AN EXTENSION THEOREM FOR A MATKOWSKI–SUTÓ PROBLEM

BY

ZOLTÁN DARÓCZY (Debrecen), GABRIELLA HAJDU (Debrecen) and CHE TAT NG (Waterloo, ON)

Abstract. Let $I$ be an interval, $0 < \lambda < 1$ be a fixed constant and $A(x, y) = \lambda x + (1 - \lambda)y$, $x, y \in I$, be the weighted arithmetic mean on $I$. A pair of strict means $M$ and $N$ is complementary with respect to $A$ if $A(M(x, y), N(x, y)) = A(x, y)$ for all $x, y \in I$. For such a pair we give results on the functional equation $f(M(x, y)) = f(N(x, y))$. The equation is motivated by and applied to the Matkowski–Sutó problem on complementary weighted quasi-arithmetic means $M$ and $N$.

1. Introduction. We call a convex subset $I$ of $\mathbb{R}$ an interval. An interval is proper when it has more than one element. We shall assume that $I$ is proper. A function $M : I^2 \to I$ is said to be a mean on $I$ if it satisfies the following conditions:

(M1) $\min \{x, y\} \leq M(x, y) \leq \max \{x, y\}$ for all $x, y \in I, x \neq y$;

(M2) $M$ is continuous on $I^2$.

A mean is called strict if the inequalities in (M1) are strict. If $M$ is a mean on $I$, then $M(x, x) = x$ for all $x \in I$. Let $\text{CM}(I)$ denote the class of all continuous and strictly monotonic real functions defined on $I$. Let $0 < \lambda < 1$ be a fixed number. A function $M : I^2 \to I$ is called a weighted quasi-arithmetic mean on $I$ (see [1]) if there exists $\varphi \in \text{CM}(I)$ such that

$$M(x, y) = \varphi^{-1}(\lambda \varphi(x) + (1 - \lambda)\varphi(y)) =: A_\varphi(x, y; \lambda)$$

for all $x, y \in I$. In this case, $\varphi \in \text{CM}(I)$ is called the generating function of the weighted quasi-arithmetic mean with weight $\lambda$. Weighted quasi-arithmetic means are strict.

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If $\varphi, \chi \in \text{CM}(I)$ then $A_\varphi(x; y; \lambda) = A_\chi(x; y; \lambda)$ for all $x, y \in I$ if and only if there exist real constants $\alpha \neq 0$ and $\beta$ such that

$$\varphi(x) = \alpha \chi(x) + \beta \quad \text{for all } x \in I.$$ 

If $\varphi, \chi \in CM(I)$ and the above equation holds for some constants $\alpha \neq 0$ and $\beta$ on a subset $J \subset I$ then we say that $\varphi$ is equivalent to $\chi$ on $J$; and, in this case, we write $\varphi \sim \chi$ on $J$. For fixed $J$, it is easy to verify that $\sim$ is indeed an equivalence relation on $\text{CM}(I)$, i.e., it is reflexive, symmetric and transitive. When $\varphi(x) = x$ for all $x \in I$, or when $\varphi$ is equivalent to the identity map $\text{id}$ on $I$, $A_\varphi(x; y; \lambda)$ is simply denoted by $A(x; y; \lambda)$ and is the well known weighted arithmetic mean

$$A(x; y; \lambda) := \lambda x + (1 - \lambda) y \quad (x, y \in I).$$

Let $M$ be a strict mean on $I$ and let $0 < \lambda \leq 1/2$. Then the function defined by

$$\widehat{M}_\lambda(x, y) := \frac{\lambda}{1 - \lambda} x + y - \frac{\lambda}{1 - \lambda} M(x, y) \quad (x, y \in I)$$

is also a strict mean on $I$ and for each $x, y \in I$, $M(x, y) = \lambda x + (1 - \lambda) y$ if and only if $\widehat{M}_\lambda(x, y) = \lambda x + (1 - \lambda) y$. The pair $M, \widehat{M}_\lambda$ satisfies

$$(1) \quad \lambda M(x, y) + (1 - \lambda) \widehat{M}_\lambda(x, y) = A(x, y; \lambda)$$

for all $x, y \in I$. In this sense, $\widehat{M}_\lambda$ is complementary to $M$ with respect to the weighted arithmetic mean.

