COLLOQUIUM MATHEMATICUM

VOL. 117 2009 NO. 2

A NOTE ON GLOBAL INTEGRABILITY OF UPPER GRADIENTS OF p-SUPERHARMONIC FUNCTIONS

BY

OUTI ELINA MAASALO (Helsinki) and ANNA ZATORSKA-GOLDSTEIN (Warszawa)

Abstract. We consider a complete metric space equipped with a doubling measure and a weak Poincaré inequality. We prove that for all p-superharmonic functions there exists an upper gradient that is integrable on H-chain sets with a positive exponent.

1. Introduction. Let (X,d) be a metric space equipped with a Borel measure μ. In a metric measure space the concept of an upper gradient serves as a substitute for the Sobolev gradient. Roughly speaking this allows us to control the growth of a function along a curve. Suppose that $1 \le p < \infty$ and let u be a real-valued function on X. A non-negative Borel measurable function g on X is said to be a *p-weak upper gradient* of u if

$$|u(\gamma(a)) - u(\gamma(b))| \le \int_\gamma g\,ds \tag{1.1}$$

for sufficiently many rectifiable paths in X. We recall that a path γ is rectifiable if $\operatorname{length}(\gamma) < \infty$. A precise formulation is that (1.1) holds for all paths except a family of zero p-modulus. The reader may consult, for example, [5], [6] and [13] for a discussion of upper gradients, and [6] or [7] for the definition of p-modulus and discussion of paths.

The Sobolev space on a metric measure space, called the Newtonian space $N^{1,p}(X)$, can be defined as the collection of equivalence classes of p-integrable functions with p-integrable upper gradients. If Ω is an open subset of X, the Newtonian space with zero boundary values, $N_0^{1,p}(\Omega)$, can be defined as the collection of functions in $N^{1,p}(X)$ that are zero outside Ω. The precise definitions and further information can be found in various references, e.g. [13].

Newtonian functions can be used to study the p-Dirichlet integral

$$\int_\Omega g_u^p\,d\mu, \tag{1.2}$$

where g_u is the minimal upper gradient of u in the sense that for any other

2010 *Mathematics Subject Classification*: 31C45, 49N60.
Key words and phrases: p-superharmonic functions, upper gradients, global integrability.

DOI: 10.4064/cm117-2-10

p-integrable p-weak upper gradient g of u we have $g_u \leq g$ μ-almost everywhere in X. There is a nonlinear potential theory related to minimizers of (1.2), which corresponds to the Euclidean theory for solutions of the p-Laplace equation. The so-called p-superharmonic functions play an important role in the theory.

A lower semicontinuous function is called *p-superharmonic* in Ω if it obeys the comparison principle with respect to p-harmonic functions, that is, continuous minimizers of the p-Dirichlet integral. A priori, a p-superharmonic function u does not belong to a Newtonian space. However, Kinnunen and Martio have proved that if $p > 1$ then u has an upper gradient, and that this upper gradient is locally integrable to a small exponent; see [9].

Maasalo [12] showed that p-superharmonic functions are globally integrable to a small exponent in H-chain sets. The purpose of this note is to extend the result of Kinnunen and Martio and show that an upper gradient of a p-superharmonic function is globally integrable on H-chain sets with a positive exponent.

For recent results on p-superharmonic functions and potential theory in the metric space setting, see, for example, [1]–[3], [9]–[11].

2. Preliminaries. Our notation is standard. Throughout the paper we assume that the measure of every nonempty open set is positive and that the measure of every bounded set is finite. The measure μ is assumed to be *doubling*, i.e. there exists a constant $c_d \geq 1$ such that

$$\mu(B(x, 2r)) \leq c_d \mu(B(x, r))$$

for every x in X and $r > 0$. Let $1 < p < \infty$. We assume that the space supports a *weak* $(1,p)$*-Poincaré inequality*, that is, there exist $c > 0$ and $\tau \geq 1$ such that

$$\fint_{B(x,r)} |u - u_{B(x,r)}| \, d\mu \leq cr \Big(\fint_{B(x,\tau r)} g^p \, d\mu \Big)^{1/p}$$

for all x in X, $r > 0$ and all pairs $\{u, g\}$ where u is a locally integrable function on X and g is a p-weak upper gradient of u. Here we use the convention

$$u_B = \fint_B u \, d\mu = \frac{1}{\mu(B)} \int_B u \, d\mu.$$

Finally, we assume that (X, d) is a complete metric space.

