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MAPPINGS OF DEGREE 5, PART I

BY

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Abstract. The class of linear (resp. quadratic) mappings over a commutative ring is determined by a set of equation-type relations. For the class of homogeneous polynomial mappings of degree $m \geq 3$ it is so over a field, and over a ring there exists a smallest equationally definable class of mappings containing the preceding one. It is proved that generating relations determining that class can be chosen to be strong relations (that is, of the same form over all commutative rings) iff $m \leq 5$. These relations are already found for $m \leq 4$. The purpose of the present paper is to find the first of two parts of generating relations (namely, all the 3-covering relations) satisfied by homogeneous polynomial mappings of degree 5. Moreover, we find some strong $(m-2)$-relations for any degree $m \geq 4$.

1. Preliminaries. Let R be a commutative ring with 1, and let X, Y be R-modules. A mapping $f : X \to Y$ is called a *homogeneous polynomial mapping* of degree m if it is obtained from a form of degree m between X and Y in the sense of N. Roby [7]. Note that such a form is an ordinary form (in n variables) if $X = R^n$ and $Y = R$. Any homogeneous polynomial mapping $f : X \to Y$ of degree m is an m-*application*, that is, it satisfies the following conditions:

(A1) $f(rx) = r^m f(x)$ for $r \in R$, $x \in X$,
(A2) $\Delta^m f : X^m \to Y$ is a (symmetric) m-linear mapping over R,

where $\Delta^n f : X^n \to Y$ $(n = 0, 1, 2, \ldots)$ is defined by the formula

$$(\Delta^n f)(x_1, \ldots, x_n) = \sum_{H \subset [1,n]} (-1)^{n-|H|} f\Big(\sum_{i \in H} x_i\Big)$$

and $[1, n] = \{1, \ldots, n\}$ (see, for example, [1]). If $m > 0$ then $f(0) = 0$ and hence $\Delta^n f$ can be defined inductively as follows: $\Delta^1 f = f$ and

$$(*) \qquad (\Delta^{n+1} f)(x_0, \ldots, x_n) = (\Delta^n f)(x_0 + x_1, x_2, \ldots, x_n) - (\Delta^n f)(x_0, x_2, \ldots, x_n) - (\Delta^n f)(x_1, x_2, \ldots, x_n).$$

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Moreover, $(\Delta^n f)(x_1,,\ldots,x_n) = 0$ if $x_i = 0$ for some i and $\Delta^n f$ is, evidently, symmetric. It is proved in [3] that any m-application which is a homogeneous polynomial mapping of degree m is also *regular*. This means that $() = \Delta^{m-1} f$ satisfies the following *regularity condition*:

$$\text{(A)} \qquad (rx, sy, -) - r(x, sy, -) - s(rx, y, -) + rs(x, y, -) = 0 \qquad \text{for } r, s \in R,\ x, y \in X$$

where $-$ stands for the remaining $m-3$ variables. The condition has some interesting consequences, listed in the following

THEOREM 1 ([3, Proposition 2.5]). *If f is a regular m-application on X then for any $r_i, r, s \in R$, $x_i, x \in X$ the mapping $() = \Delta^{m-1} f$ satisfies the following equalities*:

(1) $(r_1x_1, \ldots, r_{m-1}x_{m-1})$
$= \sum_{i=1}^{m-1} r_1 \ldots \widehat{r_i} \ldots r_{m-1}(x_1, \ldots, x_{i-1}, r_ix_i, x_{i+1}, \ldots, x_{m-1})$
$- (m-2)r_1 \ldots r_{m-1}(x_1, \ldots, x_{m-1})$,
(2) $\sum_{i=1}^{m-1}(x_1, \ldots, x_{i-1}, rx_i, x_{i+1}, \ldots, x_{m-1})$
$- (r^2 + (m-2)r)(x_1, \ldots, x_{m-1}) = 0$,
(3) $(rsx, -) = r(sx, -) + s^2(rx, -) - rs^2(x, -)$,
(4) $(r^2x, -) = (r + r^2)(rx, -) - r^3(x, -)$,
(5) $(r - r^2)(sx, -) = (s - s^2)(rx, -) + (rs^2 - r^2s)(x, -)$.

The functor of regular m-applications on X is represented by $\overline{\Delta}^m(X)$, defined as the R-module generated by elements $\overline{\delta}^m(x)$, $x \in X$, with relations meaning that $\overline{\delta}^m : X \to \overline{\Delta}^m(X)$ is a regular m-application (that is, $f = \overline{\delta}^m$ satisfies (A1), (A2) and (A)). Thus any regular m-application f on X has a unique expression in the form $f = \widetilde{f} \circ \overline{\delta}^m$ where $\widetilde{f}$ is an R-homomorphism.

In order to find all equations satisfied by homogeneous polynomial mappings of degree m, we consider the mth divided power $\Gamma^m(X)$ of X (see, for example, [7]), and the following homomorphism:

$$\overline{h}^m = \overline{h}^m(X) : \overline{\Delta}^m(X) \to \Gamma^m(X), \qquad \overline{h}^m(\overline{\delta}^m(x)) = x^{(m)}.$$

Finding generators of $\mathrm{Ker}(\overline{h}^m)$ allows us to describe generating equations, called covering relations for the functor Hom_R^m of homogeneous polynomial mappings of degree m (see [2]).

Let $\{x_1, \ldots, x_k\}$ be the standard basis of the module R^k, $k = 1, 2, \ldots$. Define

$$\Gamma^{m,k} = \Gamma^{m,k}(R) = R\Big\{x_1^{(i_1)} \cdots x_k^{(i_k)};\ \sum i_j = m,\ i_j \geq 1\Big\} \subset \Gamma^m(R^k),$$

$$\overline{\Delta}^{m,k} = \overline{\Delta}^{m,k}(R) = R\{(r_1x_1, \ldots, r_kx_k);\ r_1, \ldots, r_k \in R\} \subset \overline{\Delta}^m(R^k),$$

$$\overline{h}^{m,k} = \overline{h}^m|_{\overline{\Delta}^{m,k}(R)} : \overline{\Delta}^{m,k}(R) \to \Gamma^{m,k}(R),$$

$$\begin{aligned}\overline{h}^{m,k}(r_1x_1,\dots,r_kx_k) &= \sum_{(i)}(r_1x_1)^{(i_1)}\dots(r_kx_k)^{(i_k)}\\ &= \sum_{(i)} r_1^{i_1}\dots r_k^{i_k}x_1^{(i_1)}\dots x_k^{(i_k)}\end{aligned}$$

where $() = \Delta^k\overline{\delta}^m$ and (i) runs over all sequences of positive integers $i_1,\dots,i_k$ satisfying $i_1+\dots+i_k = m$. Since $((i_1,\dots,i_k)) := x_1^{(i_1)}\dots x_k^{(i_k)}$ form a basis of $\Gamma^{m,k}(R)$, we can also consider the "coordinate homomorphisms"

$$\Pi_{i_1,\dots,i_k} : \overline{\Delta}^{m,k}(R) \to R, \qquad \Pi_{i_1,\dots,i_k}(r_1x_1,\dots,r_kx_k) = r_1^{i_1}\dots r_k^{i_k}.$$

Since Δ^m is a functor, we can substitute any elements for the base elements $x_1,\dots,x_k$, obtaining an R-homomorphism on $\overline{\Delta}^m(R^k)$. In particular, any permutation of $x_1,\dots,x_k$ gives us an automorphism of $\overline{\Delta}^m(R^k)$, or, consequently, of $\overline{\Delta}^{m,k}(R)$. Another example is the substitution of x_1 for x_k, which gives us a homomorphism $\overline{\Delta}^{m,k}(R) \to \overline{\Delta}^{m,k-1}(R)$, as follows from $(*)$ (this is used in the proof of Theorem 2 below).

Any system of covering relations of the functor Hom_R^m is composed of partial systems of so-called k-covering relations (for any $k \geq 1$), and these relations arise from generators of $\mathrm{Ker}(\overline{h}^{m,k})$. In order to obtain relations from generators, it suffices to change the meaning of the symbols $(r_1x_1,\dots,r_kx_k)$ as follows: $()$ means now $\Delta^k f$ instead of $\Delta^k\overline{\delta}^m$ and $x_1,\dots,x_k$ become arbitrary elements of the domain of f.