The Matkowski–Sutô problem for weighted quasi-arithmetic means is the following: When will two complementary means $M$ and $\widehat{M}$ be weighted quasi-arithmetic means with the same weight $\lambda$ on $I$? In more detail, this means finding those functions $\varphi, \psi \in \text{CM}(I)$ which satisfy

$$(2) \quad \lambda \varphi^{-1}(\lambda \varphi(x) + (1 - \lambda) \varphi(y)) + (1 - \lambda) \psi^{-1}(\lambda \psi(x) + (1 - \lambda) \psi(y)) = \lambda x + (1 - \lambda) y$$

for all $x, y \in I$.

The case $\lambda = 1/2$ is the original Matkowski–Sutô problem (see [7], [8], [2], [4]), which has recently been solved in [5] completely. The case $\lambda \neq 1/2$ has been solved in [6] under the assumptions that $I$ is open and the generating functions are continuously differentiable on $I$ with nonvanishing derivatives. Under this assumption, the conclusion is that $\varphi \sim \text{id}$ and $\psi \sim \text{id}$ on $I$. Conversely, it is easy to verify that when $\varphi \sim \text{id}$ and $\psi \sim \text{id}$ on $I$, (2) is satisfied. It is natural to ask if the differentiability assumption in the forward statement can be reduced.

Without loss of generality we can suppose that $\lambda \leq 1/2$, otherwise we change the roles of $\varphi$ and $\psi$, and of $x$ and $y$. So in what follows $\lambda \leq 1/2$ is assumed.
We ask the following local versus global question. Suppose that \( \varphi, \psi \in \mathrm{CM}(I) \) satisfy (2) on \( I \) and there exists a proper interval \( J \subset I \) such that \( \varphi \sim \text{id} \) and \( \psi \sim \text{id} \) on \( J \). Is it true then that \( \varphi \sim \text{id} \) and \( \psi \sim \text{id} \) on \( I \)? In this paper we give an affirmative answer. With this result, the differentiability conditions on \( I \) used in [6] can be relaxed to their holding on some open subinterval of \( I \). In Section 2 we solve an equivariance functional equation which is later applied in Section 3 to give the main result.

2. An equivariance equation on complementary means. Let \( M \) be a strict mean on \( I \) and let \( 0 < \lambda \leq 1/2 \). A function \( f : I \to \mathbb{R} \) is called \((M, \lambda)\)-associate if it has the following property:

\((\text{MA})\) If \( x, y \in I \) satisfy \( M(x, y) = \lambda x + (1 - \lambda) y \) and \( f(x) = f(\lambda x + (1 - \lambda) y) \) then \( f(y) = f(x) \).

One can easily check that if \( f \) is \((M, \lambda)\)-associate then it is also \((\tilde{M}_\lambda, \lambda)\)-associate.

In this section we solve the equivariance functional equation

\[
f(M(x, y)) = f(\tilde{M}_\lambda(x, y)) \quad (x, y \in I),
\]

where \( 0 < \lambda \leq 1/2 \) is fixed.

**Theorem 1.** Let \( M \) be a strict mean on \( I \), \( 0 < \lambda \leq 1/2 \), and let \( f : I \to \mathbb{R} \) be a function satisfying the functional equation

\[(3) \quad f(M(x, y)) = f\left(\frac{\lambda}{1 - \lambda} x + y - \frac{\lambda}{1 - \lambda} M(x, y)\right)\]

for all \( x, y \in I \). Then

(a) For all \( x, y \in I \) where \( M(x, y) \neq A(x, y; \lambda) \), \( f \) is locally constant at \( A(x, y; \lambda) \).

(b) If \( f \) is continuous and \((M, \lambda)\)-associate then either

(i) \( f \) is constant on \( I \), or

(ii) \( f \) is injective on \( I \) and \( M(x, y) = A(x, y; \lambda) \) for all \( x, y \in I \).