2.1. *OLD-sets and H-chain sets.* Suppose that $\Omega \subsetneq X$ is open and of finite diameter R, and set

$$\Omega_r = \{x \in \Omega \colon 0 < \operatorname{dist}(x, X \setminus \Omega) < r\}.$$

Given $0 < \delta \le 1 \le \kappa < \infty$ we say that Ω is a (δ, κ)-*OLD set* if

$$\mu(\Omega_r) \le (\kappa r/R)^\delta \mu(\Omega)$$

for all $0 < r < R$. Here *OLD* stands for "outer layer decay".

Assume that Ω is connected. Suppose that $J, L \in \mathbb{R}$, $J > 1, L \ge 1$ and that $B_* = B(z_*, r_*) \subset \Omega$ with $r_* \le \operatorname{diam}(\Omega)$. We say that a chain of balls $B_i = B(z_i, r_i) \subset \Omega$, $0 \le i \le k$, is an *H-chain* of length k for $B(x, r) \subset \Omega$ with respect to Ω and with parameters J, L, B_* if

- $(z_0, r_0) = (x, r)$ and $(z_k, r_k) = (x_*, r_*)$,
- if $0 \le i < k$, then $1/J \le r_i/r_{i+1} \le J$, and $B_i \cap B_{i+1}$ contains a ball $B' = B(z_i', r_i')$, with $r_i' = (r_i + r_{i+1})/2J$,
- $B(z_i, Lr_i) \subset \Omega$, $0 < i < k$.

Let $K \ge 1$. Now Ω is an *H-chain set* with parameters J, K, L, B_* if every ball $B(x, r) \subset \Omega$ such that $r \le \operatorname{diam}(\Omega)$ has an H-chain set of length at most $K \log_2(2r_*/r)$. By a result of [4] any H-chain set with parameters $J, K, L, B(x_*, r_*)$ has diameter at most $(8K - 1)r_*$. Thus Ω is bounded and of finite measure. Furthermore, it is easy to see that the larger L the stronger the condition. In the following we call J, K, L, B_* the parameters of Ω for short.

2.2. *Superharmonic functions.* Let Ω be an open subset of X. A function $v \in N^{1,p}_{\mathrm{loc}}(\Omega)$ is called a *p-minimizer* in Ω if

$$\int_{\Omega'} g_v^p \, d\mu \le \int_{\Omega'} g_w^p \, d\mu \tag{2.3}$$

for every open set $\Omega' \subset\subset \Omega$ and all w such that $w - v \in N_0^{1,p}(\Omega')$. A function $v \in N^{1,p}_{\mathrm{loc}}(\Omega)$ is called a *p-superminimizer* in Ω if (2.3) holds for every open set $\Omega' \subset\subset \Omega$ and all w such that $w - v \in N_0^{1,p}(\Omega')$ and $w \ge v$ μ-almost everywhere in Ω'. A function $v \colon \Omega \to (-\infty, \infty)$ is called *p-harmonic* in Ω if it is a continuous p-minimizer in Ω.

Consider an open set $\Omega' \subset\subset \Omega$ and a function $v \in N^{1,p}(\Omega')$. Denote by $h_{\Omega'}(v) = h(v)$ the unique p-harmonic function in Ω' with $v - h(v) \in N_0^{1,p}(\Omega')$ (see Theorem 3.2 in [9] and [11]).

A function $u \colon \Omega \to (-\infty, \infty]$ is called *p-superharmonic* Ω if

- u is lower semicontinuous in Ω, i.e.

$$\liminf_{x \to x_0} u(x) \ge u(x_0) \qquad \text{for every } x_0 \in \Omega,$$

- u is not identically ∞ in any component of Ω,
- for every open $\Omega' \subset\subset \Omega$ the comparison principle holds: if $v \in C(\overline{\Omega'}) \cap N^{1,p}(\Omega')$ and $v \le u$ on $\overline{\Omega'}$, then $h(v) \le u$ in Ω'.

For other equivalent ways of defining p-superharmonic functions in the metric setting we refer to [1] and [10].

The difference between p-superminimizers and p-superharmonic functions is subtle. Neither of the two classes is contained in the other. We collect here some facts about them; see [9] for proofs.

- A p-superharmonic function in Ω does not a priori belong to a Newtonian space. However, if the function is also locally bounded above, it is a p-superminimizer and belongs to $N^{1,p}_{\mathrm{loc}}(\Omega)$. In particular, if u is p-superharmonic, then $\min\{u, c\}$ is a p-superminimizer and p-superharmonic for all $c \in \mathbb{R}$.
- If a function is p-superharmonic and, in addition, it belongs to $N^{1,p}_{\mathrm{loc}}(\Omega)$ (without necessarily being bounded above), it is a p-superminimizer.
- If u is a p-superminimizer such that
$$u(x) = \operatorname*{ess\,lim\,inf}_{y\to x} u(y) \qquad \text{for every } x \in \Omega$$
then u is p-superharmonic.
- Given a p-superharmonic function in Ω and a subset $\Omega' \subset\subset \Omega$, there always exists an increasing sequence of continuous p-superminimizers that converge to the p-superharmonic function pointwise everywhere in Ω'.