It follows from [2] that $\overline{h}^{m,k}$ is an isomorphism for $k = 1$ and $k \geq m-1$. In other words, (A1), (A2) and (A) form for any such k a system of k-covering relations of the functor Hom_R^m. That system is *strong*, meaning that it is of the same form over any base ring R. The only other strong k-covering systems exist for $(m,k) = (4,2),(5,3)$ and $(5,2)$ (see [2, Main Theorem 6.2]). Such a system is found for $(4,2)$ in [4], and for $(5,3)$ it is described in the present paper. The last case will be considered in the second part of the paper (in preparation).

Our goal requires investigating $\mathrm{Ker}(\overline{h}^{5,3})$ or, more generally, $\mathrm{Ker}(\overline{h}^{m,m-2})$. We assume that $m \geq 4$, so $m-2 \geq 2$. If we denote $\overline{h}^{m,m-2}$ by $\overline{h}$ then

$$\begin{aligned}&\overline{h}(r_1x_1,\dots,r_{m-2}x_{m-2})\\ &\quad= \sum_{i=1}^{m-2} r_1\dots r_{i-1}r_i^3r_{i+1}\dots r_{m-2}x_1^{(1)}\dots x_{i-1}^{(1)}x_i^{(3)}x_{i+1}^{(1)}\dots x_{m-2}^{(1)}\\ &\qquad+\sum_{1\leq i<j\leq m-2} r_1\dots r_i^2\dots r_j^2\dots r_{m-2}x_1^{(1)}\dots x_i^{(2)}\dots x_j^{(2)}\dots x_{m-2}^{(1)}.\end{aligned}$$

2. Some useful elements of $\overline{\Delta}^{m,m-2}(R)$. For any $r \in R$, consider the following elements of $\overline{\Delta}^{m,m-2}(R)$:

$$\begin{aligned}
\langle r\rangle &:= (rx_1, x_2, \ldots, x_{m-2}) + (x_1, rx_2, \ldots, x_{m-2}) + \cdots + (x_1, x_2, \ldots, rx_{m-2}),\\
[r] &:= (rx_1, x_1, x_2, \ldots, x_{m-2}) + (x_1, rx_2, x_2, \ldots, x_{m-2}) + \cdots\\
&\quad + (x_1, x_2, \ldots, rx_{m-2}, x_{m-2}) - r^2((x_1, x_1, x_2, \ldots, x_{m-2})\\
&\quad + (x_1, x_2, x_2, \ldots, x_{m-2}) + \cdots + (x_1, x_2, \ldots, x_{m-2}, x_{m-2}))\\
&\quad - 3(r - r^2)(x_1, x_2, \ldots, x_{m-2})\\
&= \langle r + 1\rangle - \langle r\rangle - \langle 1\rangle - r^2(\langle 2\rangle - 2\langle 1\rangle) - 3(r - r^2)(x_1, x_2, \ldots, x_{m-2}).
\end{aligned}$$

Observe that $\overline{h}(x_1, \ldots, x_{m-2}) = \sigma_3 + \sigma_2$ where

$$\sigma_3 = \sum_{i=1}^{m-2} x_1^{(1)} \ldots x_{i-1}^{(1)} x_i^{(3)} x_{i+1}^{(1)} \ldots x_{m-2}^{(1)},$$

$$\sigma_2 = \sum_{1\le i<j\le m-2} x_1^{(1)} \ldots x_{i-1}^{(1)} x_i^{(2)} x_{i+1}^{(1)} \ldots x_{j-1}^{(1)} x_j^{(2)} x_{j+1}^{(1)} \ldots x_{m-2}^{(1)}.$$

The proof of the following fact is a straightforward calculation.

LEMMA 1. *The following equalities hold for any* $r \in R$:

(1) $\overline{h}(\langle r\rangle) = (r^3 + (m-3)r)\sigma_3 + (2r^2 + (m-4)r)\sigma_2$,
(2) $\overline{h}([r]) = (r - r^2)\sigma_2$,

and (2) *means that*

(3) $\Pi_{1,\ldots,3,\ldots,1}([r]) = 0$, $\Pi_{1,\ldots,2,\ldots,2,\ldots,1}([r]) = r - r^2$,

where the positions of 3 *and* 2 *can be chosen arbitrarily.*

Let $P : \overline{\Delta}^{m,m-2}(R) \to R$ denote any homomorphism of the type $\Pi_{1,\ldots,3,\ldots,1} - \Pi_{1,\ldots,2,\ldots,2,\ldots,1}$. The above lemma yields

COROLLARY 1. $P([r]) = r^2 - r$ *for any* $r \in R$.

The following fact generalizes Proposition 2.2 of [4].

THEOREM 2. *The submodule* $[R] = R\{[r];\ r \in R\}$ *of* $\overline{\Delta}^{m,m-2}(R)$ *is isomorphic under* P *to the ideal* $I(R) = (r^2 - r; r \in R)$. *The inverse homomorphism splits* P, *therefore* $\overline{\Delta}^{m,m-2}(R) = \mathrm{Ker}(P) \oplus [R]$.

Proof. By the above corollary, P gives an epimorphism of $[R]$ on $I(R)$. We must check that P is mono on $[R]$. Assume that $\sum_i a_i[r_i] \in \mathrm{Ker}(P)$ for some $a_i, r_i \in R$. This means that $\sum_i a_i(r_i^2 - r_i) = 0$, and we define $s = \sum_i a_i r_i^2 = \sum_i a_i r_i$. Consider the element

$$u = \sum_i a_i(r_i x_1, x_2, \ldots, x_{m-1}) \in \overline{\Delta}^{m,m-1}(R)$$

where $\{x_1,\dots,x_{m-1}\}$ is the standard basis of R^{m-1}. Since $\overline{h}^{m,m-1}$ is mono (see Section 1) and

$$\begin{aligned}\overline{h}^{m,m-1}(u) &= \sum_i a_i(r_i^2 x_1^{(2)} x_2\dots x_k + r_i x_1 x_2^{(2)}\dots x_k + r_i x_1 x_2 \dots x_{m-1}^{(2)})\\ &= s(x_1^{(2)} x_2\dots x_k + x_1 x_2^{(2)}\dots x_k + x_1 x_2\dots x_{m-1}^{(2)})\\ &= \overline{h}^{m,m-1}(s(x_1,\dots,x_{m-1}))\end{aligned}$$

it follows that $u = s(x_1,\dots,x_{m-1})$. Returning to $\overline{\Delta}^{m,m-2}(R)$ by the substitution $x_{m-1} = x_1$ we obtain

$$\sum_i a_i(r_i x_1, x_1,\dots,x_{m-2}) = s(x_1,x_1,\dots,x_{m-2}).$$

Symmetric considerations give us also

$$\sum_i a_i(x_1,\dots,r_i x_j, x_j,\dots,x_{m-2}) = s(x_1,\dots,x_j,x_j,\dots,x_{m-2})$$

for $j = 2,\dots,m-2$, and therefore

$$\begin{aligned}\sum_i a_i[r_i] = {} & \sum_i a_i(r_i x_1, x_1,\dots,x_{m-2}) + \dots + \sum_i a_i(x_1,\dots,r_i x_{m-2}, x_{m-2})\\ & -\sum_i a_i r_i^2(x_1,x_1,\dots,x_{m-2}) - \dots - \sum_i a_i r_i^2(x_1,\dots,x_{m-2},x_{m-2})\\ & - 3\sum_i a_i(r_i - r_i^2)(x_1,\dots,x_{m-2})\\ = {} & s(x_1,x_1,\dots,x_{m-2}) + \dots + s(x_1,\dots,x_{m-2},x_{m-2})\\ & - s(x_1,x_1,\dots,x_{m-2}) - \dots - s(x_1,\dots,x_{m-2},x_{m-2}) = 0. \ \blacksquare\end{aligned}$$

COROLLARY 2. *For any $r, s \in R$ we have*

(1) $[r+s] = [r] + [s] + rs[2]$,
(2) $[rs] = r[s] + s^2[r]$,
(3) $(r^2 - r)[s] = (s^2 - s)[r]$,
(4) $2[r] = (r^2 - r)[2]$, $[2r] = (2r^2 - r)[2]$,
(5) $[r] = [1 - r]$, $[0] = [1] = 0$, $[2] = [-1]$,
(6) *if* $r^2 - r = 2s$ *then* $[r] = s[2]$,
(7) *if* s *is invertible then* $[s^{-1}] = -s^{-3}[s]$.