**Proof.** Denote by \( I_{xy} \) the closed interval joining \( M(x, y) \) and \( \tilde{M}_\lambda(x, y) \) and recall that \( A(x, y; \lambda) := \lambda x + (1 - \lambda) y \) is the weighted arithmetic mean on \( I \). We also recall that \( \lambda M(x, y) + (1 - \lambda) \tilde{M}_\lambda(x, y) = A(x, y; \lambda) \).

**Claim 1.** For all \( x_0, y_0 \in I \) two cases are possible:

(I) If \( M(x_0, y_0) \leq \tilde{M}_\lambda(x_0, y_0) \) then

\[
f(A(x_0, y_0; \lambda) - s) = f\left(A(x_0, y_0; \lambda) + \frac{\lambda}{1 - \lambda} s\right)
\]

for all \( 0 \leq s \leq A(x_0, y_0; \lambda) - M(x_0, y_0) \).
(II) If $\widetilde{M}_\lambda(x_0, y_0) < M(x_0, y_0)$ then

$$f(A(x_0, y_0; \lambda) - s) = f\left(A(x_0, y_0; \lambda) + \frac{1 - \lambda}{\lambda} s\right)$$

for all $0 \leq s \leq A(x_0, y_0; \lambda) - \widetilde{M}_\lambda(x_0, y_0)$.

Proof. The assertion is trivial when $I_{x_0y_0}$ is a singleton. Suppose $I_{x_0y_0}$ is proper. There are two cases: either $x_0 < y_0$ or $y_0 < x_0$. First let $x_0 < y_0$.

Consider $x_t := x_0 + t, y_t := y_0 - \frac{\lambda}{1 - \lambda} t$ for $0 \leq t \leq A(x_0, y_0; \lambda) - x_0$. We note that for all $t \in [0, A(x_0, y_0; \lambda) - x_0]$ we have $\lambda x_t + (1 - \lambda) y_t = A(x_0, y_0; \lambda)$, and consequently $\lambda M(x_t, y_t) + (1 - \lambda) \widetilde{M}_\lambda(x_t, y_t) = A(x_0, y_0; \lambda)$.

Now suppose $M(x_0, y_0) < \widetilde{M}_\lambda(x_0, y_0)$. This immediately implies $M(x_0, y_0) < A(x_0, y_0; \lambda)$. The function $t \mapsto M(x_t, y_t)$ is continuous and takes the values $M(x_0, y_0)$ and $A(x_0, y_0; \lambda)$. By the Intermediate Value Theorem, for each $0 \leq s \leq A(x_0, y_0; \lambda) - M(x_0, y_0)$, there exists $t \in [0, A(x_0, y_0; \lambda) - x_0]$ such that $M(x_t, y_t) = A(x_0, y_0; \lambda) - s$ and $\widetilde{M}_\lambda(x_t, y_t) = A(x_0, y_0; \lambda) + \frac{\lambda}{1 - \lambda} s$. Thus by (3),

$$f(A(x_0, y_0; \lambda) - s) = f\left(A(x_0, y_0; \lambda) + \frac{\lambda}{1 - \lambda} s\right).$$

A similar argument proves that if $\widetilde{M}_\lambda(x_0, y_0) < M(x_0, y_0)$ then for each $0 \leq s \leq A(x_0, y_0; \lambda) - \widetilde{M}_\lambda(x_0, y_0)$, there exists $t \in [0, A(x_0, y_0; \lambda) - x_0]$ such that $\widetilde{M}_\lambda(x_t, y_t) = A(x_0, y_0; \lambda) - s$ and $M(x_t, y_t) = A(x_0, y_0; \lambda) + \frac{1 - \lambda}{\lambda} s$. Then again by (3),

$$f(A(x_0, y_0; \lambda) - s) = f\left(A(x_0, y_0; \lambda) + \frac{1 - \lambda}{\lambda} s\right).$$

If $y_0 < x_0$ then let $x_t := x_0 - \frac{1 - \lambda}{\lambda} t, y_t := y_0 + t$ for $0 \leq t \leq A(x_0, y_0; \lambda) - y_0$.

