avoid completely the classical method of doubly orthogonal bases of Bergman type.

In order to show that $\hat{f}$ admits the $\mathcal{A}$-limit f at all points of W, we make use of an argument based on the Two-Constant Theorem (see [41] for more details). ∎

In conclusion, our new approach illustrates the unified character: "from local information to global extensions". In fact, "global" results (i.e. for general crosses) can be deduced from "local" ones (i.e. for boundary crosses defined over the bidisk).

8. Problem 2. In the case of crosses in the interior context (that is, $A \subset D$ and $B \subset G$), one was led to investigate cross theorems with analytic or pluripolar singularities (see, for example, [24, 25, 26, 28] and the references therein). The starting point for this kind of questions was the so-called range problem in mathematical tomography (for more details see [45]). To be more precise one had to describe the range of the exponential Radon transform R_μ, $\mu \neq 0$,

$$\mathcal{C}_c^\infty(\mathbb{R}^2, \mathbb{R}) \ni h \overset{R_\mu}{\mapsto} \int\limits_{x \cdot \omega = p} h(x) \exp(\mu x \cdot \omega^\perp)\, d\Lambda_1(x),$$

where $\omega = (\sin\alpha, \cos\alpha) \in \mathbb{S}^1$, $p \in \mathbb{R}$, $\omega^\perp = (-\sin\alpha, \cos\alpha)$, and where "$\cdot$" means the standard scalar product in $\mathbb{R}^2$ and $d\Lambda_1$ denotes the one-dimensional Lebesgue measure.

Then the natural question arises whether there also exists a general cross theorem with singularities. Namely, does there exist a general version of Theorem 2 in the spirit of Theorem 8? In other words, we want to solve Problem 2 when Z is a complex analytic space with the Hartogs extension property.

We have recently obtained, in collaboration with P. Pflug (see [43, 44]), a reasonable solution to the problem. Our idea is to follow the strategy as in the case without singularities. Namely, we investigate first the "local" case where the boundary crosses are defined over the bidisk, and then we pass from this case to the global one.

By using an idea of Jarnicki and Pflug [25, 27], applying the technique of conformal mappings (see the end of Section 7), using the technique of level sets and using the results of Chirka [9], Imomkulov–Khujamov [18] and Imomkulov [17], we obtain the following "measurable" version with singularities of Theorem 3.

THEOREM 9 ([43]). *Let $D = G = E$ and let $A \subset \partial D$, $B \subset \partial G$ be measurable subsets such that* $\operatorname{mes}(A) > 0$, $\operatorname{mes}(B) > 0$. *Suppose that D and G are equipped with the system of angular approach regions. Consider the cross $W := \mathbb{X}(A, B; D, G)$. Let M be a relatively closed subset of W such that*

- *M_a is polar* (*resp. discrete*) *in G for all $a \in A$ and M^b is polar* (*resp. discrete*) *in D for all $b \in B$* ([14]);
- *$M \cap (A \times B) = \emptyset$.*

Then there exists a relatively closed pluripolar subset (*resp. an analytic subset*) *$\widehat{M}$ of $\widehat{W}$ with the following two properties*:

([14]) In other words, M is polar (resp. discrete) in fibers over A and B.

(i) *The set of end-points of* $\widehat{W} \setminus \widehat{M}$ *contains* $((A' \times G) \cup (D \times B')) \setminus M$, *where* A' *(resp.* B'*) denotes the set of density points of* A *(resp. of* B*).*

(ii) *Let* $f : W \setminus M \to \mathbb{C}$ *be a locally bounded function such that*

- *for all* $a \in A$, $f(a, \cdot)|_{G \setminus M_a}$ *is holomorphic and admits the angular limit* $f(a, b)$ *at all points* $b \in B$;
- *for all* $b \in B$, $f(\cdot, b)|_{D \setminus M^b}$ *is holomorphic and admits the angular limit* $f(a, b)$ *at all points* $a \in A$;
- $f|_{A \times B}$ *is measurable.*

Then there is a unique function $\hat{f} \in \mathcal{O}(\widehat{W} \setminus \widehat{M}, \mathbb{C})$ *such that* $\hat{f}$ *admits the angular limit* f *at all points of* $((A' \times G) \cup (D \times B')) \setminus M$.

