COLLOQUIUM MATHEMATICUM

VOL. 117 2009 NO. 2

COMPACT HOLOMORPHICALLY PSEUDOSYMMETRIC KÄHLER MANIFOLDS

BY

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Abstract. The aim of this paper is to present the first examples of compact, simply connected holomorphically pseudosymmetric Kähler manifolds.

1. Introduction. The holomorphically pseudosymmetric Kähler manifolds were defined by Z. Olszak [O-1] in 1989 and studied in [D], [H], [Y]. A Kähler manifold (M,g,J) is called *holomorphically pseudosymmetric* if its curvature tensor R satisfies the condition

$$R.R = f\varPi.R,$$

where $R,\varPi$ act as derivations of the tensor algebra,

$$\varPi(U,V)X = \frac{1}{4}(g(V,X)U - g(U,X)V + g(JV,X)JU \\ - g(JU,X)JV - 2g(JU,V)JX)$$

is the Kähler type curvature tensor of constant holomorphic sectional curvature and $f \in C^\infty(M)$ is a smooth function. A Riemannian manifold (M,g) is called *semisymmetric* if $R.R = 0$. Until now examples of compact, holomorphically pseudosymmetric and not semisymmetric Kähler manifolds have not been known. Z. Olszak in [O-2] proved that if (M,g,J) is compact, holomorphically pseudosymmetric, has constant scalar curvature τ and $f \geq 0$ then M is locally symmetric, i.e. $\nabla R = 0$, where ∇ is the Levi-Civita connection of (M,g,J), and thus semisymmetric. He also constructed many examples of non-compact holomorphically pseudosymmetric Kähler manifolds, among others holomorphically pseudosymmetric, not semisymmetric, Kähler–Einstein manifolds with $f > 0$. The *QCH* Kähler manifolds are the Kähler manifolds admitting a smooth, two-dimensional, J-invariant distribution $\mathcal{D}$ whose holomorphic curvature $K(\pi) = R(X,JX,JX,X)$ of any J-invariant 2-plane $\pi \subset T_xM$, where $X \in \pi$ and $g(X,X) = 1$, depends only on the point x and the number $|X_{\mathcal{D}}| = \sqrt{g(X_{\mathcal{D}},X_{\mathcal{D}})}$, where $X_{\mathcal{D}}$ is the

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2010 *Mathematics Subject Classification*: 53C55, 53C25.

Key words and phrases: Kähler manifold, holomorphically pseudosymmetric curvature tensor.

DOI: 10.4064/cm117-2-6

orthogonal projection of X on $\mathcal{D}$. In this case we have

$$R(X,JX,JX,X)=\phi(x,|X_{\mathcal{D}}|)$$

where $\phi(x,t)=a(x)+b(x)t^2+c(x)t^4$ and a,b,c are smooth functions on M. Also $R=a\Pi+b\Phi+c\Psi$ for certain curvature tensors $\Pi,\Phi,\Psi\in\bigotimes^4\mathfrak{X}^*(M)$ of Kähler type. The investigation of QCH Kähler manifolds was started by G. Ganchev and V. Mihova in [G-M-1], [G-M-2]. Compact, simply connected QCH Kähler manifolds were partially classified by the author in [J]. We shall show in the present paper that any QCH Kähler manifold is holomorphically pseudosymmetric. In that way using the classification result from [J] we obtain many compact, simply connected examples of holomorphically pseudosymmetric Kähler manifolds which are not semisymmetric.

2. The curvature tensor of a QCH Kähler manifold. Let $\mathcal{E}=\mathcal{D}^{\perp}$, which is a $2(n-1)$-dimensional, J-invariant distribution. Let h denote the tensor $h=g\circ(p_{\mathcal{D}}\times p_{\mathcal{D}})$, where $p_{\mathcal{D}}$ is the orthogonal projection on $\mathcal{D}$. By $\Omega=g(J\cdot,\cdot)$ we denote the Kähler form of (M,g,J), and by ω the Kähler form of $\mathcal{D}$, i.e. $\omega(X,Y)=h(JX,Y)$. We shall recall some results proved by Ganchev and Mihova [G-M-1]. Let $R(X,Y)Z=([\nabla_X,\nabla_Y]-\nabla_{[X,Y]})Z$ and write