For any $r \in R$, let us introduce the following element of $\overline{\Delta}^{m,m-2}(R)$:

$$\begin{aligned}S(r) &:= \langle r\rangle - (r^3 + (m-3)r)(x_1,x_2,\dots,x_{m-2}) + (1-r)[r]\\ &= (rx_1,x_2,\dots,x_{m-2}) + (x_1,rx_2,\dots,x_{m-2}) + \dots + (x_1,x_2,\dots,rx_{m-2})\\ &\quad - (r^3 + (m-3)r)(x_1,x_2,\dots,x_{m-2}) + (1-r)[r].\end{aligned}$$

It follows immediately from Lemma 1 that

COROLLARY 3. $S(r) \in \mathrm{Ker}(\overline{h})$.

For $m = 4$ the element

$$S(r) = (rx_1, x_2) + (x_1, rx_2) - (r^3 + r)(x_1, x_2) + (1 - r)[r]$$

is zero ([4, Lemma 3.2]). In contrast, for $m = 5$ the element

$$S(r) = (rx_1, x_2, x_3) + (x_1, rx_2, x_3) + (x_1, x_2, rx_3) - (r^3 + 2r)(x_1, x_2, x_3) + (1 - r)[r]$$

is, in general, non-zero, because the corresponding relation is not satisfied by the following regular 5-application $f : \mathbb{Z}^3 \to \mathbb{Z}_2$ defined in [5, p. 178]:

$$\begin{aligned} f(rx_1 + sx_2 + tx_3) = r\,&\frac{s(s+1)}{2}\frac{t(t+1)}{2} + s\,\frac{r(r+1)}{2}\frac{t(t+1)}{2} \\ &+ t\frac{r(r+1)}{2}\,\frac{s(s+1)}{2} \\ &+ rst\left(1 + \frac{r(r+1)}{2} + \frac{s(s+1)}{2} + \frac{t(t+1)}{2}\right) \bmod 2. \end{aligned}$$

In fact, $(rx_1, sx_2, tx_3) = f(rx_1 + sx_2 + tx_3)$ gives us $(x_1, x_2, x_3) = 1$ and $(-x_1, x_2, x_3) = (x_1, -x_2, x_3) = (x_1, x_2, -x_3) = 0$, hence

$$\widetilde{f}(S(-1)) = (-x_1, x_2, x_3) + (x_1, -x_2, x_3) + (x_1, x_2, -x_3) + (x_1, x_2, x_3) = 1.$$

Finally, let us introduce the element

$$C_1(r, s) := (rx, sy, -) - r(x, sy, -) - s(rx, y, -) + rs(x, y, -)$$

of $\overline{\Delta}^{m,m-2}(R)$, which satisfies the following:

LEMMA 2.

$$\begin{aligned} \overline{h}(C_1(r, s)) &= (r - r^2)(s - s^2)((2, 2, 1, \ldots, 1)) \\ &= (r - r^2)(s - s^2)x_1^{(2)}x_2^{(2)}x_3^{(1)} \ldots x_{m-2}^{(1)}. \end{aligned}$$

Proof. Observe that $\Pi_{i_1,i_2,\ldots,i_k}(C_1(r, s))$ is zero if $i_1 = 1$ or $i_2 = 1$. In the remaining case, $\Pi_{2,2,1\ldots,1}(C_1(r, s)) = (r - r^2)(s - s^2)$. ∎

3. The submodule $\overline{\Delta}_1$. Let us consider the submodule

$$\overline{\Delta}_1 = \overline{\Delta}_1(R) := R\{(rx_1, x_2, \ldots, x_{m-2});\ r \in R\} \subset \overline{\Delta}^{m,m-2}(R).$$

Observe that $\overline{h}(rx_1, x_2, \ldots, x_{m-2}) = \sum_{(i)} r^{i_1} x_1^{(i_1)} \ldots x_{m-2}^{(i_{m-2})}$; then the conditions $i_j \geq 1$ and $i_1 + \cdots + i_{m-2} = m$ give $i_1 = 1, 2$ or 3. Hence we have only three different coordinate homomorphisms on $\overline{\Delta}_1$. Let $\Pi_1, \Pi_2, \Pi_3 : \overline{\Delta}_1 \to R$ denote these homomorphisms, that is, compositions of the homomorphism $\overline{h}|_{\overline{\Delta}_1}$ with suitable projections:

$$\Pi_i(rx_1, x_2, \ldots, x_{m-2}) = r^i, \quad i = 1, 2, 3.$$

In the previous notation we have, for example,

$$\Pi_1 = \Pi_{1,\ldots,1,3}|_{\overline{\Delta}_1}, \qquad \Pi_2 = \Pi_{2,2,1,\ldots,1}|_{\overline{\Delta}_1}, \qquad \Pi_3 = \Pi_{3,1,\ldots,1}|_{\overline{\Delta}_1}.$$

Define $\Pi = \Pi_1$, $P = \Pi_1 - \Pi_2$, $Q = \Pi_1 - \Pi_3$ and observe that the definition of P is compatible with the previous notation. Then

$$\mathrm{Ker}(\overline{h}|_{\Delta_1}) = \mathrm{Ker}(\Pi_1) \cap \mathrm{Ker}(\Pi_2) \cap \mathrm{Ker}(\Pi_3) = \mathrm{Ker}(\Pi) \cap \mathrm{Ker}(P) \cap \mathrm{Ker}(Q).$$

In what follows, we will write $x_1 = x$, and abbreviate $(rx_1, x_2, \ldots, x_{m-2})$ to $(rx, -)$, and $(rx_1, sx_1, x_2 \ldots, x_{m-2})$ to $(rx, sx, -)$. Then we have $\Pi, P, Q : \overline{\Delta}_1 \to R$,

$$\Pi(rx, -) = r, \qquad P(rx, -) = r - r^2, \qquad Q(rx, -) = r - r^3.$$

Since $(rx, sx, -) = ((r+s)x, -) - (rx, -) - (sx, -)$ we also have

Corollary 4.
(1) $\Pi(rx, sx, -) = 0$,
(2) $P(rx, sx, -) = -2rs$,
(3) $Q(rx, sx, -) = -3(r^2 s + rs^2)$.

4. Computation of Ker(Q)**.** Consider first the homomorphism $Q : \overline{\Delta}_1 \to R$, $Q(rx, -) = r - r^3$. The goal of this section is to prove the following generalization of [4, Theorem 3.1], using, in fact, the methods of the original proof.

Theorem 3. *The submodule* Ker(Q) *is generated by elements of the following two types*:

(1) $(rsx, -) - r(sx, -) - s^3(rx, -)$, $r, s \in R$,
(2) $3(rx, -) + (1 - r)(rx, x, -)$, $r \in R$.

It is easy to compute that the above elements belong to Ker(Q). Conversely, observe that $Q(Q(u)x, -) = Q(u) - Q(u)^3 = (1 - Q(u)^2)Q(u)$ for $u \in \overline{\Delta}_1$. If we define $T(u) = (1 - Q(u)^2)u - (Q(u)x, -)$ then the above gives us $T(u) \in \mathrm{Ker}(Q)$. Since also $T(u) = u$ for $u \in \mathrm{Ker}(Q)$ it follows that $\mathrm{Ker}(Q) = \{T(u);\, u \in \overline{\Delta}_1\}$.

Let $\sim$ and $\equiv$ denote the congruence relations modulo the elements of the type (1) and of both types of Theorem 3, respectively. We must prove that $T(u) \equiv 0$ for any $u \in \overline{\Delta}_1$. First we need some lemmas.

Lemma 3. *$Q(u)v \sim Q(v)u$ for $u, v \in \overline{\Delta}_1$.*

Proof. Since Q is linear, it suffices to assume that $u = (rx, -)$, $v = (sx, -)$. Then

$$\begin{aligned} Q(u)v - Q(v)u &= (r-r^3)(sx,-) - (s-s^3)(rx,-) \\ &= (r(sx,-) + s^3(rx,-)) - (s(rx,-) + r^3(sx,-)) \\ &\sim (rsx,-) - (srx,-) = 0. \ \blacksquare \end{aligned}$$

LEMMA 4. *For any elements $r \in R$ and $u, v \in \overline{\Delta}_1$ we have*

(1) $T(ru) \sim r^3 T(u)$,
(2) $T(u+v) \sim T(u) + T(v)$.

