COLLOQUIUM MATHEMATICUM

VOL. 117 2009 NO. 2

SEMIVARIATIONS OF AN ADDITIVE FUNCTION ON A BOOLEAN RING

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Abstract. With an additive function φ from a Boolean ring A into a normed space two positive functions on A, called semivariations of φ, are associated. We characterize those functions as submeasures with some additional properties in the general case as well as in the cases where φ is bounded or exhaustive.

1. Introduction. Let A be a Boolean ring and let φ be an additive function from A into a normed space. Associated with φ are two positive functions $\tilde{\varphi}$ and $\bar{\varphi}$ on A, both called semivariations of φ in the literature (see the beginning of Section 4). Each of them is increasing, subadditive and has zero value at the minimal element of A, i.e., it is a *submeasure*, in our terminology.

Theorem 3, which is one of the main results of this paper ([1]), exhibits necessary and sufficient conditions for a submeasure on A to be representable as $\tilde{\varphi}$ or $\bar{\varphi}$. Those conditions are multiple subadditivity of Lorentz [15] and property (G) introduced in [12]. We also deal with an analogous, but much simpler, problem of characterizing $\tilde{\varphi}$ and $\bar{\varphi}$ in the case where φ is additionally bounded or exhaustive (Theorem 4). The case where φ is σ-additive and A is σ-complete will be discussed in a subsequent paper [14].

A basic tool used in the proofs is a representation of multiply subadditive submeasures as upper envelopes of sets of positive additive functions due, in the finite case, to Lorentz [15] (see also Theorem 1 below). Motivated by this representation and some results of Dellacherie and Iwanik [2], we introduce what we call the degree of a multiply subadditive submeasure and present some relevant examples and observations. In particular, we give a precise estimate of the degree of a finite submodular submeasure on a finite Boolean algebra (Theorem 2).

2010 *Mathematics Subject Classification*: 28B05, 28A12, 28A60.

Key words and phrases: Boolean ring, nonatomic, submeasure, submodular, mulitiply subadditive, exhaustive, quasi-measure, normed space, additive function, semivariation.

([1]) Some results of the paper were presented at the 36th Winter School in Abstract Analysis (Lhota nad Rohanovem, Czech Republic, 2008).

DOI: 10.4064/cm117-2-9

The paper is divided into five sections. Sections 2 and 3 are concerned with submeasures while Section 4 presents some auxilary results on semivariations of a vector-valued additive function. The main results, Theorems 3 and 4, are contained in Section 5.

We note that the variation of an additive function from a Boolean algebra into an Abelian normed group is characterized, in the general and bounded cases, in [12] and, in the exhaustive case, in [13]. Some ideas used in [12] also play an essential role in the present paper.

2. Preliminaries on submeasures. *Throughout the paper A stands for a Boolean ring* with the operations of join, meet, difference and symmetric difference denoted by $\vee$, $\wedge$, $\setminus$ and $\triangle$, respectively. The natural ordering of A is denoted by $\leq$ and its minimal element by 0, respectively. For every $a \in A$ we denote by C_a the ideal in A generated by a, i.e.,

$$C_a = \{b \in A : b \leq a\}.$$

We say that A is *nonatomic* or *atomless* if for every nonzero $a \in A$ there are nonzero disjoint $a_1, a_2 \in A$ with $a_1 \vee a_2 = a$.

We call a function $\eta\colon A \to [0,\infty]$ a *submeasure* if it is increasing, subadditive and satisfies the condition $\eta(0) = 0$. We say that η is *exhaustive* if $\eta(a_n) \to 0$ whenever (a_n) is a sequence of pairwise disjoint elements in A. (This is an adaptation of Drewnowski's terminology [4, p. 277]; cf. also [22, Definition 2.1].) As is easily seen, a finite exhaustive submeasure on A is bounded, i.e.,

$$\sup\{\eta(a) : a \in A\} < \infty.$$

This accounts for the term *strongly bounded* used in the literature interchangeably with exhaustive.