Moreover, if $M = \emptyset$, *then* $\widehat{M} = \emptyset$.

The path from Theorem 9 to its global version is much harder than in the case without singularities. The difficulty arises when we want to show that $\hat{f}$ admits the desired $\mathcal{A}$-limit. In the case without singularities this procedure works well because we can use an argument based on the Two-Constant Theorem. But this is not available any more in the case with singularities. In [44] we have found a way to overcome this difficulty by using some special mixed cross theorems with singularities.

Recall that a subset S of a complex manifold $\mathcal{M}$ is said to be *thin* if for every point $x \in \mathcal{M}$ there are a connected neighborhood $U = U(x) \subset \mathcal{M}$ and a holomorphic function f on U, not identically zero, such that $U \cap S \subset f^{-1}(0)$. We are now ready to state our main result.

Theorem 10 ([44]). *Let* X, Y *be two complex manifolds, let* $D \subset X$, $G \subset Y$ *be two open sets, and let* A *(resp.* B*) be a subset of* $\overline{D}$ *(resp.* $\overline{G}$*). Suppose that* D *(resp.* G*) is equipped with a system of approach regions* $(\mathcal{A}_\alpha(\zeta))_{\zeta \in \overline{D},\, \alpha \in I_\zeta}$ *(resp.* $(\mathcal{A}_\beta(\eta))_{\eta \in \overline{G},\, \beta \in I_\eta}$*). Suppose in addition that* $A = A^*$ *and* $B = B^*$ (15) *and that* $\widetilde{\omega}(\cdot, A, D) < 1$ *on* D *and* $\widetilde{\omega}(\cdot, B, G) < 1$ *on* G. *Let* Z *be a complex analytic space with the Hartogs extension property. Let* M *be a relatively closed subset of* W *with the following properties*:

- M *is thin in fibers (resp. locally pluripolar in fibers) over* A *and over* B;
- $M \cap ((A \cap \partial D) \times B) = M \cap (A \times (B \cap \partial G)) = \emptyset$.

Then there exists a relatively closed analytic (resp. a relatively closed locally pluripolar) subset $\widehat{M}$ *of* $\widehat{\widetilde{W}}$ *such that* $\widehat{M} \cap \widetilde{W} \subset M$ (16), $\widetilde{W} \setminus M \subset \operatorname{End}(\widehat{\widetilde{W}} \setminus \widehat{M})$

(15) It is worth noting that this assumption is not so restrictive since we know from Subsection 3.1 that $A \setminus A^*$ and $B \setminus B^*$ are locally pluripolar for arbitrary sets $A \subset \overline{D}$, $B \subset \overline{G}$.

(16) Note that if $\widetilde{A} \cap D = \emptyset$ and $\widetilde{B} \cap G = \emptyset$, then this intersection is empty.

and for every mapping $f : W \setminus M \to Z$ *satisfying the following conditions*:

(i) $f \in \mathcal{C}_{\mathrm{s}}(W \setminus M, Z) \cap \mathcal{O}_{\mathrm{s}}(W^{\mathrm{o}} \setminus M, Z)$;
(ii) f *is locally bounded along* $\mathbb{X}(A \cap \partial D, B \cap \partial G; D, G) \setminus M$;
(iii) $f|_{(A\times B)\setminus M}$ *is continuous at all points of* $(A \cap \partial D) \times (B \cap \partial G)$,

there exists a unique mapping $\hat{f} \in \mathcal{O}(\widehat{\widetilde{W}} \setminus \widehat{M}, Z)$ *which admits the* $\mathcal{A}$*-limit* $f(\zeta, \eta)$ *at every point* $(\zeta, \eta) \in \widetilde{W} \setminus M$.