Proof. (1) Using Lemma 3 we compute that

$$\begin{aligned} &T(ru) - r^3T(u) \\ &\quad = (1 - r^2Q(u)^2)ru - (rQ(u)x,-) - r^3(1 - Q(u)^2)u + r^3(Q(u)x,-) \\ &\quad = ru - r^3Q(u)^2u - r^3u + r^3Q(u)^2u - (rQ(u)x,-) + r^3(Q(u)x,-) \\ &\quad \sim (r - r^3)u - Q(u)(rx,-) = Q(rx,-)u - Q(u)(rx,-) \sim 0. \end{aligned}$$

(2) First observe that

$$\begin{aligned} &T(u) + T(v) - T(u+v) \\ &= ((1 - Q(u)^2)u - (Q(u)x,-)) + ((1 - Q(v)^2)v - (Q(v)x,-)) \\ &\quad - (1 - Q(u+v)^2)(u+v) - (Q(u+v)x,-) \\ &= (1 - Q(u)^2)u + (1 - Q(v)^2)v - (1 - Q(u)^2 - Q(v)^2 - 2Q(u)Q(v))(u+v) \\ &\quad + (Q(u+v)x,-) - (Q(u)x,-) - (Q(v)x,-) \\ &= Q(v)^2u + Q(u)^2v + 2Q(u)Q(v)(u+v) + (Q(u)x, Q(v)x,-). \end{aligned}$$

Because of (A), Lemma 3 and Corollary 4 we get

$$\begin{aligned} &(Q(u)x, Q(v)x,-) \\ &\qquad = Q(u)(x, Q(v)x,-) + Q(v)(Q(u)x, x,-) - Q(u)Q(v)(x,x,-) \\ &\qquad \sim Q(x, Q(v)x,-)u + Q(Q(u)x,x,-)v - Q(u)Q(x,x,-)v \\ &\qquad = -3(Q(v)^2 + Q(v))u - 3(Q(u)^2 + Q(u))v + 6Q(u)v \\ &\qquad \sim -3(Q(v)+1)Q(u)v - 3Q(u)^2v + 3Q(u)v \\ &\qquad = -3Q(u)(Q(u) + Q(v))v. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} &Q(v)^2u + Q(u)^2v + 2Q(u)Q(v)(u+v) \\ &\ \sim Q(u)Q(v)v + Q(u)^2v + 2Q(u)^2v + 2Q(u)Q(v)v = 3Q(u)(Q(u)+Q(v))v \end{aligned}$$

by Lemma 3. This completes the proof. ■

LEMMA 5. *For any $r \in R$ we have*

(a) $(x,-) \sim 0$,
(b) $(r^2x,-) \sim (r + r^3)(rx,-)$,

(c) $(r^3x,-) \sim (r^2+r^4+r^6)(rx,-)$,
(d) $(-x,-) \equiv 0$,
(e) $(-rx,-) \equiv -(rx,-)$,
(f) $(rx,-rx,-) \equiv 0$,
(g) $(rx,-r^3x,-) \equiv -r^4(1-r)(rx,x,-)$,
(h) $((r-r^3)x,-) \equiv (1-r^2-r^4-r^6)(rx,-) - r^4(1-r)(rx,x,-)$.

Proof. (a) Put $r = s = 1$ in element (1) of Theorem 3.

(b) $(r^2x,-) = (r\cdot rx,-) \sim r(rx,-) + r^3(rx,-) = (r+r^3)(rx,-)$.

(c) We have $(r^3x,-) = (r^2\cdot rx,-) \sim r^2(rx,-)+r^3(r^2x,-) \sim r^2(rx,-)+r^3(r+r^3)(rx,-) = (r^2+r^4+r^6)(rx,-)$.

(d) It follows from (a) that $(-x,x,-) \sim -(-x,-)$. Then putting $r=-1$ in (2) we obtain $(-x,-)\equiv 0$.

(e) Putting $s=-1$ in (1) we get $(-rx,-) \sim r(-x,-) - (rx,-) \equiv -(rx,-)$ by (d).

(f) $(rx,-rx,-) = ((r-r)x,-) - (rx,-) - (-rx,-) \equiv -(rx,-) + (rx,-) = 0$ by (e).

(g) Using Theorem 1(3),(4) and (f) we compute that

$$\begin{aligned}
(rx,-r^3x,-) &= (rx, r^2\cdot(-r)x,-)\\
&= r^4(rx,-rx,-) - r(rx,r^2x,-) + r^5(rx,x,-)\\
&\equiv -r(rx,(-r)^2x,-) + r^5(rx,x,-)\\
&= -r((r^2-r)(rx,-rx,-) + r^3(rx,x,-)) + r^5(rx,x,-)\\
&= -r^4(1-r)(rx,x,-).
\end{aligned}$$

(h) Using (e), (g) and (c) we get

$$\begin{aligned}
((r-r^3)x,-) &= (rx,-) + (-r^3x,-) + (rx,-r^3x,-)\\
&\equiv (rx,-) - (r^3x,-) - r^4(1-r)(rx,x,-)\\
&\equiv (1-r^2-r^4-r^6)(rx,-) - r^4(1-r)(rx,x,-). \blacksquare
\end{aligned}$$

Proof of Theorem 3. Because of Lemma 4, it suffices to check that $T(rx,-) \equiv 0$ for $r\in R$. Using Lemma 5(h) we compute that

$$\begin{aligned}
T(rx,-) &= (1-(r-r^3)^2)(rx,-) - ((r-r^3)x,-)\\
&\equiv (1-r^2+2r^4-r^6)(rx,-) - (1-r^2-r^4-r^6)(rx,-)\\
&\quad + r^4(1-r)(rx,x,-) = 3r^4(rx,-) + r^4(1-r)(rx,x,-) \equiv 0
\end{aligned}$$

by (2). This completes the proof.

5. Submodules $\mathrm{Ker}(Q) \cap \mathrm{Ker}(P)$ **and** $\mathrm{Ker}(\overline{h}) \cap \overline{\Delta}_1$**.** Recall that

$$\mathrm{Ker}(\overline{h}) \cap \overline{\Delta}_1 = \bigcap_{i=1}^{3} \mathrm{Ker}(\Pi_i) = \mathrm{Ker}(\Pi) \cap \mathrm{Ker}(P) \cap \mathrm{Ker}(Q).$$

Observe that, for any $r, s \in R$,

$$\Pi((rsx, -) - r(sx, -) - s^3(rx, -)) = -rs^3,$$
$$\Pi(3(rx, -) + (1 - r)(rx, x, -)) = 3r,$$

by Corollary 4. Then, using the element $(x, -) \in \mathrm{Ker}(Q)$, we can exchange the generators of $\mathrm{Ker}(Q)$ from Theorem 2 in the following way:

Corollary 5. $\mathrm{Ker}(Q)$ *is generated by the elements*

(1′) $C_2(r, s) := (rsx, -) - r(sx, -) - s^3(rx, -) + rs^3(x, -)$, $r, s \in R$,
(2′) $C_3(r) := 3(rx, -) - 3r(x, -) + (1 - r)(rx, x, -)$, $r \in R$,
(3′) $(x, -)$.

The profit we get is the following: $C_2(r, s), C_3(r) \in \mathrm{Ker}(Q) \cap \mathrm{Ker}(\Pi)$ and $\Pi(x, -) = 1$.

Corollary 6.
(1) $\mathrm{Ker}(Q) = (\mathrm{Ker}(Q) \cap \mathrm{Ker}(\Pi)) \oplus R(x, -)$,
(2) $\mathrm{Ker}(Q) \cap \mathrm{Ker}(\Pi)$ *is generated by the elements* $C_2(r, s)$, $C_3(r)$ *for* $r, s \in R$.

It is easy to compute that

$$P(C_2(r, s)) = (r - r^2)(s^2 - s^3), \qquad P(C_3(r)) = r - r^2,$$

hence for any $r, s \in R$ the elements $C_2(r, s) - (s^2 - s^3)C_3(r)$ belong to $\mathrm{Ker}(P)$, and consequently to $\mathrm{Ker}(\overline{h})$.