Let η be a submeasure on A. We set

$$I_\eta = \{a \in A : \eta(a) < \infty\}.$$

Clearly, I_η is an ideal in A. We say that η is *semifinite* provided for every $a \in A$ we have

$$\eta(a) = \sup\{\eta(b) : b \in I_\eta \text{ and } b \leq a\}.$$

The following property the submeasure η may have is basic for our purposes:

(G) *Given $a \in A \setminus I_\eta$ and $t > 0$, there are disjoint $a_1, a_2 \in A$ with $\eta(a_1)$, $\eta(a_2) > t$ and $a_1 \vee a_2 = a$.*

For a discussion of property (G) in a less general setting see [12], especially pp. 446–447.

We denote by $\operatorname{dens}\eta$ the density character of A equipped with the topology generated by the semimetric

$$d_\eta(a,b) = \min(1, \eta(a \triangle b)) \quad \text{for all } a,\, b \in A.$$

We call a function $\eta\colon A \to [0,\infty]$ a (positive) *quasi-measure* or a *content* if it is additive and satisfies the condition $\eta(0) = 0$. Clearly, η is then a submeasure. We note that for a finite quasi-measure exhaustivity is equivalent to boundedness (see [22, Theorem 2.10]). We set

$$c(A) = \{\eta \in [0,\infty]^A : \eta \text{ is a quasi-measure}\}.$$

A function $\eta\colon A \to [0,\infty]$ is said to be *submodular* or *strongly subadditive* provided that

$$\eta(a_1 \vee a_2) + \eta(a_1 \wedge a_2) \le \eta(a_1) + \eta(a_2) \quad \text{for all } a_1,\, a_2 \in A.$$

This condition holds and, in fact, turns into equality if η is additive.

We say that $a_1, \ldots, a_n \in A$ *cover* $a \in A$ *exactly* k *times* if the following three conditions hold:

1° $a_i \le a$ for each i,

2° $a = \bigvee_{1\le i_1<\cdots<i_k\le n} \bigwedge_{j=1}^{k} a_{i_j}$;

3° $\bigwedge_{j=1}^{k+1} a_{i_j} = 0$ whenever $1 \le i_1 < \cdots < i_k < i_{k+1} \le n$.

(This definition appears in [15, p. 456], in a somewhat different wording.) We note that, in the case where A is a ring of sets, conditions 1°–3° are jointly equivalent to the following one:

$$k1_a = \sum_{i=1}^{n} 1_{a_i}.$$

Following [15, p. 455], we call a function $\eta\colon A \to [0,\infty]$ *multiply subadditive* (*m.s.*, for short) if

$$k\eta(a) \le \sum_{i=1}^{n} \eta(a_i)$$

whenever $a_1, \ldots, a_n \in A$ cover $a \in A$ exactly k times. (In fact, in [15] only finite functions are considered.) Every quasi-measure on A is m.s., with equality holding in the definition above; cf. [15, p. 457]. We shall also need the following more general result:

Lemma 1. *Every submodular function $\eta\colon A \to [0,\infty]$ is m.s.*

This lemma is essentially due to Eisenstatt and Lorentz [5, Theorem 2(β)]; see also [1, Remark 1], or [9, Lemma 3]. The converse fails to hold even for a finite submeasure η (see, e.g., [10, Example 3.2]).

The next result will be applied in the proofs of Theorem 1 in Section 3 and Theorem 3 in Section 5. For A a Boolean algebra and η a quasi-measure it is covered by [12, Proposition 1]. A part of the latter result is contained in [11, Propositions 3.1.8 and 3.1.9]. The proof below follows [11] and [12].

PROPOSITION 1. *Let η be a [m.s.] submeasure on A. Then there exist submeasures η_1 and η_2 on A such that*

(a) *η_1 is semifinite [and m.s.];*
(b) *$\eta_2(A) \subset \{0, \infty\}$;*
(c) *$\eta = \max(\eta_1, \eta_2)$* [2].

If, moreover, η has property (G), *then η_2 can be chosen with this property.*

Proof. Set

$$\eta_1(a) = \sup\{\eta(b) : b \in I_\eta \text{ and } b \leq a\}$$

for all $a \in A$. It is easily seen that η_1 is a semifinite submeasure on A. As for multiple subadditivity, it is enough to observe that, if $a_1, \ldots, a_n$ cover a exactly k times and $b \in C_a$, then $a_1 \wedge b, \ldots, a_n \wedge b$ cover b exactly k times.

