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VARIETIES OF MODULES OVER TUBULAR ALGEBRAS

BY

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Abstract. We classify the irreducible components of varieties of modules over tubular algebras. Our results are stated in terms of root combinatorics. They can be applied to understand the varieties of modules over the preprojective algebras of Dynkin type $\mathbb{A}_5$ and $\mathbb{D}_4$.

1. Introduction and main results. Let k be an algebraically closed field, and let A be a finitely generated k-algebra. We denote by $\mathrm{mod}_A(\mathbf{d})$ the affine variety of A-modules with dimension vector $\mathbf{d}$. One fundamental problem is the classification of irreducible components of $\mathrm{mod}_A(\mathbf{d})$ and their canonical decomposition. For A a tubular algebra we give a purely combinatorial answer to this problem in terms of roots and the Ringel bilinear form $\langle -,-\rangle$, similar to Schofield's work on representations of quivers [18]. See also [1] and [2] for previous work on this problem.

For irreducible components $C_i \subseteq \mathrm{mod}_A(\mathbf{d}_i)$, $1 \le i \le t$, we consider all modules of dimension vector $\mathbf{d} = \mathbf{d}_1 + \ldots + \mathbf{d}_t$ which are isomorphic to $M_1 \oplus \ldots \oplus M_t$ with the M_i in C_i, and we denote this set by $C_1 \oplus \ldots \oplus C_t$. The closure $\overline{C_1 \oplus \ldots \oplus C_t}$ is called the *direct sum* of the C_i and is again irreducible; however, it is not always an irreducible component. The subset of indecomposable modules in $\mathrm{mod}_A(\mathbf{d})$ is denoted by $\mathrm{ind}_A(\mathbf{d})$. An irreducible component C of $\mathrm{mod}_A(\mathbf{d})$ is called *indecomposable* if $C \cap \mathrm{ind}_A(\mathbf{d})$ is dense in C. Each irreducible component C of a variety of modules is a direct sum of indecomposable components and the direct summands are uniquely determined up to reordering (see [6]). This direct sum is called the *canonical decomposition* of C.

Now let $A = kQ/I$ be a tubular algebra, and let ϕ be the Coxeter matrix of A. See [17] for all unexplained notation. For a dimension vector $\mathbf{d}$ define

$$\mathrm{rk}(\mathbf{d}) = \min\{i \ge 1 \mid \mathbf{d}\phi^i = \mathbf{d})\},$$

$$\mathrm{ql}(\mathbf{d}) = \gcd\Big\{\Big(\sum_{i=1}^{\mathrm{rk}(\mathbf{d})} \mathbf{d}\phi^i\Big)_q \,\Big|\, q \in Q_0\Big\},$$

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$$\mathrm{iso}(\mathbf{d}) = \frac{1}{\mathrm{ql}(\mathbf{d})}\Big(\sum_{i=1}^{\mathrm{rk}(\mathbf{d})} \mathbf{d}\phi^i\Big).$$

Since A is tubular, we have $\phi^i = 1$ for some $i \geq 1$. Thus $\mathrm{rk}(\mathbf{d})$ is well defined.

Let $\langle -,- \rangle : \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Z}$ be the Ringel form of A, and let $q(\mathbf{d}) = \langle \mathbf{d}, \mathbf{d} \rangle$ be the associated quadratic form. A dimension vector is *indivisible* if the greatest common divisor of its entries is 1. We call a dimension vector $\mathbf{d}$ a *Schur root* if one of the following holds:

(i) $q(\mathbf{d}) = 0$ and $\mathbf{d}$ is indivisible;
(ii) $q(\mathbf{d}) = 1$ and $\mathrm{ql}(\mathbf{d}) < \mathrm{rk}(\mathbf{d})$.

In the first case, one calls $\mathbf{d}$ an *isotropic* Schur root. Our first result gives a classification of all indecomposable irreducible components of varieties of modules over tubular algebras.

THEOREM 1.1. *Let A be a tubular algebra. The map $\mathbf{d} \mapsto \overline{\mathrm{ind}_A(\mathbf{d})}$ defines a bijection between the Schur roots of A and the indecomposable irreducible components of varieties of A-modules.*

For a Schur root $\mathbf{d}$ of a tubular algebra A let $C(\mathbf{d})$ be the corresponding indecomposable irreducible component of $\mathrm{mod}_A(\mathbf{d})$.

Given irreducible components $C_1 \subseteq \mathrm{mod}_A(\mathbf{d}_1)$ and $C_2 \subseteq \mathrm{mod}_A(\mathbf{d}_2)$ define

$$\mathrm{hom}_A(C_1, C_2) = \min\{\dim \mathrm{Hom}_A(M_1, M_2) \mid M_i \in C_i\},$$
$$\mathrm{ext}_A^j(C_1, C_2) = \min\{\dim \mathrm{Ext}_A^j(M_1, M_2) \mid M_i \in C_i\} \quad \text{for } j = 1, 2.$$

Recall that in this situation there is a dense open subset $U \subseteq C_1 \times C_2$ with $\dim \mathrm{Ext}_A^j(M_1, M_2) = \mathrm{ext}_A^j(C_1, C_2)$ for all $(M_1, M_2) \in U$. If $C_i \subseteq \mathrm{mod}_A(\mathbf{d}_i)$, $1 \leq i \leq t$, are irreducible components, then $\overline{C_1 \oplus \ldots \oplus C_t}$ is an irreducible component of $\mathrm{mod}_A(\mathbf{d}_1 + \ldots + \mathbf{d}_t)$ if and only if $\mathrm{ext}_A^1(C_i, C_j) = 0$ for all $i \neq j$ (see [6]).

COROLLARY 1.2. *Any irreducible component C of $\mathrm{mod}_A(\mathbf{d})$ is of the form $C = \overline{C(\mathbf{d}_1) \oplus \ldots \oplus C(\mathbf{d}_t)}$ for a unique (up to permutation) family of Schur roots $\mathbf{d}_1, \ldots, \mathbf{d}_t$ with $\sum_{i=1}^t \mathbf{d}_i = \mathbf{d}$ and $\mathrm{ext}_A^1(C(\mathbf{d}_i), C(\mathbf{d}_j)) = 0$ for all $i \neq j$. Then*

$$\dim C = \dim \mathrm{Gl}(\mathbf{d}) - q(\mathbf{d}) + \sum_{i \neq j} \mathrm{ext}_A^2(C(\mathbf{d}_i), C(\mathbf{d}_j)).$$

Moreover, let $\mathbf{h}_0, \mathbf{h}_\infty$ be the isotropic Schur roots corresponding to the "first", respectively "last", tubular family of A. Then for any Schur roots $\mathbf{d}_i$ and $\mathbf{d}_j$ with $\mathrm{ext}_A^1(C(\mathbf{d}_i), C(\mathbf{d}_j)) = 0$,

$$\mathrm{ext}_A^2(C(\mathbf{d}_i), C(\mathbf{d}_j)) = \begin{cases} \langle \mathbf{d}_i, \mathbf{d}_j \rangle & \textit{if } \langle \mathbf{h}_s, \mathbf{d}_j \rangle \leq 0 \leq \langle \mathbf{h}_s, \mathbf{d}_i \rangle \textit{ for } s \in \{0, \infty\}, \\ 0 & \textit{else.} \end{cases}$$

Thus to compute all irreducible components it is enough to classify the indecomposable irreducible components and to know when Ext_A^1 vanishes generically between them. The following theorem solves this problem.