$$R(X,Y,Z,W)=g(R(X,Y)Z,W).$$

We shall identify $(1,3)$ tensors with $(0,4)$ tensors in this way. If R is the curvature tensor of a QCH Kähler manifold (M,g,J), then

$$R=a\Pi+b\Phi+c\Psi, \tag{2.1}$$

where $a,b,c\in C^{\infty}(M)$ and Π is the standard Kähler tensor of constant holomorphic curvature

$$\begin{aligned}\Pi(X,Y,Z,U)=\tfrac{1}{4}(&g(Y,Z)g(X,U)-g(X,Z)g(Y,U)+g(JY,Z)g(JX,U)\\&-g(JX,Z)g(JY,U)-2g(JX,Y)g(JZ,U)),\end{aligned}$$

the tensor Φ is as follows:

$$\begin{aligned}\Phi(X,Y,Z,U)=\tfrac{1}{8}(&g(Y,Z)h(X,U)-g(X,Z)h(Y,U)\\&+g(X,U)h(Y,Z)-g(Y,U)h(X,Z)+g(JY,Z)\omega(X,U)\\&-g(JX,Z)\omega(Y,U)+g(JX,U)\omega(Y,Z)-g(JY,U)\omega(X,Z)\\&-2g(JX,Y)\omega(Z,U)-2g(JZ,U)\omega(X,Y)),\end{aligned}$$

and

$$\Psi(X,Y,Z,U)=-\omega(X,Y)\omega(Z,U)=-(\omega\otimes\omega)(X,Y,Z,U).$$

Let $V=(V,g,J)$ be a real n-dimensional, $n=2k$, vector space with a scalar product g and a complex structure J such that $g(J\cdot,J\cdot)=g(\cdot,\cdot)$. Let D be a 2-dimensional, J-invariant subspace of V. By E we denote its orthogonal complement in V. Thus $V=D\oplus E$. The tensors Π,Φ,Ψ given above are of Kähler type. We have the following relations for a unit vector $X \in V$: $\Pi(X, JX, JX, X) = 1, \Phi(X, JX, JX, X) = |X_D|^2, \Psi(X, JX, JX, X) = |X_D|^4$, where X_D is the orthogonal projection of X on D and $|X| = \sqrt{g(X, X)}$. Consequently, for a tensor (2.1) defined on V we have

$$R(X, JX, JX, X) = \phi(|X_D|)$$

where $\phi(t) = a + bt^2 + ct^4$.

3. QCH Kähler manifolds are holomorphically pseudosymmetric. Let us recall that Z. I. Szabó [Sz-1], [Sz-2] classified semisymmetric Riemannian manifolds, i.e., manifolds whose curvature tensor R satisfies $R.R = 0$. In particular the products $(\mathbb{CP}^n, g_{\mathrm{can}}) \times (\Sigma, g)$, where $(\mathbb{CP}^n, g_{\mathrm{can}})$ is the complex projective space with the standard Fubini–Study metric and (Σ, g) is any Riemannian surface, are semisymmetric Kähler manifolds. It is clear that the product $(M, g) = (M(k), g_k) \times (\Sigma(l), g_l)$, where $(M(k), g_k)$ is an $(n-2)$-space, $n \geq 4$, of constant holomorphic sectional curvature k and $(\Sigma(l), g_l)$ is a 2-dimensional Riemannian surface of constant sectional curvature l, is a locally symmetric (and in particular semisymmetric) Kähler manifold. Using this fact we shall prove

Theorem 3.1. *Any QCH Kähler manifold is holomorphically pseudosymmetric, more precisely if $R = a\Pi + b\Phi + c\Psi$ is the curvature tensor of a QCH Kähler manifold (M, g, J) then*

$$R.R = (a + b/2)\Pi.R.$$

Proof. We first prove that the tensors Π, Φ, Ψ satisfy the following relations:

$$\begin{aligned} &2\Phi.\Phi = \Phi.\Pi + \Pi.\Phi, \qquad \Psi.\Psi = 0, \\ &\Psi.\Pi + \Pi.\Psi = 2(\Phi.\Psi + \Psi.\Phi). \end{aligned} \tag{3.1}$$