3. The main result. In this section we prove that each p-superharmonic function has an upper gradient which is globally integrable to a small exponent on H-chain sets. The argument is somewhat similar to the Euclidean proof by Kilpeläinen and Koskela [8]. They proved that gradients of supersolutions of degenerate elliptic PDEs are integrable to a small exponent on balls.

The starting point for our result is the observation that a p-superharmonic function itself is globally integrable on Ω to some small but positive exponent. This was proved by the first author in [12].

THEOREM 3.1. *Let u be a positive p-superharmonic function in Ω that is an H-chain set with parameters J, K, L, B_* such that $L = 2$. Then there exists $\beta_0 > 0$ such that $u \in L^{\beta}(\Omega)$ for all $0 < \beta \le \beta_0$. The exponent β_0 depends only on p, the doubling constant, the constants in the Poincaré inequality and the H-chain set parameters of Ω.*

Using this fact we prove the following:

THEOREM 3.2. *Let u be a positive p-superharmonic function in Ω that is an H-chain set with parameters J, K, L, B_* such that $L = 2$ and $2B_* \subset \Omega$. Then u has an upper gradient G_u and there exists $s > 0$ such that $G_u \in$*

$L^s(\Omega)$. *In particular*,

$$\Big(\int_{\Omega} G_u^s\,d\mu\Big)^{1/s} \le c\Big(\int_{\Omega} u^\beta\,d\mu\Big)^{1/\beta} \tag{3.4}$$

where c *depends on* c_d, p, Ω *and* β, *where* β *is as in Theorem* 3.1.

Theorem 3.1 was originally proved in [12] under the assumption that the metric space is a *length space*, i.e. for all x and y in X,

$$d(x,y) = \inf \operatorname{length}(\gamma),$$

where the infimum is taken over all rectifiable paths joining x and y. However, a complete metric space that supports a Poincaré inequality and a doubling measure is always quasiconvex and proper. Hence, it is possible to define a new geodesic metric that is bi-Lipschitz equivalent to the original one; see [6]. Therefore, the statement of Theorem 3.1 remains true under our assumptions.

Notice also that balls are H-chain sets in length spaces. In particular, Theorem 3.2 is valid for balls, and thus generalizes the Euclidean result by Kilpeläinen and Koskela.

For the proof of Theorem 3.2 we need two auxiliary results. The first is a Caccioppoli type inequality for p-superminimizers and it is independent of the Poincaré inequality. See [10, Lemma 3.1] for the proof.

LEMMA 3.3. *Suppose that* u *is a positive* p-*superminimizer in* Ω *and let* $\varepsilon > 0$. *Let* η *be a compactly supported Lipschitz continuous function in* Ω *such that* $0 \le \eta \le 1$. *Then*

$$\int_{\Omega} \eta^p g_u^p u^{-1-\varepsilon}\,d\mu \le c\int_{\Omega} u^{p-1-\varepsilon} g_\eta^p\,d\mu, \tag{3.5}$$

where $c = (p/\varepsilon)^p$.

The next lemma holds in any metric space. For the proof, see [4, Theorem 3.3].

LEMMA 3.4. *If* Ω *is an* H-*chain set with parameters* J, K, L, B_* *such that* $2B_* \subset \Omega$ *and* $L = 1$, *then* Ω *is a* (δ,κ)-*OLD set*, *and* δ *and* κ *depend only on* J, K, $r_*/\operatorname{diam}\Omega$ *and the doubling constant.*

Proof of Theorem 3.2. We divide the proof into two steps. First, we assume that u is a positive p-superminimizer in Ω.

We decompose Ω into layers in the following way. Define $\Omega_0 = \Omega$ and

$$\Omega_j = \{x \in \Omega : 0 < \operatorname{dist}(x, X \setminus \Omega) < 2^{-j} \operatorname{diam}\Omega\}$$

for $j = 1, 2, \ldots$. Set $A_j = \Omega_{j-1} \setminus \Omega_j$, $j = 1, 2, \ldots$. Then $\Omega = \bigcup_{j=1}^{\infty} A_j$.