The rest of the proof goes as above. ■

Claim 2. Suppose $I_{x_0y_0}$ is proper. Then $f$ is locally constant at the point $A(x_0, y_0; \lambda)$; i.e., there exists a neighbourhood of $A(x_0, y_0; \lambda)$ on which $f$ is constant.

Proof. We only examine case (I), when $M(x_0, y_0) < \widetilde{M}_\lambda(x_0, y_0)$. Let $x_0 < y_0$, say. For some sufficiently small $\delta > 0$, we have $[A(x_0, y_0; \lambda) - \delta, A(x_0, y_0; \lambda) + \frac{\lambda}{1 - \lambda} \delta] \subset I_{x_0y_0}$ for all $x \in [x_0, x_0 + \delta]$.

Now for all $x \in [x_0, x_0 + \delta], I_{x_0y_0}$ is proper, and by Claim 1,

$$f(A(x, y_0; \lambda) - s) = f\left(A(x, y_0; \lambda) + \frac{\lambda}{1 - \lambda} s\right).$$
whenever both arguments are in \([A(x_0, y_0; \lambda) - \delta, A(x_0, y_0; \lambda) + \frac{\lambda}{1 - \lambda} \delta]\). The point \(A(x, y_0; \lambda)\) being arbitrary in \([A(x_0, y_0; \lambda), A(x_0, y_0; \lambda) + \delta]\), this gives the constancy of \(f\) on \([A(x_0, y_0; \lambda) - \delta, A(x_0, y_0; \lambda) + \frac{\lambda}{1 - \lambda} \delta]\).

The other cases, when \(y_0 < x_0\) and (II) holds, can be proved similarly.

The above proves (a) of Theorem 1. To prove (b), in what follows we assume that \(f\) is continuous and \((M, \lambda)\)-associate.

**Claim 3.** Suppose there exist \(x_0 < y_0\) such that \(I_{x_0 y_0}\) is proper. Then \(f\) is constant on \(I\).

**Proof.** Let \(J \subset I\) be the maximal interval containing \(A(x_0, y_0; \lambda)\) on which \(f\) is constant, i.e.,

\[ J := \{ x \in I \mid f(y) = c \text{ for all } y \text{ in the closed interval joining } x \text{ and } A(x_0, y_0; \lambda) \}, \]

where \(c := f(A(x_0, y_0; \lambda))\). By the continuity of \(f\), \(J\) is closed relative to \(I\); and by Claim 2, it is a proper interval neighbourhood of \(A(x_0, y_0; \lambda)\). We shall argue that \(J = I\); thus \(f\) is constant on \(I\).

Suppose that \(\beta := \sup J\) is an interior point of \(I\). Then there exists \(\varepsilon > 0\) such that \(\beta - \varepsilon \in J\) and \(\beta + \frac{\lambda}{1 - \lambda} \varepsilon \in I\). Now for each \(y \in \] \beta, \beta + \frac{\lambda}{1 - \lambda} \varepsilon \] \) there exists a unique \(x \in [\beta - \varepsilon, \beta]\) such that \(A(x, y; \lambda) = \beta\). If the interval \(I_{xy}\) were proper then \(f\) would be constant in a neighbourhood of \(\beta\) by Claim 2 and so \(J\) would not be maximal. Therefore \(I_{xy}\) is a singleton, that is, \(M(x, y) = \tilde{M}_\lambda(x, y)\). So

\[ M(x, y) = A(x, y; \lambda) = \beta. \]

Because \(x\) and \(\beta\) belong to \(J\),

\[ f(x) = f(\beta) = c, \]

and since \(f\) is \((M, \lambda)\)-associate, we get \(f(y) = c\). As \(y \in \] \beta, \beta + \frac{\lambda}{1 - \lambda} \varepsilon \] \) is arbitrary, this implies that \(\beta + \frac{\lambda}{1 - \lambda} \varepsilon \) is in \(J\), contradicting the assumption that \(\beta = \sup J\). Thus \(\sup J = \sup I\). One can similarly prove that \(\inf J = \inf I\). Since \(J\) is closed in \(I\), we have \(J = I\). 