Let $V = T_{x_0,y_0}M$ for $(M, g) = (M(k), g_k) \times (\Sigma(l), g_l)$ where $(M(k), g_k)$ is an $(n-2)$-space, $n \geq 4$, of constant holomorphic sectional curvature k and $(\Sigma(l), g_l)$ is a 2-dimensional Riemannian surface of constant sectional curvature l, and $x_0 \in M(k)$, $y_0 \in \Sigma(l)$. If $E = T_{x_0}M$ and $D = T_{y_0}\Sigma$ then $V = E \oplus D$ is an orthogonal sum. Let R, R^k, R^l be the curvature tensors of $(M, g), (M(k), g_k), (\Sigma(l), g_l)$ respectively. If $X = Y + Z$ is a unit vector, where $X \in V$, $Y \in E$, $Z \in D$, then

$$\begin{aligned} R(X, JX, JX, X) &= R^k(Y, JY, JY, Y) + R^l(Z, JZ, JZ, Z) \\ &= (1 - |X_D|)^2 k + |X_D|^4 l, \end{aligned}$$

where $X_D = Z$ is the orthogonal projection of X on D. Thus

$$R(X, JX, JX, X) = k - 2k|X_D|^2 + (l + k)|X_D|^4.$$

Consequently,

$$R = k\varPi - 2k\varPhi + (l+k)\varPsi.$$

Let us take $l = -k$. Then $R = k(\varPi - 2\varPhi)$. Since $R.R = 0$ we obtain

$$(\varPi - 2\varPhi).(\varPi - 2\varPhi) = 0,$$

and consequently, since $\varPi.\varPi = 0$,

$$2\varPhi.\varPhi = \varPhi.\varPi + \varPi.\varPhi.$$

Now let $k = 1$ and $d = l + k \neq 0$. Then $R = \varPi - 2\varPhi + d\varPsi$. Since $R.R = 0$ we get $(\varPi - 2\varPhi + d\varPsi).(\varPi - 2\varPhi + d\varPsi) = 0$ and

$$-2\varPi.\varPhi + 4\varPhi.\varPhi + d\varPi.\varPsi - 2\varPhi.\varPi - 2d\varPhi.\varPsi + d\varPsi.\varPi - 2d\varPsi.\varPhi + d^2\varPsi.\varPsi = 0.$$

One can easily check that $\varPsi.\varPsi = 0$. Hence $\varPsi.\varPi + \varPi.\varPsi = 2(\varPhi.\varPsi + \varPsi.\varPhi)$. Note that these formulas obtained for particular Kähler manifolds are purely algebraic in $g, h, \omega, \varOmega$ and thus remain true for general tensors $\varPi, \varPhi, \varPsi$ on a complex vector space (V, J) such that $V = E \oplus D$, $JE = E$, $JD = D$, $\dim D = 2$ and a J-invariant metric g on V such that $g(E, D) = 0$ and $\omega = h(J\cdot, \cdot)$ with $h = g_{|D}$. Now let (M, g, J) be a QCH Kähler manifold with curvature tensor $R = a\varPi + b\varPhi + c\varPsi$. Then

$$\begin{aligned}
R.R &= (a\varPi + b\varPhi + c\varPsi).(a\varPi + b\varPhi + c\varPsi)\\
&= a\varPi.R + ab\varPhi.\varPi + b^2\varPhi.\varPhi + bc\varPhi.\varPsi + ac\varPsi.\varPi + bc\varPsi.\varPhi\\
&= a\varPi.R + \frac{b^2}{2}(\varPhi.\varPi + \varPi.\varPhi) + bc(\varPhi.\varPsi + \varPsi.\varPhi) + ab\varPhi.\varPi + ac\varPsi.\varPi\\
&= a\varPi.R + \frac{b^2}{2}(\varPhi.\varPi + \varPi.\varPhi) + \frac{bc}{2}(\varPsi.\varPi + \varPi.\varPsi) + a(b\varPhi + c\varPsi).\varPi\\
&= a\varPi.R + \frac{b}{2}\varPi.(b\varPhi + c\varPsi) + \frac{b}{2}(b\varPhi + c\varPsi).\varPi + aR.\varPi\\
&= \left(a + \frac{b}{2}\right)\varPi.R + \left(a + \frac{b}{2}\right)R.\varPi = \left(a + \frac{b}{2}\right)\varPi.R,
\end{aligned}$$

since $R.\varPi = 0$ in view of the equality $\nabla\varPi = 0$.