9. Some applications. In [41] the author gives various applications of Theorem 8 using three systems of approach regions. These are the canonical system, the system of angular approach regions and the system of conical approach regions. We only give here some applications of Theorem 10 for the system of conical approach regions. We leave it to the reader to treat the first two cases, that is, to translate Theorems 6 and 7 into the new context of Theorem 10.

Let X be an arbitrary complex manifold and $D \subset X$ an open subset. We say that a set $A \subset \partial D$ is *locally contained in a generic manifold* if there exist an (at most countable) index set $J \neq \emptyset$, a family of open subsets $(U_j)_{j\in J}$ of X and a family of *generic manifolds* ([17]) $(\mathcal{M}_j)_{j\in J}$ such that $A \cap U_j \subset \mathcal{M}_j$ for all $j \in J$ and $A \subset \bigcup_{j\in J} U_j$. The dimensions of $\mathcal{M}_j$ may vary.

Suppose $A \subset \partial D$ is locally contained in a generic manifold. Then we say that A is *of positive size* if under the above notation $\sum_{j\in J} \mathrm{mes}_{\mathcal{M}_j}(A \cap U_j) > 0$, where $\mathrm{mes}_{\mathcal{M}_j}$ denotes the Lebesgue measure on $\mathcal{M}_j$. A point $a \in A$ is said to be a *density point* relative to A if it is a density point relative to $A \cap U_j$ on $\mathcal{M}_j$ for some $j \in J$. Denote by A' the set of all density points relative to A.

Suppose now that $A \subset \partial D$ is of positive size. We equip D with the system of conical approach regions supported on A. Using the works of B. Coupet and B. Jöricke (see [10, 29]), one can show that ([18]) A is locally pluriregular at all density points relative to A and $A' \subset \widetilde{A}$. Consequently, it follows from Definition 3.3 that

$$\widetilde{\omega}(z, A, D) \le \omega(z, A', D), \qquad z \in D.$$

This estimate, combined with Theorem 10, implies the following result.

([17]) A $\mathcal{C}^2$-smooth submanifold $\mathcal{M}$ of a complex manifold X is said to be a *generic manifold* if for all $\zeta \in \mathcal{M}$, every complex vector subspace of $T_\zeta X$ containing $T_\zeta \mathcal{M}$ coincides with $T_\zeta X$.

([18]) A complete proof is available in [42].

Corollary 2. *Let* X, Y *be two complex manifolds, let* $D \subset X$, $G \subset Y$ *be two connected open sets, and let* A (*resp.* B) *be a subset of* ∂D (*resp.* ∂G). *Suppose that* D (*resp.* G) *is equipped with a system of conical approach regions* $(\mathcal{A}_\alpha(\zeta))_{\zeta\in\overline{D},\ \alpha\in I_\zeta}$ (*resp.* $(\mathcal{A}_\beta(\eta))_{\eta\in\overline{G},\ \beta\in I_\eta}$) *supported on* A (*resp.* B). *Suppose in addition that* A *and* B *are of positive size. Define*

$$W' := \mathbb{X}(A', B'; D, G),$$
$$\widehat{W'} := \{(z,w) \in D \times G : \omega(z, A', D) + \omega(w, B', G) < 1\},$$

where A' (*resp.* B') *is the set of density points relative to* A (*resp.* B). *Let* M *be a relatively closed subset of* W *with the following properties*:

- M *is thin in fibers* (*resp. locally pluripolar in fibers*) *over* A *and over* B;
- $M \cap (A \times B) = \emptyset$.