THEOREM 1.3. *Let* $\mathbf{d}$ *and* $\mathbf{e}$ *be Schur roots of a tubular algebra* A. *Then* $\mathrm{ext}_A^1(C(\mathbf{d}), C(\mathbf{e})) = 0$ *if and only if* $\langle \mathbf{d}, \mathbf{e} \rangle \geq 0$ *and at least one of the following holds*:

(i) $q(\mathbf{d}) = 0$ *or* $q(\mathbf{e}) = 0$;
(ii) $\mathrm{iso}(\mathbf{d}) \neq \mathrm{iso}(\mathbf{e})$ *and* $\mathrm{iso}(\mathbf{d}) \neq -\mathrm{iso}(\mathbf{e})$;
(iii) $\langle \mathbf{d}, \mathbf{e}\phi^j \rangle = 0$ *for all* j;
(iv) $\langle \mathbf{d}, \mathbf{e} \rangle > 0$;
(v) $\mathrm{ql}(\mathbf{d}) + \mathrm{ql}(\mathbf{e}) \leq \mathrm{rk}(\mathbf{d}) = \mathrm{rk}(\mathbf{e})$;
(vi) $\langle \mathbf{d}, \mathbf{e}\phi^j \rangle \neq 0$ *for some* j, $\mathrm{iso}(\mathbf{d}) = \mathrm{iso}(\mathbf{e})$ *and* $\langle \mathbf{d}, \mathbf{e}\phi^i \rangle < 0$ *where* $i = \min\{j \geq 1 \mid \langle \mathbf{d}, \mathbf{e}\phi^j \rangle \neq 0\}$;
(vii) $\langle \mathbf{d}, \mathbf{e}\phi^j \rangle \neq 0$ *for some* j, $\mathrm{iso}(\mathbf{d}) = -\mathrm{iso}(\mathbf{e})$ *and* $\langle \mathbf{d}, \mathbf{e}\phi^i \rangle > 0$ *where* $i = \min\{j \geq 1 \mid \langle \mathbf{d}, \mathbf{e}\phi^j \rangle \neq 0\}$.

Our results can be applied to understand the canonical basis of the negative part of the quantized enveloping algebras of Dynkin type $\mathbb{A}_5$ and $\mathbb{D}_4$. The cases $\mathbb{A}_2$, $\mathbb{A}_3$ and $\mathbb{A}_4$ were intensively studied before (see for example [5]). More precisely, to a Dynkin quiver Q one can associate the preprojective algebra $P(Q)$. Kashiwara and Saito proved that the canonical basis elements of the negative part of the corresponding quantized enveloping algebra correspond to the irreducible components of varieties of modules over $P(Q)$. If Q is of Dynkin type $\mathbb{A}_2$, $\mathbb{A}_3$ or $\mathbb{A}_4$, then $P(Q)$ is representation finite, thus it is easy to find the indecomposable irreducible components, which play an important role in the theory.

In case Q is of type $\mathbb{A}_5$ or $\mathbb{D}_4$, the algebra $P(Q)$ is representation infinite. In this case, $P(Q)$ has a covering which is in some sense an "iterated tubular algebra". This enables us to use our results on tubular algebras and classify the indecomposable components. Moreover, we can describe all irreducible components in terms of these irreducible components. This description is relevant for the understanding of Lusztig cones (see for example [5] for the case $\mathbb{A}_4$). In the cases $\mathbb{A}_5$ and $\mathbb{D}_4$, we prove that the irreducible components which do not contain a dense orbit are not generically reduced.

The paper is organized as follows. In Section 2 we recall known results on tubular algebras and their derived categories, which were described by Happel and Ringel in [13]. In Sections 3 and 5 we prove Theorems 1.1 and 1.3, respectively. Section 4 contains the proof of Corollary 1.2. The final Section 6 contains the above-mentioned application.

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inaccuracies. We also thank R. Marsh for several discussions concerning the application to the nilpotent variety of type $\mathbb{A}_5$.

2. Tubular algebras and their derived categories. For an algebra A let $\mathrm{mod}(A)$ be the category of finite-dimensional (right) A-modules. For an A-module M let $[M]_A = [M]$ be the corresponding element in the Grothendieck group $K_0(A)$ of $\mathrm{mod}(A)$. Thus $[M]$ is the dimension vector of M.

Let $A = kQ/I$ be a tubular algebra, where $Q = (Q_0, Q_1)$ is a quiver with set of vertices Q_0 and set of arrows Q_1, and let R be a minimal set of relations which generate the admissible ideal I. We recall some results on the representation theory of tubular algebras. For general information on tubular algebras we refer to [17]. For vertices i and j of Q let r_{ij} be the number of relations in R which start in i and end in j. Let n be the number of vertices of Q. Thus $K_0(A)$ is isomorphic to $\mathbb{Z}^n$. For an arrow α in Q let $s(\alpha)$ be its starting vertex and $e(\alpha)$ its end vertex. The *Ringel form* $\langle -, - \rangle : \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Z}$ is defined by

$$\langle \mathbf{d}, \mathbf{e} \rangle = \sum_{i \in Q_0} d_i e_i - \sum_{\alpha \in Q_1} d_{s(\alpha)} e_{e(\alpha)} + \sum_{i,j \in Q_0} r_{ij} d_i e_j.$$

This is a (not necessarily symmetric) bilinear form. By $q(\mathbf{d}) = \langle \mathbf{d}, \mathbf{d} \rangle$ we denote the corresponding quadratic form.

For a dimension vector $\mathbf{d}$ let $\mathrm{Gl}(\mathbf{d}) = \prod_{i \in Q_0} \mathrm{Gl}_{d_i}(k)$. This group operates by conjugation on $\mathrm{mod}_A(\mathbf{d})$ and the orbits $\mathcal{O}(M)$ are in 1-1 correspondence with the isomorphism classes of A-modules with dimension vector $\mathbf{d}$. Define

$$a(\mathbf{d}) = \dim \mathrm{Gl}(\mathbf{d}) - q(\mathbf{d}) = \sum_{\alpha \in Q_1} d_{s(\alpha)} d_{e(\alpha)} - \sum_{i,j \in Q_0} r_{ij} d_i d_j.$$

By Krull's principal ideal theorem it follows that each irreducible component of $\mathrm{mod}_A(\mathbf{d})$ has dimension at least $a(\mathbf{d})$ (see [4]).

Dimension vectors of indecomposable A-modules are called *roots*. If $\mathbf{d}$ is a root with $q(\mathbf{d}) = 0$, then one calls $\mathbf{d}$ an *isotropic root*. Since tubular algebras have global dimension at most two, we know that

$$\langle [M], [N] \rangle = \dim \mathrm{Hom}_A(M, N) - \dim \mathrm{Ext}^1_A(M, N) + \dim \mathrm{Ext}^2_A(M, N)$$

for all A-modules M and N (see [3] and [17]). We denote by $\phi = -C_A^{-\mathrm{t}} C_A$ the Coxeter matrix of A, where C_A is the Cartan matrix of A and $C_A^{-\mathrm{t}}$ the inverse of its transpose (see [17, 2.4]). It is known that

$$\langle \mathbf{d}, \mathbf{e} \rangle = \mathbf{d} C_A^{-\mathrm{t}} \mathbf{e}^{\mathrm{t}} = -\langle \mathbf{e}, \mathbf{d}\phi \rangle = \langle \mathbf{d}\phi, \mathbf{e}\phi \rangle.$$

Next, let $D^{\mathrm{b}}(A)$ be the derived category of bounded complexes of finite-dimensional A-modules. As a general reference on derived categories we use [11]. Note that the Grothendieck group of the triangulated category

$D^{\mathrm{b}}(A)$ can be identified with $K_0(A)$. Namely, if

$$P: \quad \ldots \to 0 \to P_n \to \ldots \to P_m \to 0 \to \ldots$$

is a complex of A-modules, i.e. an object in $D^{\mathrm{b}}(A)$, then the dimension vector $[P]$ of P is by definition $\sum_{i\in\mathbb{Z}}(-1)^i[P_i]$ (see [11] for details). Recall from [13] the following description of $D^{\mathrm{b}}(A)$ for a tubular algebra A of tubular type T.