Set

$$J = \{a \in A : \eta(b) = \eta_1(b) \text{ for every } b \in C_a\}.$$

Clearly, J is a hereditary subset of A with $I_\eta \subset J$. Moreover, if $a_1, a_2 \in J$, then $a_1 \vee a_2 \in J$. Indeed, for $b \in C_{a_1 \vee a_2}$ with $\eta(b) = \infty$ we have

$$\eta(b \wedge a_1) = \infty \quad \text{ or } \quad \eta(b \wedge a_2) = \infty,$$

and so $\eta_1(b) = \infty$. Thus $a_1 \vee a_2 \in J$, which shows that J is an ideal in A.

Set

$$\eta_2(a) = \begin{cases} 0 & \text{if } a \in J, \\ \infty & \text{if } a \in A \setminus J. \end{cases}$$

Then η_2 is a submeasure on A, and (b) and (c) hold.

The second part of the assertion can be established in exactly the same way as the corresponding part of [12, Proposition 1]. ■

3. Lorentz' theorem and the degree of an m.s. submeasure. The following result is due, for η finite, to Lorentz [15, Theorem 4]. In the general case the equivalence of (i) and (iii) is due to Plappert [17, Satz 3.5].

THEOREM 1. *For a positive function η on A the following three conditions are equivalent:*

(i) *η is an m.s. submeasure;*
(ii) *there exists a set Γ of finite quasi-measures on A such that* $\sup \Gamma = \eta$;
(iii) *there exists a set Γ of quasi-measures on A such that* $\sup \Gamma = \eta$.

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[2] Here and in what follows, the symbols max and sup applied to a set of positive functions on A mean the pointwise maximum and supremum of that set, respectively.

Proof. Obviously, (ii) implies (iii). The implication (iii)⇒(i) is clear, since every quasi-measure on A is m.s., as noted in the passage introducing Lemma 1 above. The implication (i)⇒(ii) can be reduced to the finite case as follows. Let η satisfy (i), and choose η_1 and η_2 according to Proposition 1. For all $a \in A$ and $b \in I_{\eta_1}$ set

$$(\eta_1)_b(a) = \eta_1(a \wedge b).$$

Then $(\eta_1)_b$ is a finite m.s. submeasure on A and

$$\eta_1 = \sup\{(\eta_1)_b : b \in I_{\eta_1}\}.$$

In view of Lorentz' theorem, there exists a set Γ_1 of finite quasi-measures on A such that $\sup \Gamma_1 = \eta_1$. On the other hand, η_2 is a quasi-measure on A, and so there exists a set Γ_2 of finite quasi-measures on A such that $\sup \Gamma_2 = \eta_2$ (see [11, Proposition 3.1.6]). Setting $\Gamma = \Gamma_1 \cup \Gamma_2$, we get (ii). ∎

We note that the implication (iii)⇒(ii) of Theorem 1 also follows from [11, Corollary 3.1.17].

Theorem 1 shows that an m.s. submeasure is "nowhere" pathological. Recall that a submeasure η on A is called *pathological* if for every $\gamma \in c(A)$ with $\gamma \le \eta$ we have $\gamma = 0$ (see [8, p. 203]; cf. also [18]). We also note that in [6, p. 21] this last term is given a weaker meaning, so that non-pathological submeasures of [6] coincide with m.s. ones, in view of Theorem 1.

Motivated by Theorem 1 and some results of Dellacherie and Iwanik [2], we say that an m.s. submeasure η on A *has degree* $\mathfrak{m}$ and write

$$\deg \eta = \mathfrak{m},$$

where $\mathfrak{m}$ is a cardinal number ≥ 1, provided $\mathfrak{m}$ is the smallest among the cardinalities of sets $\Gamma \subset c(A)$ for which (iii) above holds.

Clearly, $\deg \eta = 1$ if and only if $\eta \in c(A)$. According to [2, théorème 2], for A being the algebra of all subsets of $\{1, \ldots, n\}$, where n is a natural number ≥ 3, we have

$$\begin{aligned} &\deg \eta \le 2^n - n - 1 && \text{for each finite m.s. submeasure } \eta \text{ on } A, \\ &\deg \eta_0 = 2^{n-1} && \text{for some finite m.s. submeasure } \eta_0 \text{ on } A. \end{aligned}$$

We shall establish a more precise result for submodular submeasures.