We also give another, algebraic proof of relations (3.1). Note that $\varPsi(X, Y)Z = -\omega(X, Y)J \circ \pi Z$ where $\pi : V \to D$ is the orthogonal projection and J is the complex structure. Note that $\pi \circ J = J \circ \pi$. It is easy to see that $\varPsi.\varPi = \varPsi.\varPhi = \varPsi.\varPsi = 0$. We shall show that

$$\varPi.\varPsi = 2\varPhi.\varPsi.$$

We have $\varPsi = -\omega \otimes \omega$ and consequently

$$\varPi.\varPsi = -\varPi.\omega \otimes \omega - \omega \otimes \varPi.\omega.$$

Thus it is enough to show that $\Pi.\omega = 2\Phi.\omega$. Note that

$$\Pi(U,V)W = \tfrac{1}{4}(g(V,W)U - g(U,W)V + g(JV,W)JU - g(JU,W)JV - 2g(JU,V)JW)$$

and consequently

$$\begin{aligned}-4\Pi(U,V).\omega(X,Y) &= \omega(U,Y)g(V,X) + g(V,Y)\omega(X,U)\\ &\quad - g(U,X)\omega(V,Y) - g(U,Y)\omega(X,V) + g(JV,X)\omega(JU,Y)\\ &\quad + g(JV,Y)\omega(X,JU) - g(JU,X)\omega(JV,Y) - g(JU,Y)\omega(X,JV).\end{aligned}$$

Note that

$$\begin{aligned}8\Phi(U,V)W &= g(V,W)\pi U - g(U,W)\pi V + h(V,W)U - h(U,W)V\\ &\quad + g(JV,W)\pi\circ JU - g(JU,W)\pi\circ JV + \omega(V,W)JU\\ &\quad - \omega(U,W)JV - 2g(JU,V)\pi\circ JW - 2\omega(U,V)JW.\end{aligned}$$

Thus

$$\begin{aligned}-8\Phi(U,V).\omega(X,Y) &= g(V,X)\omega(U,Y) + g(V,Y)\omega(X,U)\\ &\quad - g(U,X)\omega(V,Y) - g(U,Y)\omega(X,V) + h(V,X)\omega(U,Y) + h(V,Y)\omega(X,U)\\ &\quad - h(U,X)\omega(V,Y) - h(U,Y)\omega(X,V) + g(JV,X)\omega(JU,Y)\\ &\quad + g(JV,Y)\omega(X,JU) - g(JU,X)\omega(JV,Y) - g(JU,Y)\omega(X,JV)\\ &\quad + \omega(V,X)\omega(JU,Y) + \omega(V,Y)\omega(X,JU) - \omega(U,X)\omega(JV,Y)\\ &\quad - \omega(U,Y)\omega(X,JV)\\ &= g(V,X)\omega(U,Y) + g(V,Y)\omega(X,U)\\ &\quad - g(U,X)\omega(V,Y) - g(U,Y)\omega(X,V) + g(JV,X)\omega(JU,Y)\\ &\quad + g(JV,Y)\omega(X,JU) - g(JU,X)\omega(JV,Y) - g(JU,Y)\omega(X,JV).\end{aligned}$$

Hence

$$\Pi(U,V).\omega(X,Y) = 2\Phi(U,V).\omega(X,Y).$$

It is easy to see that $\Pi(U,V).h(X,Y) = 2\Phi(U,V).h(X,Y)$ and by easy calculations we get

$$\Phi.g = 0, \qquad \Phi.J = 0, \qquad \Phi.\Omega = 0,$$

and consequently $\Phi.\Pi = 0$. Also by direct calculations one can see that $\Pi.g = 0$, $\Pi.J = 0$, $\Pi.\Omega = 0$. Since $(\Pi - 2\Phi).\omega = 0$ and $(\Pi - 2\Phi).h = 0$ it follows that

$$(\Pi - 2\Phi).\Phi = 0.$$

Thus $\Pi.\Phi = 2\Phi.\Phi$. Summarizing, we have

$$\Pi.\Pi = \Phi.\Pi = \Psi.\Pi = \Psi.\Phi = \Psi.\Psi = 0,$$
$$\Pi.\Phi = 2\Phi.\Phi, \qquad \Pi.\Psi = 2\Phi.\Psi. \ \blacksquare$$