(1) Let $\mathbb{Q}_\infty = \mathbb{Q}\cup\{\infty\}$. We have the decomposition

$$D^{\mathrm{b}}(A) = \bigvee_{i\in\mathbb{Z}} \mathcal{H}[i] \quad \text{where} \quad \mathcal{H}[i] = \bigvee_{q\in\mathbb{Q}_\infty} \bigvee_{\lambda\in\mathbb{P}^1(k)} \mathcal{T}_{\lambda,q}[i]$$

and for each $(q,i)\in\mathbb{Q}_\infty\times\mathbb{Z}$ we see that $(\mathcal{T}_{\lambda,q}[i])_{\lambda\in\mathbb{P}^1(k)}$ is a tubular family of type T, and moreover each $\mathcal{T}_{\lambda,q}[i]$ is a standard tube, and $\mathcal{T}_{\lambda,q}[i+1]$ contains precisely the objects $X[1]$ for $X\in\mathcal{T}_{\lambda,q}[i]$. Note that the primary decomposition of $D^{\mathrm{b}}(A)$ into degrees comes from the fact that A is piecewise hereditary in the sense of [12].

(2) Let $X,Y\in D^{\mathrm{b}}(A)$ be indecomposable with $X\in\mathcal{T}_{\lambda,q}[i]$ and $Y\in\mathcal{T}_{\mu,r}[j]$. If $\operatorname{Hom}_{D^{\mathrm{b}}(A)}(X,Y)\neq 0$ then one of the following holds:

(i) $j=i$, $r=q$, $\mu=\lambda$;
(ii) $j=i$, $r>q$;
(iii) $j=i+1$, $r=q$, $\mu=\lambda$;
(iv) $j=i+1$, $r<q$.

(3) We may view $\operatorname{mod}(A)$ as a full subcategory of $D^{\mathrm{b}}(A)$, concentrated in degrees 0 and 1, i.e. each indecomposable A-module corresponds to an object in $\mathcal{H}[0]\cup\mathcal{H}[1]$. If X and Y are indecomposable A-modules, then $\operatorname{iso}([X])=\operatorname{iso}([Y])$ if and only if $X\in\mathcal{T}_{\lambda,q}[l]$ and $Y\in\mathcal{T}_{\mu,q}[l]$ for some λ and μ, and some $l=0,1$.

(4) We recall the following facts for algebras of finite global dimension.

(i) There exist globally Auslander–Reiten triangles, and for the corresponding translation τ we have $[\tau X]=[X]\phi$.
(ii) The Ringel form extends to the derived category by

$$\langle [X],[Y]\rangle = \sum_{i\in\mathbb{Z}}(-1)^i \dim \operatorname{Hom}_{D^{\mathrm{b}}(A)}(X,Y[i]),$$

and for A-modules M and N we have

$$\operatorname{Ext}^i_A(M,N) = \operatorname{Hom}_{D^{\mathrm{b}}(A)}(M,N[i])$$

for $i\geq 0$.
(iii) The Auslander–Reiten formula becomes

$$\operatorname{Hom}_{D^{\mathrm{b}}(A)}(Y,\tau X) = D\operatorname{Hom}_{D^{\mathrm{b}}(A)}(X,Y[1]).$$

(5) With $\phi^m = 1$ we get

$$\mathrm{iso}(\mathbf{d}) = \frac{\mathrm{rk}(\mathbf{d})}{m \cdot \mathrm{ql}(\mathbf{d})} \sum_{i=1}^{m} \mathbf{d}\phi^i,$$

an isotropic Schur root for any $\mathbf{d} \in \mathbb{Z}^n$. Thus $\mathrm{iso}(\mathbf{d}_1) = \mathrm{iso}(\mathbf{d}_2)$ implies $\mathrm{iso}(\mathbf{d}_1+\mathbf{d}_2) = \mathrm{iso}(\mathbf{d}_1)$. Now, if $X, Y \in D^{\mathrm{b}}(A)$ are indecomposable and belong to the same Auslander–Reiten component of $D^{\mathrm{b}}(A)$ then $\mathrm{iso}([X]) = \mathrm{iso}([Y])$, thus $\mathrm{iso}([X]) = \mathrm{iso}([X \oplus Y])$.

We obtain the following direct consequences.

LEMMA 2.1. *Let A be a tubular algebra, and let $X = \bigoplus_{i\in I} M_i$ and $Y = \bigoplus_{j\in J} N_j$ be objects in $D^{\mathrm{b}}(A)$ such that $M_i \in \mathcal{T}$ and $N_j \in \mathcal{T}'$ for all i, j and some Auslander–Reiten components $\mathcal{T}$ and $\mathcal{T}'$ of $D^{\mathrm{b}}(A)$.*

If $\mathrm{Hom}_{D^{\mathrm{b}}(A)}(X, Y[i]) \neq 0 \neq \mathrm{Hom}_{D^{\mathrm{b}}(A)}(X, Y[j])$, $i < j$, then $j = i + 1$ and $\mathcal{T} = \mathcal{T}'[i]$. In particular, $\mathrm{iso}([X]) = \mathrm{iso}([Y[i]])$.

LEMMA 2.2. *Let A be a tubular algebra. If M and N are indecomposable A-modules such that M and N, and also M and $N[1]$, lie in different AR-components of $D^{\mathrm{b}}(A)$, then $\mathrm{Ext}^1_A(M, N) = 0$ if and only if $\langle [M], [N]\rangle \geq 0$.*

Proof. If $\langle [M], [N]\rangle < 0$, then $\mathrm{Ext}^1_A(M, N) \neq 0$ since

$$\langle [M], [N]\rangle = \dim \mathrm{Hom}_A(M, N) - \dim \mathrm{Ext}^1_A(M, N) + \dim \mathrm{Ext}^2_A(M, N).$$

Thus assume $\langle [M], [N]\rangle \geq 0$. If $\mathrm{Ext}^1_A(M, N) = \mathrm{Hom}_{D^{\mathrm{b}}(A)}(M, N[1]) \neq 0$, then either $\mathrm{Hom}_{D^{\mathrm{b}}(A)}(M, N) \neq 0$ or $\mathrm{Hom}_{D^{\mathrm{b}}(A)}(M, N[2]) \neq 0$. By Lemma 2.1 we see that in the first case M and N, and in the second case M and $N[1]$, lie in the same AR-component of $D^{\mathrm{b}}(A)$, a contradiction. ∎

LEMMA 2.3. *Let A be a tubular algebra, and let M and N be indecomposable A-modules. If* $\mathrm{iso}([M]) = \mathrm{iso}([N])$ *or* $\mathrm{iso}([M]) = -\mathrm{iso}([N])$, *and $\langle [M], [N]\phi^j\rangle \neq 0$ for some $j \geq 1$, then M and N, or M and $N[1]$, respectively, lie in the same AR-component of $D^{\mathrm{b}}(A)$.*

Proof. Define $X = \bigoplus_{i=1}^{\mathrm{rk}([M])} \tau^i M$ and $Y = \bigoplus_{j=1}^{\mathrm{rk}([N])} \tau^j N$ where τ is the Auslander–Reiten translation in $D^{\mathrm{b}}(A)$. We have $\mathrm{iso}([M]) = \mathrm{iso}([N])$ or $\mathrm{iso}([M]) = -\mathrm{iso}([N])$. Thus the dimension vectors of X and Y are integer multiples of the indivisible isotropic root $\mathrm{iso}([M])$. This implies $\langle [X], [Y]\rangle = 0$. We know that

$$\langle [X], [Y]\rangle = \sum_{i=0}^{2} (-1)^i \dim \mathrm{Hom}_{D^{\mathrm{b}}(A)}(X, Y[i]).$$