Corollary 7. $\mathrm{Ker}(Q)$ *is generated by the following elements*:

(1) $B_2(r, s) := C_2(r, s) - (s^2 - s^3)C_3(r)$, $r, s \in R$,
(2) $C_3(r), r \in R$,
(3) $(x, -)$

and $B_2(r, s) \in \mathrm{Ker}(\overline{h}) \cap \overline{\Delta}_1$. *In other words*,

$$\mathrm{Ker}(Q) = B_2 + C_3 + R(x, -)$$

where

$$B_2 = R\{B_2(r, s);\ r, s \in R\} \subset \mathrm{Ker}(\overline{h}) \cap \overline{\Delta}_1, \qquad C_3 = R\{C_3(r);\ r \in R\}.$$

It follows from the above that $B_2(r, s)$ and $C_3(r)$ generate $\mathrm{Ker}(Q) \cap \mathrm{Ker}(\Pi)$. We will show that the elements $B_2(r, s)$ and $(x, -)$ generate $\mathrm{Ker}(Q) \cap \mathrm{Ker}(P)$, and the elements $B_2(r, s)$ generate $\mathrm{Ker}(\overline{h}) \cap \overline{\Delta}_1$, that is, $\mathrm{Ker}(\overline{h}) \cap \overline{\Delta}_1 = B_2$. First we prove some auxiliary facts.

Lemma 6. *For any $r, s \in R$, we have*

(1) $2(rx, sx, -) = 2rs(x, x, -) + (rs^2 + r^2 s - 2rs)(x, x, x, -)$,
(2) $C_3(2) = (x, x, -) - (x, x, x, -)$,
(3) $(-rx, -) + (rx, -) = r^2 C_3(2)$,
(4) $C_2(r, -1) = (r^2 - r)C_3(2)$.

Proof. (1) Theorem 1(5), for $r = 2$, gives us

$$2(sx, x, -) = (s^2 - s)(2x, x, -) + (4s - 2s^2)(x, x, -).$$

Hence using (A) we obtain

$$\begin{aligned}
2(rx, sx, -) &= 2(r(x, sx, -) + s(rx, x, -) - rs(x, x, -)) \\
&= r((s^2 - s)(2x, x, -) + (4s - 2s^2)(x, x, -)) \\
&\quad + s((r^2 - r)(2x, x, -) + (4r - 2r^2)(x, x, -)) - 2rs(x, x, -) \\
&= (rs^2 + r^2 s - 2rs)(2x, x, -) + (6rs - 2rs^2 - 2r^2 s)(x, x, -) \\
&= (rs^2 + r^2 s - 2rs)(x, x, x, -) + (2rs^2 + 2r^2 s - 4rs)(x, x, -) \\
&\quad + (6rs - 2rs^2 - 2r^2 s)(x, x, -) \\
&= (rs^2 + r^2 s - 2rs)(x, x, x, -) + 2rs(x, x, -).
\end{aligned}$$

(2) We have $C_3(2) = 3(2x, -) - 3 \cdot 2(x, -) - (2x, x, -) = 3(x, x, -) - (2(x, x, -) + (x, x, x, -)) = (x, x, -) - (x, x, x, -)$.

(3) Using (A), (A2), (1) and (2) we compute that

$$\begin{aligned}
(-rx, -) + (rx, -) &= -(-rx, rx, -) = (rx, rx, -) + (-rx, rx, rx, -) \\
&= 2r(rx, x, -) - r^2(x, x, -) - r^3(x, x, x, -) \\
&= 2r^2(x, x, -) + (r^3 - r^2)(x, x, x, -) - r^2(x, x, -) \\
&\quad - r^3(x, x, x, -) \\
&= r^2((x, x, -) - (x, x, x, -)) = r^2 C_3(2).
\end{aligned}$$

(4) Since $C_2(r, -1) = (-rx, -) - r(-x, -) + (rx, -) - r(x, -)$ we can apply (3) to write it as $r^2 C_3(2) - rC_3(2)$. ∎

Proposition 1. *For any $r, s \in R$ we have*

(1) $C_3(r + s) = C_3(r) + C_3(s) + rsC_3(2)$,
(2) $C_3(rs) - rC_3(s) - s^2 C_3(r) = 3B_2(r, s) - (s^2 - s^3)B_2(r, -1)$.

Proof. (1) Observe that

$$\begin{aligned}
&C_3(r + s) - C_3(r) - C_3(s) \\
&\qquad = 3((r + s)x, -) - 3(r + s)(x, -) + (1 - (r + s))((r + s)x, x, -)
\end{aligned}$$

$$
\begin{aligned}
&- 3(rx,-) + 3r(x,-) - (1-r)(rx,x,-)\\
&- 3(sx,-) + 3s(x,-) - (1-s)(sx,x,-)\\
={}& 3(rx,sx,-) + (1-(r+s))((rx,x,-) + (sx,x,-) + (rx,sx,x,-))\\
&- (1-r)(rx,x,-) - (1-s)(sx,x,-)\\
={}& 3(rx,sx,-) - r(sx,x,-) - s(rx,x,-) + (1-r-s)(rx,sx,x,-).
\end{aligned}
$$

Using (A2), Lemma 6 and (A) we find that the above is equal to

$$
\begin{aligned}
&2(rx,sx,-) + ((rx,sx,-) - r(sx,x,-) - s(rx,x,-))\\
&\qquad + (rs - r^2s - rs^2)(x,x,x,-)\\
&\qquad = 2rs(x,x,-) + (rs^2 + r^2s - 2rs)(x,x,x,-) - rs(x,x,-)\\
&\qquad\quad + (rs - r^2s - rs^2)(x,x,x,-)\\
&\qquad = rs(x,x,-) - rs(x,x,x,-) = rsC_3(2).
\end{aligned}
$$

(2) Observe that

$$
\begin{aligned}
&C_3(rs) - rC_3(s) - s^3C_3(r) - 3C_2(r,s)\\
&\qquad = 3(rsx,-) - 3rs(x,-) + (1-rs)(rsx,x,-)\\
&\qquad\quad - r(3(sx,-) - 3s(x,-) + (1-s)(sx,x,-))\\
&\qquad\quad - s^3(3(rx,-) - 3r(x,-) + (1-r)(rx,x,-))\\
&\qquad\quad - 3((rsx,-) - r(sx,-) - s^3(rx,-) + rs^3(x,-))\\
&\qquad = (1-rs)(rsx,x,-) - r(1-s)(sx,x,-) - (1-r)s^3(rx,x,-).
\end{aligned}
$$

By Theorem 1(3), (5), the above is equal to

$$
\begin{aligned}
&(1-rs)(r(sx,x,-) + s^2(rx,x-) - rs^2(x,x,-)) + (rs-r)(sx,x,-)\\
&\qquad + (rs^3 - s^3)(rx,x,-)\\
&\qquad = s(r-r^2)(sx,x,-) + (s^2-s^3)(rx,x,-) - (1-rs)rs^2(x,x,-)\\
&\qquad = s((s-s^2)(rx,x,-) + (rs^2 - r^2s)(x,x,-)) + (s^2-s^3)(rx,x,-)\\
&\qquad\quad + (r^2s^3 - rs^2)(x,x,-)\\
&\qquad = 2(s^2-s^3)(rx,x,-) - (s^2-s^3)(r+r^2)(x,x,-).
\end{aligned}
$$

Using Lemma 6(1) and (2) we deduce that the last element is equal to

$$
\begin{aligned}
&(s^2-s^3)(2r(x,x,-) + (r^2-r)(x,x,x,-)) - (s^2-s^3)(r+r^2)(x,x,-)\\
&\qquad = (s^2-s^3)(r-r^2)((x,x,-) - (x,x,x,-)) = (s^2-s^3)(r-r^2)C_3(2).
\end{aligned}
$$

Finally, the above and Lemma 6(4) give us

$$
\begin{aligned}
&C_3(rs) - rC_3(s) - s^2C_3(r)\\
&\qquad = 3C_2(r,s) - (s^2 - s^3)C_3(r) - (s^2 - s^3)(r^2 - r)C_3(2)\\
&\qquad = 3(C_2(r,s) - (s^2 - s^3)C_3(r)) - (s^2 - s^3)(C_2(r,-1) - 2C_3(r))\\
&\qquad = 3B_2(r,s) - (s^2 - s^3)B_2(r,-1). \blacksquare
\end{aligned}
$$

We combine the above fact with the following one, which is the main result of [6]:

THEOREM 4. *Let $C(R)$ be the R-module generated by the elements $c(r)$, $r \in R$, with relations*

(1) $c(r+s) = c(r) + c(s) + rsc(2)$, $r, s \in R$,
(2) $c(rs) = rc(s) + s^2c(r)$, $r, s \in R$.

Then there exists an R-isomorphism $p : C(R) \to I(R)$ such that $p(c(r)) = r - r^2$ for $r \in R$.

This gives us a key result of our investigation, similar to Theorem 2.