Let $0 < \delta \le 1$ be as in Lemma 3.4 and $\beta > 0$ as in Theorem 3.1. We choose an exponent s such that

$$0 < s < \min\left\{\frac{\delta\beta}{\delta+\beta}, p, \frac{\beta p}{p+\beta}\right\}.$$

Then $p/s > 1$, and we have

$$\begin{aligned}\int_{A_j} g_u^s\,d\mu &= \int_{A_j} g_u^s u^{-s} u^s\,d\mu \\ &\le \Big(\int_{A_j} g_u^p u^{-p}\,d\mu\Big)^{s/p}\Big(\int_{A_j} u^{ps/(p-s)}\,d\mu\Big)^{1-s/p}\end{aligned} \tag{3.6}$$

for all $j = 1, 2, \ldots$ by the Hölder inequality.

To handle the first integral on the right-hand side of (3.6), fix j and let η_j be a Lipschitz function such that $0 \le \eta_j \le 1$,

$$\eta_j = \begin{cases} 1 & \text{on } A_j, \\ 0 & \text{outside } A_{j-1} \cup A_j \cup A_{j+1}. \end{cases}$$

Furthermore, let $g_{\eta_j} \le c/2^{-j}$ and notice that the set where g_{η_j} is not zero μ-almost everywhere is included in Ω_{j-2} (1).

Since u is a positive p-superminimizer it satisfies the conclusion of Lemma 3.3 with $\varepsilon = p - 1 > 0$, and for $j \ge 2$ we get

$$\begin{aligned}\Big(\int_{A_j} g_u^p u^{-p}\,d\mu\Big)^{s/p} &\le \Big(\int_{\Omega} g_u^p u^{-p}\eta_j^p\,d\mu\Big)^{s/p} \le c\Big(\int_{\Omega} g_{\eta_j}^p\,d\mu\Big)^{s/p} \\ &\le c\Big(\int_{\Omega_{j-2}} \frac{c}{2^{-jp}}\,d\mu\Big)^{s/p} \le c\Big(\int_{\Omega_{j-2}} 2^{jp}\,d\mu\Big)^{s/p} \\ &\le c2^{js}\mu(\Omega_{j-2})^{s/p}.\end{aligned}$$

For A_1 it is easy to get

$$\Big(\int_{A_1} g_u^p u^{-p}\,d\mu\Big)^{s/p} \le c2^s\mu(\Omega)^{s/p}$$

by the Caccioppoli inequality.

(1) It is enough to set

$$\eta_j(x) = \begin{cases} 1 & \text{on } A_j, \\ 1 - c2^{j-1}\operatorname{dist}(x, A_j) & \text{on } A_{j-1}, \\ 1 - c2^{j+1}\operatorname{dist}(x, A_j) & \text{on } A_{j+1}, \end{cases}$$

and zero elsewhere, where $c = 1/\operatorname{diam}\Omega$. An easy calculation shows that η_j is Lipschitz with global Lipschitz constant $c2^{j+1}$.

Consider the second integral in (3.6). By the Hölder inequality we get, for $j \geq 1$,

$$\Big(\int_{A_j} u^{ps/(p-s)}\,d\mu\Big)^{(p-s)/p} \leq \Big(\int_{A_j} u^{\beta}\,d\mu\Big)^{\frac{p-s}{p}\frac{ps}{\beta(p-s)}} \mu(A_j)^{(1-\frac{ps}{\beta(p-s)})\frac{p-s}{p}}$$
$$= \Big(\int_{A_j} u^{\beta}\,d\mu\Big)^{s/\beta} \mu(A_j)^{(p-s)/p-s/\beta}.$$

By combining these estimates we get

$$\int_{A_j} g_u^s\,d\mu \leq c2^{js}\mu(\Omega_{j-2})^{s/p}\Big(\int_{A_j} u^{\beta}\,d\mu\Big)^{s/\beta}\mu(A_j)^{(p-s)/p-s/\beta}$$
$$\leq c2^{js}\mu(\Omega_{j-2})^{1-s/\beta}\Big(\int_{\Omega} u^{\beta}\,d\mu\Big)^{s/\beta},$$

since obviously $A_j \subset \Omega_{j-2}$ and $(p-s)/p - s/\beta > 0$. For $j = 1$ this holds true with Ω_{j-2} replaced by Ω.