Note that

$$\langle [X], [Y]\rangle = \sum_{\substack{1\leq i\leq \mathrm{rk}([M])\\ 1\leq j\leq \mathrm{rk}([N])}} \langle [\tau^i M], [\tau^j N]\rangle$$

and

$$\operatorname{Hom}_{D^{\mathrm{b}}(A)}(X, Y[l]) = \bigoplus_{\substack{1 \le i \le \operatorname{rk}([M]) \\ 1 \le j \le \operatorname{rk}([N])}} \operatorname{Hom}_{D^{\mathrm{b}}(A)}(\tau^i M, \tau^j N[l])$$

for all l. Note that $\tau^{\operatorname{rk}([M])} M = M$. We have $\langle [M], [\tau^j N] \rangle = \langle [M], [N]\phi^j \rangle \neq 0$. So one of the summands in the second of the above formulas is non-zero. Since $\langle [X], [Y] \rangle = 0$, there exist i, j such that $\langle [\tau^i M], [\tau^j N] \rangle < 0$, thus $\operatorname{Hom}_{D^{\mathrm{b}}(A)}(\tau^i M, \tau^j N[1]) \neq 0$ and we get $\operatorname{Hom}_{D^{\mathrm{b}}(A)}(X, Y[1]) \neq 0$. This implies that $\operatorname{Hom}_{D^{\mathrm{b}}(A)}(X, Y)$ or $\operatorname{Hom}_{D^{\mathrm{b}}(A)}(X, Y[2])$ must be non-zero. Now we use Lemma 2.1 to conclude that M and N, or M and $N[1]$, respectively, lie in the same AR-component of $D^{\mathrm{b}}(A)$. ∎

3. Proof of Theorem 1.1. Let A be a tubular algebra, and let M be an indecomposable A-module. Then $\operatorname{Ext}_A^2(M, M) = 0$, and thus we get

$$q([M]) = \dim \operatorname{End}_A(M) - \dim \operatorname{Ext}_A^1(M, M).$$

Also, either $q([M]) = 0$ or $q([M]) = 1$ (see [17]). If $q([M]) = 1$, then

$$\dim \mathcal{O}(M) = \dim \operatorname{Gl}([M]) - \dim \operatorname{End}_A(M) = a([M]) + 1 - \dim \operatorname{End}_A(M).$$

Thus the following are equivalent:

(1) $\dim \mathcal{O}(M) \geq a([M])$;
(2) $\dim \mathcal{O}(M) = a([M])$;
(3) $\operatorname{End}_A(M) = k$;
(4) $\operatorname{Ext}_A^1(M, M) = 0$;
(5) $\mathcal{O}(M)$ is open in $\operatorname{mod}_A([M])$;
(6) the closure of $\mathcal{O}(M)$ is an irreducible component.

For (4)⇒(5) we use Voigt's Lemma (see [20, 3.4]). Since $q([M]) = 1$, we know that $\mathcal{O}(M) = \operatorname{ind}_A([M])$. Moreover, M lies in a tube of rank $\operatorname{rk}([M])$ in the Auslander–Reiten quiver of $D^{\mathrm{b}}(A)$, and the quasi-length of M in this tube is $\operatorname{ql}([M])$. Using the mesh category one easily checks that the above conditions are equivalent to $\operatorname{ql}([M]) < \operatorname{rk}([M])$. For an illustration of calculations in the mesh category we refer to [19, Proposition 3.5].

Next assume $q([M]) = 0$. Let n be the greatest common divisor of the entries of $[M]$, and let r be the rank of the tube of the Auslander–Reiten quiver of $D^{\mathrm{b}}(A)$ in which M is contained. Thus M has quasi-length nr in this tube. Again by using the mesh category one gets $\dim \operatorname{End}_A(M) = \dim \operatorname{Ext}_A^1(M, M) = n$. Thus

$$\dim \mathcal{O}(M) = \dim \operatorname{Gl}([M]) - n = \dim \operatorname{Gl}([M]) - q([M]) - n = a([M]) - n.$$

Now there exists an affine line L in $\operatorname{mod}_A([M])$ which intersects almost all, i.e. all but finitely many, orbits in $\operatorname{ind}_A([M])$ (see for example [8]). Thus there is a morphism

$$\theta_L : \operatorname{Gl}([M]) \times L \to \operatorname{mod}_A([M])$$

which is induced by the conjugation action of $\mathrm{Gl}([M])$ on $\mathrm{mod}_A([M])$, and the image $\mathrm{Im}(\theta_L)$ of θ_L contains almost all orbits in $\mathrm{ind}_A([M])$. A simple fibre dimension argument yields

$$\dim \mathrm{Im}(\theta_L) = \dim \mathrm{Gl}([M]) - n + 1 = a([M]) - n + 1.$$

Furthermore, we know that $\mathrm{Im}(\theta_L)$ is irreducible, since $\mathrm{Gl}([M]) \times L$ is irreducible. Now $\mathrm{ind}_A([M])$ is the union of $\mathrm{Im}(\theta_L)$ and of finitely many orbits, say $\mathcal{O}_1, \ldots, \mathcal{O}_t$. The $\mathcal{O}_i$ all have dimension strictly smaller than $a([M])$. For $n \geq 2$, the dimension of $\mathrm{Im}(\theta_L)$ is also strictly smaller than $a([M])$. Thus the closure $\overline{\mathrm{ind}_A([M])}$ cannot contain an irreducible component of $\mathrm{mod}_A([M])$.

An orbit $\mathcal{O}(N)$ in $\mathrm{mod}_A([N])$ is called *maximal* if it is not contained in the closure of another orbit. Observe that the set of maximal orbits is dense in $\mathrm{mod}_A([N])$.

We know that $[M]$ is of the form $n\mathbf{h}$ with $\mathbf{h}$ an indivisible isotropic root. Now assume that $\mathcal{O}(M)$ is a maximal orbit in $\mathrm{mod}_A([M])$. Thus M is isomorphic to $\bigoplus_{i=1}^t M_i$ with M_i indecomposable and $\mathrm{Ext}_A^1(M_i, M_j) = 0$ for all $i \neq j$. For convenience, we repeat an observation from [1, Proposition 5.4]: We have

$$0 = \langle [M], [M] \rangle = \sum_{1 \leq i,j \leq t} \langle [M_i], [M_j] \rangle = \sum_{i=1}^t \langle [M_i], [M] \rangle.$$

Because of the vanishing of Ext_A^1 all summands in the above equation must be 0. Since A is a tubular algebra, isotropic roots $\mathbf{e}$ and $\mathbf{f}$ with $\langle \mathbf{e}, \mathbf{f} \rangle = 0$ are integer multiples of each other. Thus $[M_i] = n_i\mathbf{h}$ for some n_i with $\sum_{i=1}^t n_i = n$. For $n = 1$ we deduce that each maximal orbit must be the orbit of an indecomposable module. Thus $\mathrm{mod}_A([M]) = \overline{\mathrm{ind}_A([M])}$. Furthermore, $\mathrm{ind}_A([M]) \setminus \mathrm{Im}(\theta_L)$ is a finite union of orbits which all have dimension strictly smaller than $a([M])$. This implies that they must be contained in the closure of $\mathrm{Im}(\theta_L)$. Thus $\overline{\mathrm{ind}_A([M])}$ is an irreducible component. This finishes the proof.