COROLLARY 8. *The homomorphism induced by P,*

$$\overline{P} : \overline{\Delta}_1/B_2 \to R, \qquad \overline{P}((rx,-) + B_2) = r - r^2,$$

is a monomorphism on the submodule $\overline{C}_3$ generated by the elements $C_3(r) + B_2$, $r \in R$, and therefore $C_3 \cap \mathrm{Ker}(P) \subset B_2$.

Proof. Since $B_2 = R\{B_2(r,s); r, s \in R\} \subset \mathrm{Ker}(P)$ the induced homomorphism $\overline{P}$ does exist. By Proposition 1, the elements $C_3(r) + B_2$ satisfy relations (1) and (2) of Theorem 4. Hence there exists a homomorphism $i : C(R) \to \overline{C}_3$ such that $i(c(r)) = C_3(r) + B_2$. Then $\overline{P}|_{\overline{C}_3} \circ i = p$. Since i is epi and p is mono, it follows that $\overline{P}|_{\overline{C}_3}$ is mono. $\blacksquare$

We now prove the main result of this section.

THEOREM 5. *The following equalities hold true*:

(1) $\mathrm{Ker}(Q) \cap \mathrm{Ker}(P) = B_2 \oplus R(x,-)$,
(2) $\mathrm{Ker}(\overline{h}) \cap \overline{\Delta}_1 = B_2$,

where B_2 is the submodule generated by the elements $B_2(r,s)$, $r, s \in R$. In particular,

(3) $\mathrm{Ker}(Q) \cap \mathrm{Ker}(P) = (\mathrm{Ker}(h) \cap \overline{\Delta}_1) \oplus R(x,-)$.

Proof. If $v \in \mathrm{Ker}(Q) \cap \mathrm{Ker}(P)$ then, by Corollary 7, we can write $v = b + \sum_i a_iC_3(r_i) + r(x,-)$ where $b \in B_2$ and $a_i, r_i, r \in R$. Since $v, b, (x,-) \in \mathrm{Ker}(P)$ it follows that $\sum_i a_iC_3(r_i)$ is in $\mathrm{Ker}(P)$, and hence belongs to B_2 by Corollary 8. Hence $v \in B_2 + R(x,-)$. If, moreover, $v \in \mathrm{Ker}(\overline{h})$, then $v = b + r(x,-)$ where $b \in B_2$ and $r \in R$; but $0 = \Pi(v) = r$, so $v = b \in B_2$. The same argument for $v = 0$ shows that B_2 and $R(x,-)$ form a direct sum. $\blacksquare$

6. The case of degree 4. The above result lets us give a short proof of the main theorem of [4]. This section can also be regarded as an introduction to the much more complicated case of degree 5.

Let $m = 4$ and let $\{x, y\}$ be the standard basis of R^2. Then the homomorphism $\overline{h} : \overline{\Delta}^{4,2}(R) \to \Gamma^{4,2}(R)$ is defined by the formula

$$\overline{h}(rx, sy) = r^3 s((3,1)) + r^2 s^2((2,2)) + rs^3((1,3))$$

where $((i,j)) = x^{(i)} y^{(j)}$. The notation $\Pi = \Pi_{1,3}, P = \Pi_{1,3} - \Pi_{2,2}, Q = \Pi_{1,3} - \Pi_{3,1}$ is compatible with other similar notations in the paper. Moreover,

$$\Pi(rx, sy) = rs^3, \qquad P(rx, sy) = rs^3 - r^2 s^2, \qquad Q(rx, sy) = rs^3 - r^3 s$$

and

$$\begin{aligned} \operatorname{Ker}(\overline{h}) &= \operatorname{Ker}(\Pi_{3,1}) \cap \operatorname{Ker}(\Pi_{1,3}) \cap \operatorname{Ker}(\Pi_{2,2}) \\ &= \operatorname{Ker}(\Pi) \cap \operatorname{Ker}(P) \cap \operatorname{Ker}(Q). \end{aligned}$$

For any $r, s \in R$, we have the following elements:

$$\begin{aligned} C_1(r,s) &= (rx, sy) - r(x, sy) - s(rx, y) + rs(x, y), \\ C_2(r,s) &= (rsx, y) - r(sx, y) - s^3(rx, y) + rs^3(x, y), \\ C_3(r) &= 3(rx, y) - 3r(x, y) + (1 - r)(rx, x, y), \\ [r] &= (rx, x, y) + (x, ry, y) - r^2((x, x, y) + (x, y, y)) - 3(r - r^2)(x, y). \end{aligned}$$

Recall that

$$\overline{h}(C_1(r,s)) = (r - r^2)(s - s^2)((2,2))$$

for $r, s \in R$ by Lemma 2. Moreover,

$$\overline{h}([r]) = (r - r^2)\sigma_2 = (r - r^2)((2,2))$$

by Lemma 1(2) and hence $\Pi([r]) = Q([r]) = 0$ and $P([r]) = r^2 - r$.

LEMMA 7. *$\overline{h}(C_3(r)) = (r^2 - r)((2,2))$ for $r \in R$.*

Proof. Since $C_3(r) \in \operatorname{Ker}(Q) \cap \operatorname{Ker}(\Pi) = \operatorname{Ker}(\Pi_{1,3}) \cap \operatorname{Ker}(\Pi_{3,1})$, and $P(C_3(r)) = r - r^2$, it follows that $\Pi_{2,2}(C_3(r)) = r^2 - r$. ∎

The above and Corollary 7 give us

COROLLARY 9. *For any $r, s \in R$ the elements*

(1) $C_1(r,s) + (s - s^2)C_3(r)$
$\quad = (rx, sy) - r(x, sy) - s(rx, y) + rs(x, y) + (s - s^2)C_3(r)$,
(2) $C_2(r,s) - (s^2 - s^3)C_3(r)$
$\quad = (rsx, y) - r(sx, y) - s^3(rx, y) + rs^3(x, y) - (s^2 - s^3)C_3(r)$,
(3) $C_3(r) + [r]$

belong to $\operatorname{Ker}(\overline{h})$.

We prove that the submodule K generated by the above elements is equal to $\mathrm{Ker}(\overline{h})$. Recall that $\overline{\Delta}_1 = R\{(rx, y);\ r \in R\}$; by symmetry we define $\overline{\Delta}_2 = R\{(x, ry);\ r \in R\}$. Note that (as pointed out in Section 2)

$$S(r) = (rx, y) + (x, ry) - (r^3 + r)(x, y) + (1 - r)[r] = 0.$$

LEMMA 8.

$$\overline{\Delta}^{4,2}(R) = \overline{\Delta}_1 + \overline{\Delta}_2 + K + [R] = \overline{\Delta}_1 + K + [R],$$
$$\mathrm{Ker}(Q) \subset K + [R] + R(x, y).$$

Proof. Using elements (1) and (3) of Corollary 9 we find that any generator (rx, sy) of $\overline{\Delta}^{4,2}(R)$ can be written modulo K as a linear combination of $(rx, y), (x, sy), (x, y)$ and $[r]$. Moreover, $(x, sy) \in \overline{\Delta}_1 + [R]$ because $S(s) = 0$. This proves the required presentations of $\overline{\Delta}^{4,2}(R)$. Let now $v \in \mathrm{Ker}(Q)$. Then $v = v_1 + k + \overline{r}$ where $v_1 \in \overline{\Delta}_1$, $k \in K$, $\overline{r} \in [R]$. Since $k, \overline{r} \in \mathrm{Ker}(Q)$ it follows that $v_1 \in \mathrm{Ker}(Q) \cap \overline{\Delta}_1$. Then Corollary 7 shows that $v_1 \in K + C_3 + R(x, y)$. Using (3) of Corollary 9 we conclude that v_1, and consequently v, belongs to $K + [R] + R(x, y)$. ∎

THEOREM 6 ([4, Corollary 4.3]). $\mathrm{Ker}(\overline{h}) = K$, *that is*, $\mathrm{Ker}(\overline{h})$ *is generated by the elements*

(1) $C_1(r, s) + (s - s^2)C_3(r)$,
(2) $C_2(r, s) - (s^2 - s^3)C_3(r)$,
(3) $C_3(r) + [r]$,

where $r, s \in R$, *or*, *equivalently*, *by the following elements of* [4]:

(1′) $C_1(r, s) - (s - s^2)[r]$,
(2′) $C_2(r, s) + (s^2 - s^3)[r]$,
(3′) $C_3(r) + [r]$.