THEOREM 2. *Let A be the algebra of all subsets of $\{1, \ldots, n\}$ where $n \ge 1$. For every finite submodular submeasure η on A we have*

$$\deg \eta \le \binom{n}{[n/2]},$$

and this estimate is best possible.

Proof. Given a chain D of elements of A and a finite submodular submeasure η on A, there exists $\gamma \in c(A)$ with

$$\gamma \le \eta \quad \text{and} \quad \gamma|D = \eta|D$$

(see [9, Example 3]). On the other hand, by a combination of classical results due to Dilworth and Sperner (see, e.g., [21, Theorems 2.1 and 4.1]), A can be covered by $\binom{n}{[n/2]}$ chains in A. Therefore, the first part of the assertion follows. To prove the remaining part, we fix $n \ge 2$ and define, for natural $1 \le k \le n$ and $a \in A$,

$$\eta_k(a) = \begin{cases} \dfrac{1}{k}\operatorname{card} a & \text{if } \operatorname{card} a < k, \\ 1 & \text{if } \operatorname{card} a \ge k. \end{cases}$$

Clearly, $\eta_k(0) = 0$ and η_k is increasing. We shall check the inequality

$$\eta_k(a_1 \vee a_2) + \eta_k(a_1 \wedge a_2) \le \eta_k(a_1) + \eta_k(a_2)$$

for $a_1, a_2 \in A$. It is enough to consider the case where $\operatorname{card} a_i < k$ for $i = 1, 2$. If $\operatorname{card}(a_1 \vee a_2) < k$, the inequality in question turns into equality. Otherwise, we have

$$\begin{aligned} \eta_k(a_1 \vee a_2) + \eta_k(a_1 \wedge a_2) &= \frac{1}{k}\,(k + \operatorname{card}(a_1 \wedge a_2)) \\ &\le \frac{1}{k}\,(\operatorname{card}(a_1 \vee a_2) + \operatorname{card}(a_1 \wedge a_2)) \\ &= \frac{1}{k}\,(\operatorname{card} a_1 + \operatorname{card} a_2) = \eta_k(a_1) + \eta_k(a_2). \end{aligned}$$

We claim that $\deg \eta_k \ge \binom{n}{k}$. Indeed, take $\Gamma \subset c(A)$ with $\sup \Gamma = \eta$. We may assume that Γ is finite. Denote by E_k the family of all k-element subsets of $\{1, \ldots, n\}$, and choose, for each $c \in E_k$, an element γ_c of Γ with $\gamma_c(c) = 1$. Since for different $c_1, c_2 \in E_k$ we have $\operatorname{card}(c_1 \wedge c_2) < k$, the map $c \mapsto \gamma_c$ is injective. Thus, the claim is established, which completes the proof. ∎

It is worth noting that the submeasure η_k defined in the proof of Theorem 2 is symmetric in the sense of [2, p. 2], i.e., $\eta_k(a)$ depends only on the cardinality of a. Moreover, for $n = 4$, η_2 coincides with the submeasure c_1 of [10, Example 3.2].

The following simple example shows that $\deg \eta$, where η is a finite m.s. submeasure, can be an arbitrary cardinal number ≥ 1. This is still so if η is defined on a Boolean σ-algebra and is order continuous (see [14, Example 1]).

Example 1. Let S be a set of cardinality $\mathfrak{m} \ge 1$ and let A stand for the ring of finite subsets of S. Set

$$\eta(0) = 0 \quad \text{and} \quad \eta(a) = 1 \text{ for } a \in A \setminus \{0\}.$$

Clearly, η is a submodular submeasure on A and $\eta = \sup\{\delta_s : s \in S\}$, where δ_s stands for the Dirac quasi-measure on A concentrated at s. Hence

$\deg \eta \le \mathfrak{m}$. To establish the other inequality, take $\Gamma \subset c(A)$ with $\sup \Gamma = \eta$. For each $s \in S$ there exists $\gamma_s \in \Gamma$ with $\gamma_s(\{s\}) > 1/2$. It follows that the map $s \mapsto \gamma_s$ is injective. This completes the argument.

In our next example we only give some estimates for $\deg \eta$. To determine its precise value might be impossible in ZFC.