4. Proof of Corollary 1.2. Consider the natural map

$$\varphi\colon\ \mathrm{Gl}(\mathbf{d}) \times C(\mathbf{d}_1) \times \ldots \times C(\mathbf{d}_t) \to \mathrm{mod}_A(\mathbf{d}).$$

It is easy to determine the dimensions of its fibres, and thus from Chevalley's theorem we get

$$\begin{aligned} \dim \mathrm{Im}(\varphi) = &\ \dim \mathrm{Gl}(\mathbf{d}) + \sum_{i=1}^t \dim C(\mathbf{d}_i) \\ &- \Big(\sum_{i=1}^t \dim \mathrm{Gl}(\mathbf{d}_i) + \sum_{i \neq j} \hom_A(C(\mathbf{d}_i), C(\mathbf{d}_j)) \Big). \end{aligned}$$

Since the $\mathbf{d}_i$ are Schur roots, we have

$$q(\mathbf{d}_i) = \dim \mathrm{Gl}(\mathbf{d}_i) - \dim C(\mathbf{d}_i).$$

This follows from the proof of Theorem 1.1. Next, note that $\mathrm{gl.dim}\, A = 2$ implies, for any irreducible component $C_i \subseteq \mathrm{mod}_A(\mathbf{e}_i)$ and any dimension vectors $\mathbf{e}_1, \mathbf{e}_2$, that

$$\langle \mathbf{e}_1, \mathbf{e}_2 \rangle = \hom_A(C_1, C_2) - \mathrm{ext}^1_A(C_1, C_2) + \mathrm{ext}^2_A(C_1, C_2).$$

For an irreducible component $C \subseteq \mathrm{mod}_A(\mathbf{d})$ we get

$$\begin{aligned} q(\mathbf{d}) &= \hom_A(C, C) - \mathrm{ext}^1_A(C, C) + \mathrm{ext}^2_A(C, C) \\ &= \sum_{i=1}^{t} q(\mathbf{d}_i) + \sum_{i \neq j} \hom_A(C(\mathbf{d}_i), C(\mathbf{d}_j)) + \sum_{i \neq j} \mathrm{ext}^2_A(C(\mathbf{d}_i), C(\mathbf{d}_j)) \end{aligned}$$

where the second equality holds if $C = \overline{C(\mathbf{d}_1) \oplus \ldots \oplus C(\mathbf{d}_t)}$. Recall that this implies $\mathrm{ext}^1_A(C(\mathbf{d}_i), C(\mathbf{d}_j)) = 0$ for all $i \neq j$. Since in this case $\mathrm{Im}(\varphi)$ is dense in C, from the above equalities we get

$$\begin{aligned} \dim C &= \dim \mathrm{Gl}(\mathbf{d}) - \sum_{i=1}^{t} q(\mathbf{d}_i) - \sum_{i \neq j} \hom_A(C(\mathbf{d}_i), C(\mathbf{d}_j)) \\ &= \dim \mathrm{Gl}(\mathbf{d}) - q(\mathbf{d}) + \sum_{i \neq j} \mathrm{ext}^2_A(C(\mathbf{d}_i), C(\mathbf{d}_j)). \end{aligned}$$

With the notation from [17, 5.2], the following hold:

$$\begin{aligned} M \in \mathcal{P}_0 \vee \mathcal{T}_0 &\Rightarrow \mathrm{proj.dim}\, M \leq 1, \\ M \in \mathcal{Q}_0 \cap \mathcal{P}_\infty &\Rightarrow \mathrm{proj.dim}\, M = \mathrm{inj.dim}\, M = 1, \\ M \in \mathcal{T}_\infty \vee \mathcal{Q}_\infty &\Rightarrow \mathrm{inj.dim}\, M \leq 1. \end{aligned}$$

Let X and Y be indecomposable A-modules. If $\mathrm{Ext}^2_A(X, Y) \neq 0$, then $X \in \mathcal{T}_\infty \vee \mathcal{Q}_\infty$ and $Y \in \mathcal{P}_0 \vee \mathcal{T}_0$. On the other hand, if $\mathrm{Ext}^1_A(X, Y) = 0$, then $\mathrm{Hom}_A(X, Y) = 0$ or $\mathrm{Ext}^2_A(X, Y) = 0$, since A is quasi-tilted, and thus

$$\dim \mathrm{Ext}^2_A(X, Y) = \begin{cases} \langle [X], [Y] \rangle & \text{if } \langle \mathbf{h}_i, [X] \rangle \leq 0 \leq \langle \mathbf{h}_i, [Y] \rangle \text{ for } i \in \{0, \infty\}, \\ 0 & \text{else,} \end{cases}$$

by the table from [17, 5.2(1)].

For Schur roots $\mathbf{d}_i$, $1 \leq i \leq t$, with $\mathrm{ext}^1_A(C(\mathbf{d}_i), C(\mathbf{d}_j)) = 0$ for all $i \neq j$ this implies our formula for ext^2_A.

5. Proof of Theorem 1.3. Let M and N be indecomposable A-modules with $\mathbf{d} = [M]$ and $\mathbf{e} = [N]$ Schur roots. Assume that $\langle [M], [N] \rangle \geq 0$.

If M and N, and also M and $N[1]$, lie in different components of the Auslander–Reiten quiver of $D^{\mathrm{b}}(A)$, then Lemma 2.2 yields $\mathrm{Ext}^1_A(M, N) = 0$.

This implies $\operatorname{ext}^1_A(C([M]),C([N]))=0$. We will use this observation several times in our proof.

Assume now that condition (i) in Theorem 1.3 holds. Thus $q([M])=0$ or $q([N])=0$. Recall that $\operatorname{ind}_A(\mathbf{f})$ contains modules from infinitely many Auslander–Reiten components of $D^{\mathrm{b}}(A)$ provided $q(\mathbf{f})=0$. Thus we can assume that M and N, and also M and $N[1]$, lie in different components. This implies $\operatorname{ext}^1_A(C([M]),C([N]))=0$.

Thus, from now on we can assume that $q([M])=q([N])=1$, and that either M and N or M and $N[1]$ lie in the same Auslander–Reiten component of $D^{\mathrm{b}}(A)$. Note that this implies $\operatorname{rk}([M])=\operatorname{rk}([N])$.

Assume that (ii) holds. Thus we have $\operatorname{iso}([M])\neq\operatorname{iso}([N])$ and $\operatorname{iso}([M])\neq -\operatorname{iso}([N])$. By Lemma 2.1 we deduce that M and N, and also M and $N[1]$, lie in different Auslander–Reiten components of $D^{\mathrm{b}}(A)$. The same follows if (iii) holds. Namely, since by assumption M and N, or M and $N[1]$, lie in the same AR-component, we can use the mesh category to get $\langle [M],[N]\phi^j\rangle\neq 0$ for some j. Here we use the fact that $[M]\phi^i=[\tau^i M]$ for all indecomposable A-modules M and all i. For details we refer to the explanations below; see also Figure 1. In both cases we get a contradiction.

Now assume that (iv) holds. Since $[M]$ and $[N]$ are Schur roots, the quasi-lengths of M, N and $N[1]$ are strictly smaller than $\operatorname{rk}([M])=\operatorname{rk}([N])$. Using the mesh category, we get $\dim\operatorname{Hom}_{D^{\mathrm{b}}(A)}(M,N)\leq 1$ if M and N lie in the same AR-component, and $\dim\operatorname{Hom}_{D^{\mathrm{b}}(A)}(M,N[2])\leq 1$ otherwise. In the second case, we use the Auslander–Reiten formula

$$\operatorname{Hom}_{D^{\mathrm{b}}(A)}(M,N[2])\simeq D\operatorname{Hom}_{D^{\mathrm{b}}(A)}(N[1],\tau M).$$

In both cases, Lemma 2.1 and $\langle [M],[N]\rangle>0$ imply $\operatorname{Ext}^1_A(M,N)=0$.

Thus, from now on we additionally assume $\langle [M],[N]\rangle=0$.

Before proceeding, we need some properties of the mesh category of a tube. Let $\mathcal{T}$ be a tube of rank r in the Auslander–Reiten quiver of $D^{\mathrm{b}}(A)$. Let $M(i,j)$, $1\leq i\leq r$, $j\geq 1$, be the indecomposable objects in $\mathcal{T}$. We have arrows $M(i,j)\to M(i,j+1)$ for all i and j, and $M(i,j+1)\to M(i+1,j)$ for $1\leq i\leq r-1$ and $j\geq 1$, and $M(r,j+1)\to M(1,j)$ for $j\geq 1$. Thus the object $M(i,j)$ has quasi-length j in $\mathcal{T}$.