**Claim 4.** If \(f\) is nonconstant on \(I\), then \(M(x, y) = A(x, y; \lambda)\) for all \(x, y \in I\).

**Proof.** By Claim 3, \(I_{xy}\) is a singleton for all \(x, y \in I\), that is, \(M(x, y) = \tilde{M}_\lambda(x, y)\). As \(A(x, y; \lambda) = \lambda M(x, y) + (1 - \lambda) \tilde{M}_\lambda(x, y)\), we get \(A(x, y; \lambda) = M(x, y)\). 

Claim 5. If \( f \) is nonconstant, then it is injective on \( I \).

Proof. By Claim 4, the continuous and nonconstant function \( f : I \to \mathbb{R} \) satisfies the condition (MA):

\[
(4) \quad f(y) = f(x) \text{ whenever } f(x) = f(A(x, y; \lambda)), \quad x, y \in I.
\]

(I) Case 1. Suppose \( \lambda = 1/2 \). This has been dealt with in [3], where the proof of injectivity of \( f \) on all closed \( [a, b] \subset I \) is given. So \( f \) is injective on \( I \).

(II) Case 2. Suppose \( 0 < \lambda < 1/2 \). Let \( \rho := \frac{\lambda}{1-\lambda} \). Since \( 0 < \lambda < 1/2 \), we have \( 0 < \rho < 1 \). We rewrite (4) in the form

\[
(5) \quad f(u) = f(v) \quad \text{implies} \quad f(u + \rho(u - v)) = f(u) = f(v),
\]

\[ u, v, u + \rho(u - v) \in I. \]

Suppose to the contrary that \( f \) is not injective. Then there exist \( x_1 < x_2 \) in the interior of \( I \) which are as close as we wish so that \( f(x_1) = f(x_2) \).

Let them be chosen close enough that \( x_2 + \rho(x_2 - x_1) \) stays in \( I \). We shall now argue that

\[
(6) \quad f \text{ is constant on } [x_1, x_2].
\]

If this were not true, then there would exist a proper connected component interval \( ]x_3, x_4[ \) of the nonempty open set \( \{ t \in ]x_1, x_2[ \mid f(t) \neq f(x_1) \} \) for which

\[
(7) \quad f(x_1) = f(x_2) = f(x_3) = f(x_4), \text{ but } f(t) \neq f(x_1) \text{ for all } t \in ]x_3, x_4[.
\]

Since \( x_2 + \rho(x_2 - x_1) \in I \), this implies \( x_5 := x_4 + \rho(x_4 - x_3) \in I \). Applying (5) once we get \( f(x_5) = f(x_4) = f(x_3) \). Let \( x_6 := x_4 + \rho(x_4 - x_5) \). Then \( x_6 = x_4 + \rho(-\rho(x_4 - x_3)) = x_4 - \rho^2(x_4 - x_3) \) where \( 0 < \rho^2 < 1 \); we have \( x_6 \in ]x_3, x_4[ \). Applying (5) once more we get \( f(x_6) = f(x_4) = f(x_5) \), i.e. \( f(x_6) = f(x_1) \) while \( x_6 \in ]x_3, x_4[ \). This is a contradiction to (7). This proves (6).