EXAMPLE 2. Let A stand for the algebra of all subsets of $[0,1]$ and let η be the Lebesgue outer measure on A. It is well known that η is submodular, and so m.s. (see Lemma 1). Clearly, $\deg \eta \le 2^{2^{\aleph_0}}$. Let $C \subset A$ be such that $\eta(c) = 1$ for each $c \in C$ and $\eta(c_1 \wedge c_2) = 0$ whenever $c_1, c_2 \in C$ and $c_1 \ne c_2$. The argument used in Example 1 shows that $\deg \eta \ge \operatorname{card} C$. Now, according to classical results, we can find sets C with these properties whose cardinality is $2^{\aleph_0}$ (in ZFC; see [16]) or $2^{2^{\aleph_0}}$ (under CH; see [20]). In particular, we have

$$2^{\aleph_0} \le \deg \eta \le 2^{2^{\aleph_0}},$$

and it is consistent with ZFC that $\deg \eta = 2^{2^{\aleph_0}}$.

REMARK 1. For every m.s. submeasure η on A we have $\deg \eta \le \operatorname{dens} \eta$. Indeed, if η_0 is a submeasure on A such that $\eta_0 \le \eta$ and the set

$$\{a \in A : \eta_0(a) = \eta(a)\}$$

is dense in (A, d_η), then $\eta_0 = \eta$.

4. Preliminaries on vector-valued additive functions. Throughout this section X stands for a normed vector space over the scalar field $\mathbb{R}$ or $\mathbb{C}$. We set

$$a(A,X) = \{\varphi \in X^A : \varphi \text{ is additive}\},$$
$$ba(A,X) = \{\varphi \in a(A,X) : \varphi \text{ is bounded}\},$$
$$ea(A,X) = \{\varphi \in a(A,X) : \varphi \text{ is exhaustive}\}.$$

Recall that $\varphi \in a(A,X)$ is called *exhaustive* or *strongly bounded* or *strongly additive* provided $\varphi(a_n) \to 0$ whenever (a_n) is a sequence of pairwise disjoint elements in A (see [3, pp. 7 and 32]), [4, p. 277] and [22, Definition 2.1]). As is well known, $ea(A,X) \subset ba(A,X)$ (see, e.g., [22, Corollary 2.7]).

With each $\varphi \in a(A,X)$ we associate three positive functions on A defined by the formulas:

$$|\varphi|(a) = \sup\Big\{\sum_{i=1}^n \|\varphi(a_i)\| : a_i \in A \text{ are pairwise disjoint and } \bigvee_{i=1}^n a_i = a\Big\},$$

$$\tilde{\varphi}(a) = \sup\Big\{\Big\|\sum_{i=1}^n t_i\varphi(a_i)\Big\| : a_i \in A \text{ are pairwise disjoint and } \bigvee_{i=1}^n a_i = a, \text{ and } t_i \text{ are scalars with } |t_i| \le 1\Big\},$$

$$\bar{\varphi}(a) = \sup\{\|\varphi(b)\| : b \in C_a\}$$

for $a \in A$. The first one is a quasi-measure and is called the *variation* of φ. The others are submeasures. The notation $\|\varphi\|$ is often used for $\tilde{\varphi}$. Both $\tilde{\varphi}$ and $\bar{\varphi}$ are called *semivariations* of φ in the literature (see [3, p. 2 and Proposition I.1.11] and [22, Example 1.2]). In [4, p. 273], the term *submeasure majorant for* φ is used for $\bar{\varphi}$.

The next proposition collects some properties of $|\varphi|$, $\tilde{\varphi}$ and $\bar{\varphi}$ which will be needed later.

PROPOSITION 2. *If* $\varphi \in a(A, X)$, *then*

(a) $\bar{\varphi} \le \tilde{\varphi} \le |\varphi|$;
(b) $\tilde{\varphi} \le 4\bar{\varphi}$;
(c) $\tilde{\varphi} = \sup\{|x^*\varphi| : x^* \in M \text{ and } \|x^*\| \le 1\}$, *where* M *is an arbitrary* 1-*norming subset of* X^*;
(d) φ *is bounded* [*resp.*, *exhaustive*] *if and only if* $\bar{\varphi}$ *is bounded* [*resp.*, *exhaustive*] *if and only if* $\tilde{\varphi}$ *is bounded* [*resp.*, *exhaustive*].

Part (a) is straightforward. Part (b) and a special case of (c) with $M = X^*$ are presented in [3, Proposition I.1.11]. The proof given there works in the general case. Finally, the first equivalence of (d) is straightforward in both cases and the rest follows from (a) and (b).