Now it is easy to prove

THEOREM 3.2. *There exist infinitely many mutually nonhomothetic holomorphically pseudosymmetric compact, simply connected Kähler manifolds which are not semisymmetric. The class of compact simply connected QCH Kähler manifolds is included in the class of holomorphically pseudosymmetric Kähler manifolds.*

Proof. Note that in the case of a QCH Kähler manifold (M, g, J) constructed in [J] we have $M_k = P_k \times_{S^1} \mathbb{CP}^1, k \in \mathbb{N}$, where $p : P_k \to \mathbb{CP}^{n-1}$ is the S^1-bundle over $\mathbb{CP}^{n-1}$ with a connection form θ_k and $n \geq 2$. Hence M is a holomorphic $\mathbb{CP}^1$-bundle over $\mathbb{CP}^{n-1}$. An open and dense subset of M_k is isometric to the manifold $(0, L) \times P_k$ with the metric

$$g_k = dt^2 + f(t)^2\theta_k^2 + r(t)^2 p^* h, \tag{3.2}$$

where $f = 2rr'/s$, h is the metric of constant holomorphic sectional curvature on $\mathbb{CP}^{n-1}$, $s = 2k/n$, $k \in \mathbb{N}$. The metric (3.2) extends onto the whole of M_k if the function r is positive and smooth on $(0, L)$, even at the points $0, L$, with $r' > 0$ on $(0, L)$, $r'(0) = r'(L) = 0$ and

$$2r(0)r''(0) = s, \qquad 2r(L)r''(L) = -s.$$

There exist uncountably many functions r satisfying these conditions. The curvature tensor of (M_k, g_k, J) is of the form $R = a\Pi + b\Phi + c\Psi$ and

$$a + \frac{b}{2} = 4\left(\frac{(r')^2}{r^2} + \frac{f'r'}{rf}\right) = 4\,\frac{r'}{r}\left(\frac{r'}{r} + \frac{f'}{f}\right) = 4\,\frac{r'}{r}\,(\ln fr)' = 4\,\frac{r'}{r}\,(\ln(r^2 r'))'.$$

Since

$$\lim_{t\to 0+} \ln(r^2 r') = \lim_{t\to L-} \ln(r^2 r') = -\infty,$$

it is clear that the function $a + b/2$ changes sign on M. From the results of Szabó [Sz-2] it is clear that (M_k, g_k, J) is not semisymmetric. One can also verify directly that $\Pi.R \neq 0$ for (M_k, g_k, J) and hence $R.R \neq 0$. ∎

Acknowledgments. The author would like to thank Professor Andrzej Derdziński for his valuable remarks which improved the paper.

REFERENCES

[D] R. Deszcz, *On pseudosymmetric spaces*, Bull. Soc. Math. Belg. Sér. A 44 (1992), 1–34.

[G-M-1] G. Ganchev and V. Mihova, *Kähler manifolds of quasi-constant holomorphic sectional curvatures*, Cent. Eur. J. Math. 6 (2008), 43–75.

[G-M-2] —, —, *Warped product Kähler manifolds and Bochner–Kähler metrics*, J. Geom. Phys. 58 (2008), 803–824.

[H] M. Hotloś, *On holomorphically pseudosymmetric Kählerian manifolds*, in: Geometry and Topology of Submanifolds, VII (Leuven and Brussels, 1994), World Sci., River Edge, NJ, 1995, 139–142.

[J] W. Jelonek, *Kähler manifolds with quasi-constant holomorphic curvature*, Ann. Global Anal. Geom. 36 (2009), 143–159; Erratum, 35 (2009), 443.

[O-1] Z. Olszak, *Bochner flat Kählerian manifolds with a certain condition on the Ricci tensor*, Simon Stevin 63 (1989), 295–303.

[O-2] —, *On compact holomorphically pseudosymmetric Kählerian manifolds*, preprint, 2009.

[Sz-1] Z. I. Szabó, *Structure theorems on Riemannian spaces satisfying $R(X,Y).R = 0$. I. The local version*, J. Differential Geom. 17 (1982), 531–582.

[Sz-2] —, *Structure theorems on Riemannian spaces satisfying $R(X,Y).R = 0$. II. Global versions*, Geom. Dedicata 19 (1985), 65–108.

[Y] S. Yaprak, *Pseudosymmetry type curvature conditions on Kähler hypersurfaces*, Math. J. Toyama Univ. 18 (1995), 107–136.

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Received 19 February 2009;
revised 20 March 2009 (5165)