Proof. As we know, $K \subset \mathrm{Ker}(\overline{h})$. We prove that $\mathrm{Ker}(\overline{h}) \subset K$. Let $v \in \mathrm{Ker}(\overline{h})$. Then obviously $v \in \mathrm{Ker}(Q)$, and therefore Lemma 8 gives us $v = k + \overline{r} + r(x, y)$ where $k \in K$, $\overline{r} \in [R]$ and $r \in R$. Since $\Pi(v) = \Pi(\overline{r}) = 0$ and $\Pi(x, y) = 1$ we obtain $r = 0$, and hence $v = k + \overline{r}$. Moreover, the equalities $P(v) = P(k) = 0$ give us $\overline{r} \in \mathrm{Ker}(P) \cap [R]$. Observe that P is of the type $\Pi_{1,\ldots 3,\ldots 1} - \Pi_{1,\ldots,2,\ldots,2\ldots,1}$, therefore $P|_{[R]}$ is mono by Theorem 2, and hence $\mathrm{Ker}(P) \cap [R] = 0$. This gives $\overline{r} = 0$, and consequently $v = k \in K$. ∎

7. The case of degree 5. Let $m = 5$ and let $\{x, y, z\}$ be the standard basis of R^3. The homomorphism $\overline{h} : \overline{\Delta}^{5,3}(R) \to \mathit{\Gamma}^{5,3}(R)$ is defined by the formula

$$\begin{aligned}\overline{h}(rx, sy, tz) = {} & r^3st((3,1,1)) + rs^3t((1,3,1)) + rst^3((1,1,3)) \\ & + r^2s^2t((2,2,1)) + r^2st^2((2,1,2)) + rs^2t^2((1,2,2))\end{aligned}$$

where $((i,j,k)) = x^{(i)}y^{(j)}z^{(k)}$. Let us also define

$$\Pi = \Pi_{1,1,3}, \quad P = \Pi_{1,1,3} - \Pi_{2,2,1},$$
$$P_1 = \Pi_{1,2,2} - \Pi_{2,2,1}, \quad P_2 = \Pi_{2,1,2} - \Pi_{2,2,1},$$
$$Q_1 = \Pi_{1,1,3} - \Pi_{3,1,1}, \quad Q_2 = \Pi_{1,1,3} - \Pi_{1,3,1}.$$

Thus

$$\begin{aligned}\operatorname{Ker}(\overline{h}) &= \bigcap_{\substack{i+j+k=5\\ i,j,k\ge 1}} \operatorname{Ker}(\Pi_{i,j,k})\\ &= \operatorname{Ker}(\Pi) \cap \operatorname{Ker}(P) \cap \operatorname{Ker}(P_1) \cap \operatorname{Ker}(P_2) \cap \operatorname{Ker}(Q_1) \cap \operatorname{Ker}(Q_2)\end{aligned}$$

and

$$\begin{aligned}\Pi(rx,sy,tz) &= rst^3,\ P(rx,sy,tz) = rst^3 - r^2s^2t,\\ P_1(rx,sy,tz) &= rs^2t^2 - r^2s^2t,\ P_2(rx,sy,tz) = r^2st^2 - r^2s^2t,\\ Q_1(rx,sy,tz) &= rst^3 - r^3st,\ Q_2(rx,sy,tz) = rst^3 - rs^3t.\end{aligned}$$

For any $r,s,t \in R$ define the elements

$$\begin{aligned}B(r,s,t) &= (rx,sy,tz) - r(x,sy,tz) - s(rx,y,tz) - t(rx,sy,z)\\ &\quad + rs(x,y,tz) + rt(x,sy,z) + st(rx,y,z) - rst(x,y,z),\\ C_1'(r,s) &= (x,ry,sz) - r(x,y,sz) - s(x,ry,z) + rs(x,y,z).\end{aligned}$$

We also have the elements

$$C_3(r) = 3(rx,y,z) - 3r(x,y,z) + (1-r)(rx,x,y,z).$$

Lemma 9.
(1) $B(r,s,t) \in \operatorname{Ker}(\overline{h})$.
(2) *The expression $B(r,s,t)$ is additive with respect to r,s and t.*

Proof. (1) By symmetry, it suffices to observe that

$$\begin{aligned}\Pi_{3,1,1}(B(r,s,t)) &= r^3st - r(st) - s(r^3t) - t(r^3s)\\ &\quad + (rs)t + (rt)s + (st)r^3 - rst = 0,\\ \Pi_{2,2,1}(B(r,s,t)) &= r^2s^2t - r(s^2t) - s(r^2t) - t(r^2s^2)\\ &\quad + (rs)t + (rt)s^2 + (st)r^2 - rst = 0.\end{aligned}$$

(2) It is easy to check that

$$\begin{aligned}&B(r+r',s,t) - B(r,s,t) - B(r',s,t)\\ &\quad = (rx,r'x,sy,tz) - s(rx,r'x,y,tz) - t(rx,r'x,sy,z) + st(rx,r'x,y,z),\end{aligned}$$

and this is zero by the regularity condition (A). The rest follows by symmetry. ∎

Lemma 10.

$$\overline{h}(C_1'(r,s)) = (r-r^2)(s-s^2)((1,2,2)), \qquad r,s \in R,$$
$$\overline{h}(C_3(r)) = (r^2-r)(((2,1,2)) + ((2,2,1))), \qquad r \in R.$$

Proof. The first equality follows from a symmetric version of Lemma 2. Recall that, in the notation of Section 5, we have $Q(C_3(r)) = \Pi(C_3(r)) = 0$ and $P(C_3(r)) = r - r^2$. This means that

$$\Pi_{1,1,3}(C_3(r)) = \Pi_{1,3,1}(C_3(r)) = \Pi_{1,2,2}(C_3(r)) = \Pi_{3,1,1}(C_3(r)) = 0,$$
$$\Pi_{2,1,2}(C_3(r)) = \Pi_{2,2,1}(C_3(r)) = r^2 - r.$$

Hence $\overline{h}(C_3(r)) = (r^2-r)(((2,1,2)) + ((2,2,1)))$. ∎

Recall that for any $r \in R$ we have an element

$$[r] = (rx,x,y,z) + (x,ry,y,z) + (x,y,rz,z)$$
$$- r^2((x,x,y,z) + (x,y,y,z) + (x,y,z,z)) - 3(r-r^2)(x,y,z).$$

Since

$$\overline{h}([r]) = (r-r^2)\sigma_2 = (r-r^2)(((1,2,2)) + ((2,1,2)) + ((2,2,1)))$$

it follows that

$$\overline{h}(C_3(r) + [r]) = (r-r^2)((1,2,2)).$$

Consequently, we have

Corollary 10. *$C_1'(r,s) - (s-s^2)(C_3(r) + [r]) \in \mathrm{Ker}(\overline{h})$ for $r,s \in R$.*

Let us introduce the following notation:

$$C_1^{2,3}(r,s) = C_1'(r,s) = (x,ry,sz) - r(x,y,sz) - s(x,ry,z) + rs(x,y,z),$$
$$C_1^{1,3}(r,s) = (rx,y,sz) - r(x,y,sz) - s(rx,y,z) + rs(x,y,z),$$
$$C_1^{1,2}(r,s) = C_1(r,s) = (rx,sy,z) - r(x,sy,z) - s(rx,y,z) + rs(x,y,z),$$
$$C_2^{1}(r,s) = C_2(r,s) = (rsx,y,z) - r(sx,y,z) - s^3(rx,y,z) + rs^3(x,y,z),$$
$$C_2^{2}(r,s) = (x,rsy,z) - r(x,sy,z) - s^3(x,ry,z) + rs^3(x,y,z),$$
$$C_2^{3}(r,s) = (x,y,rsz) - r(x,y,sz) - s^3(x,y,rz) + rs^3(x,y,z),$$
$$C_3^{1}(r) = C_3(r) = 3(rx,y,z) - 3r(x,y,z) + (1-r)(rx,x,y,z),$$
$$C_3^{2}(r) = 3(x,ry,z) - 3r(x,y,z) + (1-r)(x,ry,y,z),$$
$$C_3^{3}(r) = 3(x,y,rz) - 3r(x,y,z) + (1-r)(x,y,rz,z),$$

for $r,s,t \in R$. We will prove the following

Theorem 7. $\mathrm{Ker}(\overline{h}) = K$ *where K is the submodule generated by the following elements*:

(1) $B(r,s,t)$,
(2) $B_1^{2,3}(r,s) = C_1^{2,3}(r,s) - (s-s^2)(C_3^1(r) + [r])$,
(3) $B_1^{1,3}(r,s) = C_1^{1,3}(r,s) - (s-s^2)(C_3^2(r) + [r])$,
(4) $B_1^{1,2}(r,s) = C_1^{1,2}(r,s) - (s-s^2)(C_3^3(r) + [r])$,
(5) $B_2^1(r,s) = C_2^1(r,s) - (s^2-s^3)C_3^1(r)$,
(6) $B_2^2(r,s) = C_2^2(r,s) - (s^2-s^3)C_3^2(r)$,
(7) $S(r) = (rx,y,z)+(x,ry,z)+(x,y,rz)-(r^3+2r)(x,y,z)+(1-r)[r]$,

where $r,s,t \in R$.