Given $\varphi \in a(A, \mathbb{R})$, we set

$$\varphi_+(a) = \sup\{\varphi(b) : b \in C_a\} \quad \text{and} \quad \varphi_-(a) = \sup\{-\varphi(b) : b \in C_a\}$$

for $a \in A$. Both φ_+ and φ_- are quasi-measures on A. The following simple proposition shows how φ_+ and φ_- are related to the previously defined functions $|\varphi|$, $\tilde{\varphi}$ and $\bar{\varphi}$.

PROPOSITION 3. *If* $\varphi \in a(A, \mathbb{R})$, *then*

(a) $|\varphi| = \varphi_+ + \varphi_-$ *and* $\bar{\varphi} = \max(\varphi_+, \varphi_-)$;
(b) $|\varphi| \le 2\bar{\varphi}$;
(c) $|\varphi| = \tilde{\varphi}$.

The next two lemmas will be used in the proofs of Theorems 3 and 4 in Section 5.

LEMMA 2. *If* $\varphi \in a(A, X)$, *then both* $\tilde{\varphi}$ *and* $\bar{\varphi}$ *are m.s. and have property* (G).

Proof. To establish the first part of the assertion, we apply Theorem 1, (iii)⇒(i). In the case of $\tilde{\varphi}$ we use additionally Proposition 2(c). In the case of $\bar{\varphi}$ and X over $\mathbb{R}$ we also make use of the formula

$$\bar{\varphi} = \sup\{(x^*\varphi)_+,\ (x^*\varphi)_- : x^* \in X^* \text{ and } \|x^*\| \le 1\},$$

which follows from Proposition 3(a) via the Hahn–Banach theorem. If the scalar field of X is $\mathbb{C}$, we consider X to be a normed space over $\mathbb{R}$ (with the same norm) and note that this does not affect $\bar{\varphi}$.

To establish the second part of the assertion, fix $a \in A$ with $\bar{\varphi}(a) = \infty$ and $t > 0$. We can then find $b \in C_a$ with

$$\|\varphi(b)\| > \|\varphi(a)\| + t.$$

This implies $\bar{\varphi}(b), \bar{\varphi}(a \setminus b) > t$. Thus, $\bar{\varphi}$ has property (G). Since $\bar{\varphi} \le \tilde{\varphi} \le 4\bar{\varphi}$, by Proposition 2(a),(b), it follows that $\tilde{\varphi}$ also has property (G). ∎

In view of Lemma 2, one might ask whether $\deg \tilde{\varphi}$ and $\deg \bar{\varphi}$ are related, for arbitrary $\varphi \in a(A, X)$, in some way. The author only knows the following negative answer to this question. For $\varphi \in a(A, \mathbb{R})$ we have $\deg \tilde{\varphi} = 1$ while $\deg \bar{\varphi} = 2$ unless $\bar{\varphi} = |\varphi|$, by Proposition 3(c) and Propositions 2(a) and 3(a), respectively. On the other hand, the inequality $\deg \bar{\varphi} < \deg \tilde{\varphi}$ is also possible, as the next simple example shows.

Example 3. Let A be the algebra of all subsets of the set $\{1, 2, 3\}$. Consider $\varphi \in a(A, l_\infty^{(4)})$, which is uniquely determined by the equalities

$$\varphi(\{1\}) = (2, 0, 0, 1), \quad \varphi(\{2\}) = (0, 2, 0, -1) \quad \text{and} \quad \varphi(\{3\}) = (0, 0, 2, 1).$$

We then have

$$\tilde{\varphi}(a) = \bar{\varphi}(a) = 2 \text{ if } \operatorname{card} a \le 2, \quad \tilde{\varphi}(\{1, 2, 3\}) = 3 \quad \text{and} \quad \bar{\varphi}(\{1, 2, 3\}) = 2.$$

Hence

$$\tilde{\varphi} = \max\{2\delta_1, 2\delta_2, 2\delta_3, \delta_1 + \delta_2 + \delta_3\} \quad \text{and} \quad \bar{\varphi} = \max\{2\delta_1, 2\delta_2, 2\delta_3\},$$

where δ_i stands for the Dirac quasi-measure on A concentrated at i. As is easily seen, $\deg \tilde{\varphi} = 4$ (cf. [2, p. 3]), while $\deg \bar{\varphi} = 3$, according to Example 1.