Let \( K \) be the maximal interval containing \( ]x_1, x_2[ \) on which \( f \) is constant. Then \( K \) is proper, and is closed relative to \( I \). It is easy to see that \( K \) must be equal to \( I \). For otherwise, say \( k := \sup K \) is an interior point in \( I \); then by (5), \( f \) will remain constant on \( [k, k + \rho(k - x_1)] \cap I \) and \( K \) will not be maximal. This contradiction shows that \( \sup K = \sup I \). Similarly, \( \inf K = \inf I \) holds. \( K \) being closed in \( I \), this gives \( K = I \). Thus \( f \) is constant on \( I \), and this is a contradiction. ■

This completes the proof of Theorem 1. ■
3. The extension theorem

Lemma 1. Let $\varphi, \psi \in CM(I)$ satisfy (2) on $I$ and let $J \subset I$ be a proper subinterval on which $\varphi \sim id$ and $\psi \sim id$. Then there exist $\tilde{\varphi}, \tilde{\psi} \in CM(I)$ satisfying (2) such that $\varphi \sim \tilde{\varphi}$ and $\psi \sim \tilde{\psi}$ on $I$ and

$$\tilde{\varphi}(x) = x, \quad \tilde{\psi}(x) = x \quad \text{for all } x \in J.$$

Proof. There exist constants $\alpha_i \neq 0$ and $\beta_i$ $(i = 1, 2)$ such that

$$\alpha_1 \varphi(x) + \beta_1 = x, \quad \alpha_2 \psi(x) + \beta_2 = x$$

for all $x \in J$. Then $\tilde{\varphi} := \alpha_1 \varphi + \beta_1$ and $\tilde{\psi}(x) := \alpha_2 \psi + \beta_2$ have the asserted properties. $\blacksquare$

Theorem 2. Let $\varphi, \psi \in CM(I)$ satisfy (2) for all $x, y \in I$ and let $J$ be a proper subinterval of $I$ such that $\varphi \sim id$ and $\psi \sim id$ on $J$. Then $\varphi \sim id$ and $\psi \sim id$ on $I$.

Proof. According to Lemma 1, we can suppose that

$$\varphi(x) = x, \quad \psi(x) = x \quad (x \in J)$$

and we need to show that $\varphi = \psi = id$ on the full interval $I$. Let $K \subset I$ be the maximal interval containing $J$ such that

$$\varphi(x) = x, \quad \psi(x) = x \quad (x \in K).$$

We are going to show that $K = I$. By the continuity of $\varphi$ and $\psi$, $K$ is closed in $I$. Suppose to the contrary that $K \neq I$; then either $\inf K$ or $\sup K$ is an interior point of $I$. Say, $a := \inf K$ is an interior point of $I$.

Choose another element $b \in K$ which is above $a$, i.e. $a < b$. Then $]a, b[$ is an open neighbourhood of $A_\varphi(a, b; \lambda)$ and $A_\psi(a, b; \lambda)$ because the two means are strict. By the continuity of $A_\varphi(\cdot, b; \lambda)$ and $A_\psi(\cdot, b; \lambda)$, and the fact that $a$ is an interior point of $I$, there exists $\delta > 0$ such that $]a - \delta, a[ \subset I$ and $A_\varphi(x, b; \lambda)$ and $A_\psi(x, b; \lambda)$ are both in $]a, b[$ for all $x \in ]a - \delta, a[$.

Let $x \in [a - \delta, a]$. Then from (2) and (8) we have

$$\lambda(\lambda \varphi(x) + (1 - \lambda)b) + (1 - \lambda)(\lambda \psi(x) + (1 - \lambda)b) = \lambda x + (1 - \lambda)b,$$

which implies $\lambda \varphi(x) + (1 - \lambda)\psi(x) = x$. The latter also holds true for $x \in [a, b]$ where $\varphi(x) = \psi(x) = x$ and so we have

$$\lambda \varphi(x) + (1 - \lambda)\psi(x) = x \quad \text{for all } x \in [a - \delta, b].$$