If R and S are indecomposable objects in $\mathcal{T}$ with $\langle [R],[S]\rangle=0$, then Lemma 2.1 shows that $\operatorname{Hom}_{D^{\mathrm{b}}(A)}(R,S[2])=0$. We get

$$\langle [R],[S]\rangle=\dim\operatorname{Hom}_{D^{\mathrm{b}}(A)}(R,S)-\dim\operatorname{Hom}_{D^{\mathrm{b}}(A)}(R,S[1]).$$

Assume now that $\operatorname{ql}([R])<r$ and $\operatorname{ql}([S])<r$. We can assume $R=M(1,l)$ for some $1\leq l\leq r-1$. We have to compute $\dim\operatorname{Hom}_{D^{\mathrm{b}}(A)}(R,S)$ and $\dim\operatorname{Hom}_{D^{\mathrm{b}}(A)}(R,S[1])$. Using the mesh category, it is easy to show that $\dim\operatorname{Hom}_{D^{\mathrm{b}}(A)}(R,S)=1$ if and only if $S=M(j,l+i-j+1)$ where $1\leq j\leq l$ and $0\leq i\leq r-l-2+j$. Otherwise, $\dim\operatorname{Hom}_{D^{\mathrm{b}}(A)}(R,S)=0$. Using the

mesh category again and the Auslander–Reiten formula

$$\mathrm{Hom}_{D^{\mathrm{b}}(A)}(R, S[1]) \simeq D\,\mathrm{Hom}_{D^{\mathrm{b}}(A)}(S, \tau R),$$

we get $\dim \mathrm{Hom}_{D^{\mathrm{b}}(A)}(R, S[1]) = 1$ if and only if $S = M(r-i, l-j+i)$ where $0 \le j \le l-1$ and $0 \le i \le r-l-1+j$. The picture in Figure 1 describes the situation. Here we write $i - j$ to mean $\dim \mathrm{Hom}_{D^{\mathrm{b}}(A)}(R, S) = i$ and

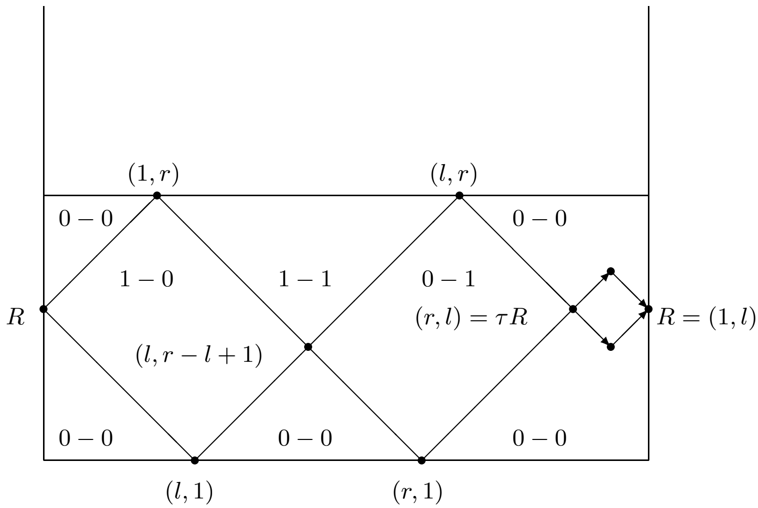

Fig. 1

$\dim \mathrm{Hom}_{D^{\mathrm{b}}(A)}(R, S[1]) = j$ for any module S in the marked region. Thus $\langle [R], [S] \rangle = i - j$. Furthermore, we write (i, j) instead of $M(i, j)$. Note that the picture does not tell what values the Ringel form takes on the objects sitting on the lines. For example, the above precise description shows that the objects on the lower two sides of the $1-1$ triangle also take values $1-1$.

Next, assume that (v) holds. Thus $\mathrm{ql}([M]) + \mathrm{ql}([N]) \le \mathrm{rk}([M]) = \mathrm{rk}([N])$. Assume first that M and N lie in the same Auslander–Reiten component. Thus M and N are objects in a tube of rank $r = \mathrm{rk}([M])$ and $\mathrm{ql}([M]) < r$ and $\mathrm{ql}([N]) < r$. Thus we can apply the above considerations. Without loss of generality we can assume $M = M(1, l)$. But the assumption $\mathrm{ql}([M]) + \mathrm{ql}([N]) \le r$ implies that N has quasi-length strictly smaller than $r - l + 1$. Since we assumed before that $\langle [M], [N] \rangle = 0$, we deduce that $\mathrm{Hom}_{D^{\mathrm{b}}(A)}(M, N[1]) = \mathrm{Ext}^1_A(M, N) = 0$. Similarly one argues in the case when M and $N[1]$ are in the same component.

From now on we can assume additionally $\mathrm{ql}([M]) + \mathrm{ql}([N]) > \mathrm{rk}([M]) = \mathrm{rk}([N])$.

Now assume that (vi) holds. Thus $\langle [M], [N]\phi^j \rangle \neq 0$ for some j. Since $\mathrm{iso}(\mathbf{d}) = \mathrm{iso}(\mathbf{e})$, we know that M and N lie in the same AR-component. Recall that all AR-components are tubes. Thus we can assume that M and

N lie in a tube of rank r with $M = M(1,l)$. By assumption $\langle [M],[N]\phi^i\rangle < 0$ with $i = \min\{j \geq 1 \mid \langle [M],[N]\phi^j\rangle \neq 0\}$. Recall that we have also assumed $\langle [M],[N]\rangle = 0$. Now we use the considerations which led to Figure 1 and get $\mathrm{Hom}_{D^{\mathrm{b}}(A)}(M,N[1]) = \mathrm{Ext}^1_A(M,N) = 0$.

Finally, we assume that (vii) holds. This implies that M and $N[1]$ lie in the same AR-component of $D^{\mathrm{b}}(A)$. Having in mind that $\langle [M],N[1]\rangle = -\langle [M],[N]\rangle$ we can proceed as before to get again $\mathrm{Ext}^1_A(M,N) = 0$.

This finishes the proof of one direction of the theorem. Now we prove the other direction.

Let $\mathbf{d}$ and $\mathbf{e}$ be Schur roots with $\langle \mathbf{d},\mathbf{e}\rangle < 0$; then $\mathrm{Ext}^1_A(M,N) \neq 0$ for all M in $C(\mathbf{d})$ and N in $C(\mathbf{e})$. This implies $\mathrm{ext}^1_A(C(\mathbf{d}),C(\mathbf{e})) \neq 0$. Thus assume that $\langle \mathbf{d},\mathbf{e}\rangle \geq 0$ and none of (i)–(vii) holds. Thus we are in the following situation:

(1) $q(\mathbf{d}) = q(\mathbf{e}) = 1$;
(2) $\langle \mathbf{d},\mathbf{e}\rangle = 0$;
(3) $\mathrm{ql}(\mathbf{d}) + \mathrm{ql}(\mathbf{e}) > \mathrm{rk}(\mathbf{d}) = \mathrm{rk}(\mathbf{e})$;
(4) one of the following holds:
  (a) $\mathrm{iso}(\mathbf{d}) = \mathrm{iso}(\mathbf{e})$, $\langle \mathbf{d},\mathbf{e}\phi^j\rangle \neq 0$ for some j, and $\langle \mathbf{d},\mathbf{e}\phi^i\rangle > 0$ where $i = \min\{j \geq 1 \mid \langle \mathbf{d},\mathbf{e}\phi^j\rangle \neq 0\}$,
  (b) $\mathrm{iso}(\mathbf{d}) = -\,\mathrm{iso}(\mathbf{e})$, $\langle \mathbf{d},\mathbf{e}\phi^j\rangle \neq 0$ for some j, and $\langle \mathbf{d},\mathbf{e}\phi^i\rangle < 0$ where $i = \min\{j \geq 1 \mid \langle \mathbf{d},\mathbf{e}\phi^j\rangle \neq 0\}$.