Lemma 3. *If η is a semifinite m.s. submeasure on A, then there exist $\Gamma \subset c(I_\eta)$ and $\varphi \in a(A, l_\infty(\Gamma))$ such that $\tilde{\varphi} = \bar{\varphi} = \eta$.*

Proof. By Theorem 1, (i)⇒(ii), applied to $\eta | I_\eta$, there exists $\Gamma \subset c(I_\eta)$ such that

$$\eta(a) = \sup\{\gamma(a) : \gamma \in \Gamma\} \quad \text{for all } a \in I_\eta.$$

Define $\varphi_0 \colon I_\eta \to l_\infty(\Gamma)$ by $\varphi_0(a)(\gamma) = \gamma(a)$ for $a \in I_\eta$ and $\gamma \in \Gamma$. Clearly, $\varphi_0 \in a(I_\eta, l_\infty(\Gamma))$ and, by Proposition 2(c), we have

$$\tilde{\varphi}_0 = \bar{\varphi}_0 = \eta | I_\eta.$$

Choose $\varphi \in a(A, l_\infty(\Gamma))$ to be an arbitrary extension of φ_0 (cf. Lemma 1 of [12] and its proof). Since I_η is an ideal in A, we have

$$\tilde{\varphi} | I_\eta = \tilde{\varphi}_0 \quad \text{and} \quad \bar{\varphi} | I_\eta = \bar{\varphi}_0,$$

and so $\tilde{\varphi}$, $\bar{\varphi}$ and η coincide on I_η. Since η is semifinite, by assumption, and both $\tilde{\varphi}$ and $\bar{\varphi}$ are increasing, we conclude that φ is as desired. ∎

As an example, we note that, in view of Lemma 1, Lemma 3 applies to the Lebesgue outer measure on $\mathbb{R}$.

The following lemma will be used in the proof of Theorem 3 below.

LEMMA 4. *If A is nonatomic, then there exists $\varphi \in a(A, \mathbb{R})$ with $\varphi(A) \subset \mathbb{Q}$ and $\tilde{\varphi}(a) = \bar{\varphi}(a) = \infty$ for every nonzero $a \in A$.*

In the case where A is a Boolean algebra, this is a reformulation of [12, Lemma 3] (see Proposition 3(b),(c) above). The general case follows, since every [nonatomic] Boolean ring can be embedded as an ideal into a [nonatomic] Boolean algebra. We note that, by using the natural embedding of $\mathbb{R}$ into $\mathbb{C}$, we can deduce from Lemma 4 its complex version where we have $\varphi \in a(A, \mathbb{C})$.

REMARK 2. For A additionally assumed to be countable, Lemma 4 can be improved to the effect that φ is integer-valued and $\varphi(a) \neq 0$ for every nonzero $a \in A$ (cf. [7, Proposition 13(b)]). In this connection, we also note that [12, Remark 5] is related to [7, Proposition 6].

REMARK 3. In the special case where A is, in addition, complete and admits a strictly positive finite measure μ, Lemma 4 can also be proved as follows. Let $f\colon \mathbb{R} \to \mathbb{Q}$ be a nonzero additive function, and set $\varphi = f \circ \mu$. The additional assumptions imply that

$$\mu(C_a) = [0, \mu(a)],$$

and so $\varphi(C_a)$ is unbounded for every nonzero $a \in A$. The idea of this proof is due to Sierpiński [19, pp. 245–246].

5. Main results. Recall that, as before, A stands for an arbitrary Boolean ring.

THEOREM 3. *For $\eta\colon A \to [0, \infty]$ the following four conditions are equivalent*:

(i) *η is an m.s. submeasure and has property* (G);
(ii) *there exist a normed space X and $\varphi \in a(A, X)$ with $\tilde{\varphi} = \eta$*;
(iii) *there exist a normed space X and $\varphi \in a(A, X)$ with $\bar{\varphi} = \eta$*;
(iv) *there exist a normed space X and $\varphi \in a(A, X)$ with $\tilde{\varphi} = \bar{\varphi} = \eta$.*

Proof. Clearly, (iv) implies (iii) and (ii). In view of Lemma 2, each of the conditions (iii) and (ii) implies (i).