Then there exists a relatively closed analytic (*resp. a relatively closed locally pluripolar*) *subset* $\widehat{M}$ *of* $\widehat{W'}$ *such that for every mapping* $f : W \setminus M \to Z$ *satisfying the following conditions*:

(i) $f \in \mathcal{C}_{\mathrm{s}}(W \setminus M, Z) \cap \mathcal{O}_{\mathrm{s}}(W^{\mathrm{o}} \setminus M, Z)$;
(ii) f *is locally bounded along* $\mathbb{X}(A, B; D, G) \setminus M$;
(iii) $f|_{A\times B}$ *is continuous*,

there exists a unique mapping $\hat{f} \in \mathcal{O}(\widehat{W'} \setminus \widehat{M}, Z)$ *which admits the* $\mathcal{A}$-*limit* $f(\zeta, \eta)$ *at every point* $(\zeta, \eta) \in (W \cap W') \setminus M$.

The second application is a very general mixed cross theorem.

Corollary 3. *Let* X, Y *be two complex manifolds, let* $D \subset X$, $G \subset Y$ *be connected open sets, let* A *be a subset of* ∂D, *and let* B *be a subset of* G. *Suppose that* D *is equipped with the system of conical approach regions* $(\mathcal{A}_\alpha(\zeta))_{\zeta\in\overline{D},\, \alpha\in I_\zeta}$ *supported on* A *and* G *is equipped with the canonical system of approach regions* $(\mathcal{A}_\beta(\eta))_{\eta\in\overline{G},\, \beta\in I_\eta}$. *Suppose in addition that* A *is of positive size and that* $B = B^* \neq \emptyset$. *Define*

$$W' := \mathbb{X}(A', B; D, G),$$
$$\widehat{W'} := \{(z,w) \in D \times G : \omega(z, A', D) + \omega(w, B, G) < 1\},$$

where A' *is the set of density points relative to* A. *Let* M *be a relatively closed subset of* W *with the following properties*:

- M *is thin in fibers* (*resp. locally pluripolar in fibers*) *over* A *and over* B;
- $M \cap (A \times B) = \emptyset$.

Then there exists a relatively closed analytic (*resp. a relatively closed locally pluripolar*) *subset* $\widehat{M}$ *of* $\widehat{W'}$ *such that* $W' \setminus M \subset \mathrm{End}(\widehat{W'} \setminus \widehat{M})$ *and for every mapping* $f : W \setminus M \to Z$ *satisfying the following conditions*:

(i) $f \in \mathcal{C}_{\mathrm{s}}(W \setminus M, Z) \cap \mathcal{O}_{\mathrm{s}}(W^{\mathrm{o}} \setminus M, Z)$;

(ii) *f is locally bounded along* $(A \times G) \setminus M$,

there exists a unique mapping $\hat{f} \in \mathcal{O}(\widehat{W'} \setminus \widehat{M}, Z)$ *which admits the* $\mathcal{A}$*-limit* $f(\zeta, \eta)$ *at every point* $(\zeta, \eta) \in W' \setminus M$.

Recently, Sadullaev and Imomkulov (see [55]) have obtained some similar, but less general results. In fact, they introduced the inner plurisubharmonic measure for boundary sets and formulated their results using this function.

10. Concluding remarks and open questions. We collect here some open questions which seem to be of interest for the future developments of the theory of separately holomorphic mappings.

QUESTION 1. *Study the optimality of Theorems* 8 *and* 10.

QUESTION 2. *Investigate Problem* 1 *and* 2 *when the "target space"* Z *does not have the Hartogs extension property.*

QUESTION 3. *Study Problem* 1 *when* D *and* G *are not necessarily open subsets of* X *and* Y. *Here* $\mathcal{O}(D, Z)$ *denotes the set of all holomorphic mappings* $f : U \to Z$, *where* $U = U_f$ *is an open neighborhood of* D *in* X *that depends on* f.

Some results concerning Question 2 could be found in [20, 21, 22]. Question 3 has some connections with Sibony's work in [58].

We think that new tools and new ideas need to be introduced in order to solve these questions.

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