Note that conditions (1) and (4) imply that $\mathrm{rk}(\mathbf{d}) = \mathrm{rk}(\mathbf{e})$.

Let M and N be the (unique up to isomorphism) A-modules in $\mathrm{ind}_A(\mathbf{d})$ and $\mathrm{ind}_A(\mathbf{e})$, respectively. By Lemma 2.3 and (4), either M and N, or M and $N[1]$, lie in the same Auslander–Reiten component of $D^{\mathrm{b}}(A)$. Again, let us first consider the case where M and N are in a tube of rank r, and without loss of generality let $M = M(1,l)$. Thus (4)(a) holds. By (3) the quasi-length of N is at least $r-l+1$. Then (2) and (4)(a) imply that N must lie in the $1-1$ triangle of Figure 1. This implies $\mathrm{Ext}^1_A(M,N) \neq 0$. Since $\mathbf{d}$ and $\mathbf{e}$ are Schur roots with $q(\mathbf{d}) = q(\mathbf{e}) = 1$, the orbits $\mathcal{O}(M)$ and $\mathcal{O}(N)$ are open. Thus $\mathrm{Ext}^1_A(M,N) \neq 0$ implies that $\mathrm{ext}^1_A(C(\mathbf{d}),C(\mathbf{e})) \neq 0$. The case when M and $N[1]$ are in the same component yields the same result. This finishes the proof.

6. Application. Let Λ denote the preprojective algebra for a quiver of type $\mathbb{A}_5$, i.e. $\Lambda = kQ/I$ where Q is the quiver

$$1 \underset{a_1}{\overset{\overline{a}_1}{\rightleftarrows}} 2 \underset{a_2}{\overset{\overline{a}_2}{\rightleftarrows}} 3 \underset{a_3}{\overset{\overline{a}_3}{\rightleftarrows}} 4 \underset{a_4}{\overset{\overline{a}_4}{\rightleftarrows}} 5$$

and the ideal I is generated by the following set of relations:

$$\{\overline{a}_1a_1, a_1\overline{a}_1 - \overline{a}_2a_2, a_2\overline{a}_2 - \overline{a}_3a_3, a_3\overline{a}_3 - \overline{a}_4a_4, a_4\overline{a}_4\}.$$

Then Λ has a Galois covering $F\colon \widetilde{\Lambda} \to \Lambda$ with Galois group $G = (\mathbb{Z}, +)$ where $\widetilde{\Lambda}$ is given by the infinite quiver with relations described in Figure 2.

$$\begin{tikzcd}[column sep=small]
1^{(2)} \arrow[dr, "\overline{a}_1^{(2)}"] & \vdots & 3^{(2)} \arrow[dl, "a_2^{(1)}"'] \arrow[dr, "\overline{a}_3^{(2)}"] & \vdots & 5^{(2)} \arrow[dl, "a_4^{(1)}"'] \\
 & 2^{(1)} \arrow[dl, "a_1^{(1)}"'] \arrow[dr, "\overline{a}_2^{(1)}"] & & 4^{(1)} \arrow[dl, "a_3^{(1)}"'] \arrow[dr, "\overline{a}_4^{(1)}"] & \\
1^{(1)} \arrow[dr, "\overline{a}_1^{(1)}"] & & 3^{(1)} \arrow[dl, "a_2^{(0)}"'] \arrow[dr, "\overline{a}_3^{(1)}"] & & 5^{(1)} \arrow[dl, "a_4^{(0)}"'] \\
 & 2^{(0)} \arrow[dl, "a_1^{(0)}"'] \arrow[dr, "\overline{a}_2^{(0)}"] & & 4^{(0)} \arrow[dl, "a_3^{(0)}"'] \arrow[dr, "\overline{a}_4^{(0)}"] & \\
1^{(0)} & & 3^{(0)} & & 5^{(0)} \\
 & \vdots & & \vdots &
\end{tikzcd}$$

Fig. 2. The quiver $\widetilde{\Lambda}$ with relations $\{\overline{a}_1^{(i+1)} a_1^{(i)},\ a_1^{(i)}\overline{a}_1^{(i)} - \overline{a}_2^{(i)} a_2^{(i-1)},\ a_2^{(i)}\overline{a}_2^{(i)} - \overline{a}_3^{(i+1)} a_3^{(i)},\ a_3^{(i)}\overline{a}_3^{(i)} - \overline{a}_4^{(i)} a_4^{(i-1)},\ a_4^{(i)}\overline{a}_4^{(i)} \mid i \in \mathbb{Z}\}$

The generator of G acts as $x^{(i)} \mapsto x^{(i+1)}$. See [10, Section 3] for basic information on Galois coverings. We need several subcategories of $\widetilde{\Lambda}$. Firstly, Γ_0 will be the full subcategory with objects

$$\{1^{(0)}, 3^{(0)}, 5^{(0)}, 2^{(0)}, 4^{(0)}, 1^{(1)}, 3^{(1)}, 5^{(1)}\}$$

(this is tame concealed of type $\widetilde{\mathbb{E}}_7$), and Δ_0 with objects

$$\{2^{(0)}, 4^{(0)}, 1^{(1)}, 3^{(1)}, 5^{(1)}, 2^{(1)}, 4^{(1)}\}$$

(this is tame concealed of type $\widetilde{\mathbb{D}}_6$). Next, we have Γ with objects

$$\{1^{(0)}, 3^{(0)}, 5^{(0)}, 2^{(0)}, 4^{(0)}, 1^{(1)}, 3^{(1)}, 5^{(1)}, 2^{(1)}, 4^{(1)}\}$$

and Δ with objects

$$\{2^{(0)}, 4^{(0)}, 1^{(1)}, 3^{(1)}, 5^{(1)}, 2^{(1)}, 4^{(1)}, 1^{(2)}, 3^{(2)}, 5^{(2)}\};$$

these are both tubular of type $(6, 3, 2)$. Note that $\widetilde{\Lambda}$ is the repetitive algebra of Γ or of Δ. As a consequence the AR-quiver of $\widetilde{\Lambda}$ consists only of stable (and standard) tubes, with the exception of the tubes which contain the projective-injective modules. These are obtained (up to translation by G) from (co-)ray insertions at stable tubes over the tame concealed algebras Γ_0 or Δ_0. All indecomposable modules which do not belong to these tubes have (up to shift by G) support in Γ or $\Delta^{(-1)}$. Thus the push-down functor F_λ associated to F is dense by [9, Section 2]. See also the brief discussion in [7, Section 6], and for example [14, Section 4].

For convenience, we display the dimension vectors of the tubes coming from Γ_0 and Δ_0 below. For Γ_0, the dimension vectors of homogeneous modules are multiples of

$$\mathbf{h}_0 := \begin{smallmatrix} 1 & & 2 & & 1 \\ & 3 & & 3 & \\ 1 & & 2 & & 1 \end{smallmatrix}.$$

The dimension vectors at the "mouth" of the non-homogeneous tubes are described in Figure 3. Denote by $\mathcal{T}_0$ the family of indecomposable, non-projective $\widetilde{\Lambda}$-modules with dimension vector being a multiple of $\mathbf{h}_0$ or corresponding to the three non-homogeneous tubes described in Figure 3.