Suppose (i) holds. To establish (iv) with X over $\mathbb{R}$, let η_1 and η_2 be given by Proposition 1. In view of Lemma 3, there exist a set Γ and $\varphi_1 \in a(A, l_\infty(\Gamma))$ with $\tilde{\varphi}_1 = \bar{\varphi}_1 = \eta_1$. Since η_2 has property (G), the quotient Boolean ring A/I_{η_2} is nonatomic. Denote by h the canonical homomorphism of A onto A/I_{η_2}. By Lemma 4, there exists

$$\psi \in a(A/I_{\eta_2}, \mathbb{R}) \quad \text{with} \quad \tilde{\psi}(h(a)) = \bar{\psi}(h(a)) = \infty \text{ for every } a \in A \setminus I_{\eta_2}.$$

Setting $\varphi_2 = \psi \circ h$, we get $\varphi_2 \in a(A, \mathbb{R})$ with $\tilde{\varphi}_2 = \bar{\varphi}_2 = \eta_2$. Let X stand for the l_∞-sum of the Banach spaces $l_\infty(\Gamma)$ and $\mathbb{R}$, and set $\varphi = (\varphi_1, \varphi_2)$. We have $\varphi \in a(A, X)$ and

$$\tilde{\varphi} = \max(\tilde{\varphi}_1, \tilde{\varphi}_2) = \max(\bar{\varphi}_1, \bar{\varphi}_2) = \bar{\varphi} = \eta.$$

Thus, (iv) holds in the real case. In the complex case, we only have to replace "$\mathbb{R}$" by "$\mathbb{C}$" throughout the argument. ■

REMARK 4 (cf. [12, Remark 6]). The space X constructed in the proof of Theorem 3, (i)⇒(iv), is, in fact, linearly isometric to an l_∞-space. There is, however, no point in including this in the formulation of condition (iv), since every normed space is linearly isometric to a subspace of $l_\infty(\Gamma)$ for some set Γ, as a consequence of the Hahn–Banach theorem.

REMARK 5. In Theorem 3 we cannot restrict the size of X, keeping A arbitrary. (This is in contrast with both [12, Theorem 1] and [13, Theorems 1 and 2].) Indeed, for every $\varphi \in a(A, X)$ and every 1-norming subset M of X^* we have

$$\deg \tilde{\varphi} \le \operatorname{card} M \quad \text{and} \quad \deg \bar{\varphi} \le 2 \operatorname{card} M,$$

by Propositions 2(c) and 3(a), respectively. On the other hand, $\deg \eta$, where η is a finite m.s. submeasure, can be an arbitrary cardinal number ≥ 1 (see Example 1).

From Theorem 3 we immediately get the following corollary.

COROLLARY. *Let X be a normed space and let $\varphi \in a(A, X)$.*

(a) *There exist a normed space Y and $\chi \in a(A, Y)$ with $\tilde{\chi} = \bar{\chi} = \tilde{\varphi}$.*
(b) *There exist a normed space Z and $\psi \in a(A, Z)$ with $\tilde{\psi} = \bar{\psi} = \bar{\varphi}$.*

THEOREM 4. *For $\eta\colon A \to [0, \infty)$ the following four conditions are equivalent*:

(i) *η is a bounded [resp., exhaustive] m.s. submeasure*;
(ii) *there exist a normed space X and $\varphi \in ba(A, X)$ [resp., $\varphi \in ea(A, X)$] with $\tilde{\varphi} = \eta$*;
(iii) *there exist a normed space X and $\varphi \in ba(A, X)$ [resp., $\varphi \in ea(A, X)$] with $\bar{\varphi} = \eta$*;
(iv) *there exist a normed space X and $\varphi \in ba(A, X)$ [resp., $\varphi \in ea(A, X)$] with $\tilde{\varphi} = \bar{\varphi} = \eta$.*

Proof. Clearly, (iv) implies (iii) and (ii). In view of Lemma 2 and Proposition 2(d), each of the conditions (iii) and (ii) implies (i). That (i) implies (iv) follows from Lemma 3. ■

In closing, we note that Theorem 4 implies an analogue of the Corollary above for $\varphi \in ba(A, X)$ [resp., $\varphi \in ea(A, X)$].

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Received 3 November 2008;
revised 31 March 2009 (5119)