That is,

$$\psi(x) = -\frac{\lambda}{1 - \lambda} \varphi(x) + \frac{x}{1 - \lambda}.$$
Since
\[ \lambda A_\varphi(x, y; \lambda) + (1 - \lambda)A_\psi(x, y; \lambda) = \lambda x + (1 - \lambda)y, \]
\[ \varphi(A_\varphi(x, y; \lambda)) = \lambda \varphi(x) + (1 - \lambda)\varphi(y), \]
\[ \psi(A_\psi(x, y; \lambda)) = \lambda \psi(x) + (1 - \lambda)\psi(y), \]
equation (9) yields
\[ \varphi(A_\psi(x, y; \lambda)) - A_\varphi(x, y; \lambda) = \varphi(A_\varphi(x, y; \lambda)) - A_\varphi(x, y; \lambda) \]
for all \( x, y \in [a - \delta, b] \). Now let \( f(t) := \varphi(t) - t \). Then
\[
(10) \quad f(A_\psi(x, y; \lambda)) = f(A_\varphi(x, y; \lambda)) \quad \text{for all } x \in [a - \delta, b].
\]
We show that \( f \) is \((A_\varphi(x, y; \lambda), \lambda)\)-associate. Let \( x, y \in [a - \delta, a] \) be such that \( A_\varphi(x, y; \lambda) = \lambda x + (1 - \lambda)y \) and \( f(x) = f(\lambda x + (1 - \lambda)y) \). Then
\[ \lambda \varphi(x) + (1 - \lambda)\varphi(y) = \varphi(\lambda x + (1 - \lambda)y) \]
and
\[ \varphi(x) - x = \varphi(\lambda x + (1 - \lambda)y) - (\lambda x + (1 - \lambda)y). \]
These equations imply
\[ \varphi(y) - y = \varphi(x) - x, \]
that is, \( f \) is \((A_\varphi(x, y; \lambda), \lambda)\)-associate.

By Theorem 1, either \( f \) is constant or \( A_\varphi(x, y; \lambda) = A(x, y; \lambda) \) for all \( x, y \in [a - \delta, b] \). In both cases \( \varphi(x) = \alpha x + \beta \) for all \( x \in [a - \delta, b] \) follows for some \( \alpha \neq 0 \) and \( \beta \). Comparing this with \( \varphi(x) = x \) for all \( x \in [a, b] \), we get \( \alpha = 0 \) and \( \beta = 0 \). This in turn implies \( \varphi(x) = x \) for all \( x \in [a - \delta, b] \). Putting this in (8) we also have \( \psi(x) = x \) for all \( x \in [a - \delta, b] \). Thus \([a - \delta, b] \cup K\) is an interval larger than \( K \) on which (8) holds, and this is a contradiction to the maximality of \( K \). Similarly, \( \sup K \) cannot be an interior point of \( I \).
This proves that \( K = I \).

The results of [6] and Theorem 2 yield the following corollary.

**Corollary 1.** Suppose \( \lambda \neq 1/2 \). Let \( \varphi, \psi \in \text{CM}(I) \) satisfy (2) for all \( x, y \in I \) and let \( K \) be a proper open subinterval of \( I \) such that \( \varphi \) and \( \psi \) are continuously differentiable on \( K \). Then \( \varphi \sim \text{id} \) and \( \psi \sim \text{id} \) on \( I \).

**Proof.** Let \( H := \{ x \mid x \in K, \varphi'(x) = 0 \} \), which is a closed set in \( K \). Then \( H \neq K \), because \( \varphi \in \text{CM}(I) \). Therefore there exists a proper open interval \( K_1 \subset K \) such that \( \varphi'(x) \neq 0 \) if \( x \in K_1 \). Similarly, let \( H_1 := \{ x \mid x \in K_1, \psi'(x) = 0 \} \). Then there exists a proper open interval \( K_2 \subset K_1 \) such that \( \psi'(x) \neq 0 \) if \( x \in K_2 \). Thus \( \varphi'(x) \neq 0 \) and \( \psi'(x) \neq 0 \) if \( x \in K_2 \). By [6], \( \varphi \sim \text{id} \) and \( \psi \sim \text{id} \) on \( K_2 \). Now Theorem 2 implies \( \varphi \sim \text{id} \) and \( \psi \sim \text{id} \) on \( I \).
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Institute of Mathematics and Informatics
Lajos Kossuth University
H-4010 Debrecen, Pf. 12
Hungary
E-mail: daroczy@math.klte.hu

Department of Pure Mathematics
University of Waterloo
H-4010 Debrecen, Pf. 12
Hungary
E-mail: hajdug@math.klte.hu

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