Now, by summing over j we have

$$\int_{\Omega} g_u^s\,d\mu \leq \sum_{j=1}^{\infty}\int_{A_j} g_u^s\,d\mu \leq c\sum_{j=2}^{\infty} 2^{js}\mu(\Omega_{j-2})^{1-s/\beta}\Big(\int_{\Omega} u^{\beta}\,d\mu\Big)^{s/\beta}.$$

By Lemma 3.4, Ω is an *OLD*-set and thus $\mu(\Omega_{j-2}) \leq c(\kappa 2^{-(j-2)})^{\delta}\mu(\Omega)$. This implies

$$\int_{\Omega} g_u^s\,d\mu \leq c\sum_{j=2}^{\infty} 2^{-j(-s+\delta-\delta s/\beta)}\Big(\int_{\Omega} u^{\beta}\,d\mu\Big)^{s/\beta}.$$

Since we have chosen $s < \delta\beta/(\delta+\beta)$, the series converges and hence

$$\int_{\Omega} g_u^s\,d\mu \leq c\Big(\int_{\Omega} u^{\beta}\,d\mu\Big)^{s/\beta}. \tag{3.7}$$

This completes the proof under the assumption that u is a superminimizer.

Assume now that u is a p-superharmonic function in Ω. Then $u_k = \min(u,k)$ is in $N^{1,p}(\Omega)$ and is a p-superminimizer for every $k = 1, 2, \ldots$ by Corollary 7.8 in [9]. Furthermore, if $j > k$, then $g_{u_j} = g_{u_k}$ μ-almost everywhere on the set $\{x \in \Omega\colon u(x) \leq k\}$ (see [13]). It follows that the sequence (g_{u_k}) is increasing and

$$G_u = \lim_{k\to\infty} g_{u_k} \tag{3.8}$$

is well defined μ-almost everywhere on the set $\{u(x) \leq \infty\}$. By Theorem 3.1, u is integrable to a small exponent in Ω, and thus finite almost everywhere. This implies that G_u is well defined μ-almost everywhere.

By Theorem 5.5 in [10], G_u is a p-weak upper gradient of u. Now (3.4) follows directly from the definition of G_u and (3.7) by the monotone convergence theorem. By Theorem 3.1, $G_u \in L^s(\Omega)$. ∎

We point out that in the proof of the quantitative estimate (3.4) we do not use directly the property of Ω being an H-chain set. The important fact is that Ω has the *OLD*-set property. However, the H-chain assumption is necessary since it is required for global integrability of u itself.

Acknowledgements. This research has been partially supported by the Polish Ministry of Science, grant no. N N201 397737 (years 2009–2012).

REFERENCES

[1] A. Björn, *Characterizations of p-superharmonic functions on metric spaces*, Studia Math. 169 (2005), 45–62.

[2] A. Björn, J. Björn, and N. Shanmugalingam, *The Dirichlet problem for p-harmonic functions on metric spaces*, J. Reine Angew. Math. 556 (2003), 173–203.

[3] J. Björn, *Boundary continuity for quasiminimizers on metric spaces*, Illinois J. Math. 46 (2002), 383–403.

[4] S. M. Buckley, *Inequalities of John–Nirenberg type in doubling spaces*, J. Anal. Math. 79 (1999), 215–240.

[5] L. Cheeger, *Differentiability of Lipschitz functions on metric measure spaces*, Geom. Funct. Anal. 9 (1999), 428–517.

[6] J. Heinonen, *Lectures on Analysis on Metric Spaces*, Springer, 2001.

[7] J. Heinonen and P. Koskela, *Quasiconformal maps in metric spaces with controlled geometry*, Acta Math. 181 (1998), 1–61.

[8] T. Kilpeläinen and P. Koskela, *Global integrability of the gradients of solutions to partial differential equations*, Nonlinear Anal. 23 (1994), 899–909.

[9] J. Kinnunen and O. Martio, *Nonlinear potential theory on metric spaces*, Illinois J. Math. 46 (2002), 857–883.

[10] —, —, *Sobolev space properties of superharmonic functions on metric spaces*, Results Math. 44 (2003), 114–129.

[11] J. Kinnunen and N. Shanmugalingam, *Regularity of quasi-minimizers on metric spaces*, Manuscripta Math. 105 (2001), 401–423.

[12] O. E. Maasalo, *Global integrability of p-superharmonic functions on metric spaces*, J. Anal. Math. 106 (2008), 191–207.

[13] N. Shanmugalingam, *Newtonian spaces: An extension of Sobolev spaces to metric measure spaces*, Rev. Mat. Iberoamer. 16 (2000), 243–279.

Institute of Mathematics
Helsinki University of Technology
P.O. Box 1100
FI-02015 TKK, Finland
E-mail: outi.elina.maasalo@tkk.fi

Institute of Applied Mathematics and Mechanics
University of Warsaw
Banacha 2
PL-02-097 Warszawa, Poland
E-mail: azator@mimuw.edu.pl

Received 19 September 2008;
revised 15 April 2009 (5099)