Recall that $\overline{\Delta}_1 = R\{(rx,y,z);\ r\in R\}$. By symmetry we define $\overline{\Delta}_2 = R\{(x,ry,z);\ r\in R\}$, $\overline{\Delta}_3 = R\{(x,y,rz);\ r\in R\}$.

LEMMA 11.
$$\overline{\Delta}^{5,3}(R) = \overline{\Delta}_1 + \overline{\Delta}_2 + \overline{\Delta}_3 + K + [R] = \overline{\Delta}_1 + \overline{\Delta}_2 + K + [R].$$

Proof. Consider a generator (rx,sy,tz) of $\overline{\Delta}^{5,3}(R)$. Using $B(r,s,t)$ we can write that element modulo K as a linear combination of elements of the types (rx,sy,z), (rx,y,tz), (x,sy,tz). Using (2), (3) and (4) of Theorem 7 we can write these elements modulo K as linear combinations of elements of the types $(rx,y,z),(x,sy,z),(x,y,tz),C_3^i(r)$ and $[r]$. Since $C_3^i(r)\in\overline{\Delta}_i$, this gives the first equality. Then the second follows from (7). ∎

Let
$$K_0 = \mathrm{Ker}(P_1)\cap\mathrm{Ker}(P_2)\cap\mathrm{Ker}(Q_1)\cap\mathrm{Ker}(Q_2).$$
Then $\mathrm{Ker}(\overline{h}) = K_0\cap\mathrm{Ker}(P)\cap\mathrm{Ker}(\Pi)$.

LEMMA 12. *$K_0\cap(\overline{\Delta}_1+\overline{\Delta}_2) = K_0\cap\overline{\Delta}_1 + K_0\cap\overline{\Delta}_2$.*

Proof. First observe that $P_2|_{\overline{\Delta}_1} = Q_2|_{\overline{\Delta}_1} = 0$, $P_1|_{\overline{\Delta}_2} = Q_1|_{\overline{\Delta}_2} = 0$. Let $v\in K_0\cap(\overline{\Delta}_1+\overline{\Delta}_2)$ and $v=v_1+v_2$ where $v_1\in\overline{\Delta}_1, v_2\in\overline{\Delta}_2$. Since $P_1(v)=0=P_1(v_2)$ we also have $P_1(v_1)=0$, and $P_2(v)=0=P_2(v_1)$ gives us $P_2(v_2)=0$. Similar considerations show that $Q_1(v_1)=Q_2(v_2)=0$, hence $v_1,v_2\in K_0$, and finally $v\in K_0\cap\overline{\Delta}_1+K_0\cap\overline{\Delta}_2$. The inverse inclusion is evident. ∎

Observe that all the generators of K belong to $\mathrm{Ker}(\overline{h})$, and consequently $K\subset\mathrm{Ker}(h)\subset K_0$. Moreover,

LEMMA 13. *$[R]\subset K_0$.*

Proof. By Lemma 1 (2) we have
$$\Pi_{1,2,2}([r]) = \Pi_{2,1,2}([r]) = \Pi_{2,2,1}([r]) = r-r^2,$$
$$\Pi_{1,1,3}([r]) = \Pi_{1,3,1}([r]) = \Pi_{3,1,1}([r]) = 0,$$
and hence $P_1([r]) = P_2([r]) = Q_1([r]) = Q_1([r]) = 0$. ∎

PROPOSITION 2. *$K_0\subset K+[R]+R(x,y,z)$.*

Proof. Let $v \in K_0$. Then Lemma 11 gives $v = v_1 + v_2 + k + \overline{r}$ where $v_1 \in \overline{\Delta}_1$, $v_2 \in \overline{\Delta}_2$, $k \in K$, $\overline{r} \in [R]$. Since $v, k, \overline{r} \in K_0$ we get $v_1 + v_2 \in K_0$, and hence $v_1, v_2 \in K_0$ by Lemma 12. Therefore $v_i \in \overline{\Delta}_i \cap K_0, i = 1, 2$.

First consider $\overline{\Delta}_1 \cap K_0$. We have $P_2|_{\overline{\Delta}_1} = Q_2|_{\overline{\Delta}_1} = 0$ and, in the previous notation (Section 3), $P_1|_{\overline{\Delta}_1} = P$, $Q_1|_{\overline{\Delta}_1} = Q$. Hence $\overline{\Delta}_1 \cap K_0$ is the submodule $\mathrm{Ker}(P) \cap \mathrm{Ker}(Q)$ considered in Section 3, and then Theorem 5(1) shows that $\overline{\Delta}_1 \cap K_0 = B_2 \oplus R(x, y, z) \subset K + R(x, y, z)$. Therefore $v_1 \in K + R(x, y, z)$ and, by symmetry, $v_2 \in K + R(x, y, z)$. Hence $v \in K + [R] + R(x, y, z)$. ∎

Proof of Theorem 7. It suffices to prove that $\mathrm{Ker}(\overline{h}) \subset K$. Let $v \in \mathrm{Ker}(\overline{h})$. Then $v \in K_0$, and, by Proposition 2, $v = k + \overline{r} + r(x, y, z)$ where $k \in K \subset \mathrm{Ker}(\overline{h})$, $\overline{r} \in [R]$ and $r \in R$. Since $\Pi(v) = \Pi(k) = \Pi(\overline{r}) = 0$ and $\Pi(x, y, z) = 1$ it follows that $r = 0$. Then $\overline{r} \in \mathrm{Ker}(h) \cap [R] \subset \mathrm{Ker}(P) \cap [R]$. The homomorphism P is of the type $\Pi_{1,\ldots,3,\ldots,1} - \Pi_{1,\ldots,2,\ldots,2,\ldots,1}$, and so Theorem 2 shows that the restriction $P|_{[R]}$ is mono. This means that $\mathrm{Ker}(P) \cap [R] = 0$, therefore $\overline{r} = 0$. Finally, $v = k \in K$. ∎

Theorem 7 gives us, according to [2] (or Section 1), the main result of the paper. Observe that the symmetric versions of the following relations can obviously be omitted.

THEOREM 8. *The following relations constitute a complete* 3*-covering system for the functor* Hom^5_R : (A1), (A2), (A) *and*

$$
\begin{aligned}
\text{(B)}\quad B(r,s,t) &:= (rx, sy, tz) - r(x, sy, tz) - s(rx, y, tz) - t(rx, sy, z)\\
&\quad + rs(x, y, tz) + rt(x, sy, z) + st(x, sy, tz) - rst(x, y, z)\\
&= 0,\\
\text{(B1)}\quad B_1(r,s) &:= (x, ry, sz) - r(x, y, sz) - s(x, ry, z) + rs(x, y, z)\\
&\quad - (s - s^2)(C_3(r) + [r]) = 0,\\
\text{(B2)}\quad B_2(r,s) &:= (rsx, y, z) - r(sx, y, z) - s^3(rx, y, z) + rs^3(x, y, z)\\
&\quad - (s^2 - s^3)C_3(r) = 0,\\
\text{(S)}\quad S(r) &:= (rx, y, z) + (x, ry, z) + (x, y, rz) - (r^3 + 2r)(x, y, z)\\
&\quad + (1 - r)[r] = 0
\end{aligned}
$$

where $r, s, t \in R$, x, y, z *are arbitrary elements from the domain of a* 5*-application and*

$$
\begin{aligned}
C_3(r) &= 3(rx, y, z) - 3r(x, y, z) + (1 - r)(rx, x, y, z),\\
[r] &= (rx, x, y, z) + (x, ry, y, z) + (x, y, rz, z)\\
&\quad - r^2((x, x, y, z) + (x, y, y, z) + (x, y, z, z)) - 3(r - r^2)(x, y, z).
\end{aligned}
$$

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