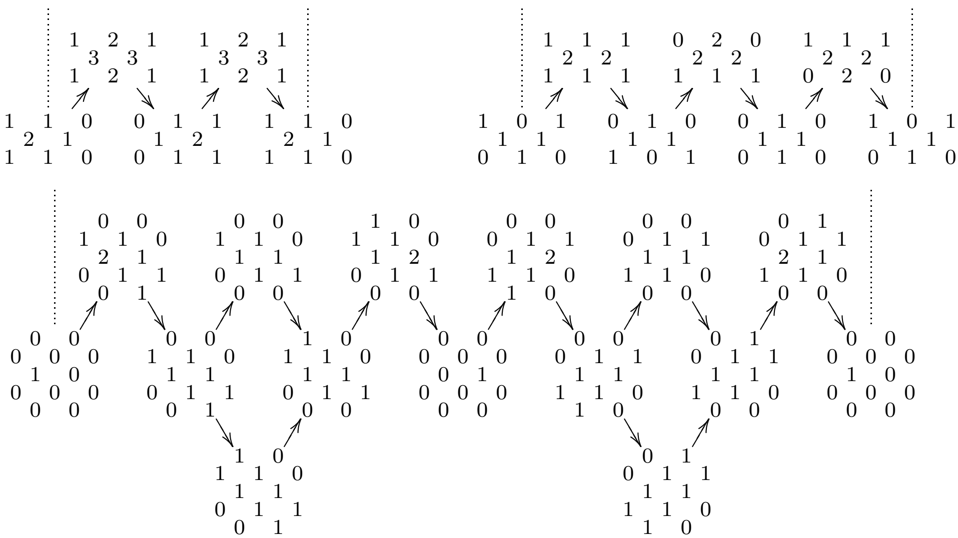

Fig. 3. Non-homogeneous tubes in $\mathcal{T}_0$

For Δ_0, the dimension vectors of homogeneous modules are multiples of

$$\mathbf{h}_\infty := \begin{smallmatrix} & 1 & & 1 & \\ 1 & & 2 & & 1 \\ & 1 & & 1 & \end{smallmatrix}.$$

The dimension vectors at the "mouth" of the non-homogeneous tubes are described in Figure 4. Denote by $\mathcal{T}_\infty$ the family of indecomposable, non-projective $\widetilde{\Lambda}$-modules with dimension vector being a multiple of $\mathbf{h}_\infty$ or corresponding to the three non-homogeneous tubes described in Figure 4.

Lemma 6.1. *The translation functor* $?^{(+1)}\colon \widetilde{\Lambda} \to \widetilde{\Lambda}$ *induces a functor for the stable category* $\underline{\mathrm{mod}}(\widetilde{\Lambda})$, *and we have*

$$\tau_{\widetilde{\Lambda}}(?^{(+1)}) \cong ?[1]$$

where $?[1]$ *is the translation functor for the triangulated category* $\underline{\mathrm{mod}}(\widetilde{\Lambda})$, *and* $\tau_{\widetilde{\Lambda}}$ *is the Auslander–Reiten translate which is a functor for the stable category* (*since* $\widetilde{\Lambda}$ *is selfinjective*).

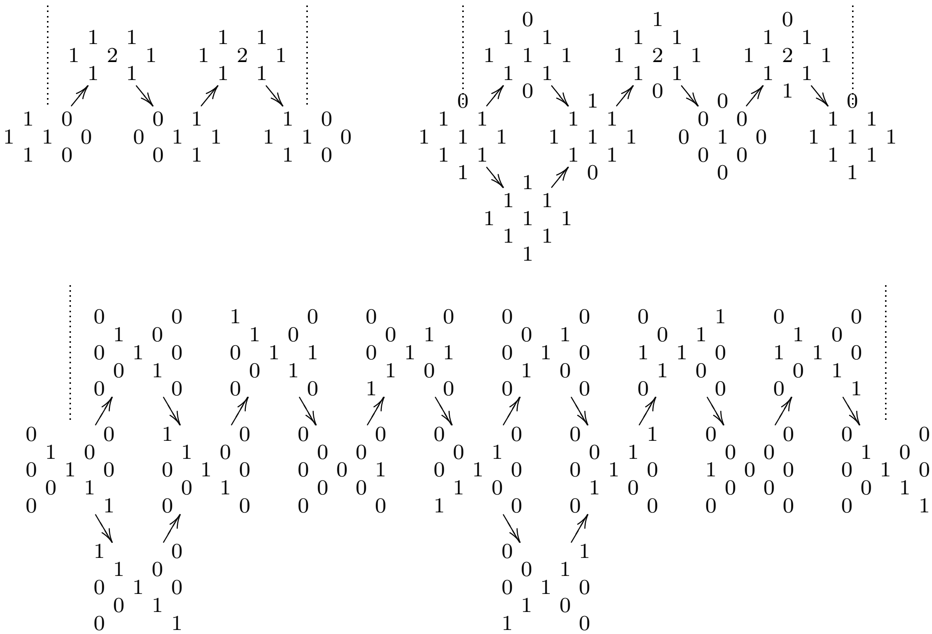

Fig. 4. Non-homogeneous tubes in $\mathcal{T}_\infty$

Proof. From our calculations above it is clear that the isomorphism holds for simple $\widetilde{\Lambda}$-modules, thus it holds for all finitely generated modules. ■

If X is an indecomposable homogeneous $\widetilde{\Lambda}$-module of quasi-length 1, then it has (up to translation by G) support in Γ or $\Delta^{(-1)}$. In the first case $[X]_{\widetilde{\Lambda}} = a\mathbf{h}_0 + b\mathbf{h}_\infty$, in the second $[X]_{\widetilde{\Lambda}} = b\mathbf{h}_\infty^{(-1)} + a\mathbf{h}_0$, where $\gcd(a,b) = 1$ in both cases.

Lemma 6.2. *With the above notation we have in both cases*

$$\dim \operatorname{Hom}_{\widetilde{\Lambda}}(X, X^{(+1)}) = 6a^2 + 6ab + 2b^2 \quad \textit{and} \quad \operatorname{Hom}_{\widetilde{\Lambda}}(X, X^{(-1)}) = 0.$$

As a consequence

$$\dim \operatorname{End}_\Lambda(F_\lambda X) = 1 + 6a^2 + 6ab + 2b^2$$

where F_λ is the push-down functor associated to the Galois covering F.

Proof. Since X is homogeneous, by Lemma 6.1 we get $X^{(+1)} \cong X[1]$. Thus we have a short exact sequence

$$0 \to X \to I_X \to X^{(+1)} \to 0$$

with I_X projective-injective. Applying $\operatorname{Hom}_{\widetilde{\Lambda}}(X, -)$ to this short exact sequence, we find that

$$\dim \operatorname{Hom}_{\widetilde{\Lambda}}(X, I_X) = \dim \operatorname{Hom}_{\widetilde{\Lambda}}(X, X^{(+1)})$$

since $\mathrm{Hom}_{\widetilde{\Lambda}}(X,X)$ and $\mathrm{Ext}^1_{\widetilde{\Lambda}}(X,X)$ are both of dimension 1. On the other hand, $\widetilde{\Lambda}$ is triangular, thus we can easily determine the decomposition of I_X into direct summands using $[I_X] = [X] + [X^{(+1)}]$, and our first claim follows. The same calculation shows that the supports of $\mathrm{top}\, X^{(+1)}$ and X are disjoint, and the second claim follows. Note that $\mathrm{top}\, X^{(+1)} \cong \mathrm{top}\, I_X$. Finally

$$\mathrm{Hom}_{\Lambda}(F_\lambda X, F_\lambda X) = \bigoplus_{g\in\mathbb{Z}} \mathrm{Hom}_{\widetilde{\Lambda}}(X, X^{(g)})$$

and our last claim follows from the previous calculations since only for $g \in \{-1,0,1\}$ are the supports of X and $X^{(g)}$ not disjoint. ∎

Recall that Γ, being a tubular algebra, is quasi-tilted. Since $\widetilde{\Lambda}$ is the repetitive algebra of Γ, we can find in $\underline{\mathrm{mod}}(\widetilde{\Lambda})$ a full hereditary subcategory $\underline{\mathcal{H}}$ such that

$$\underline{\mathrm{mod}}(\widetilde{\Lambda}) = \bigvee_{i\in\mathbb{Z}} \underline{\mathcal{H}}[i].$$

We describe a choice of a subcategory $\mathcal{H} \subset \mathrm{mod}(\widetilde{\Lambda})$ which induces such an $\underline{\mathcal{H}}$. Denote by $\mathcal{T}_0$ (respectively $\mathcal{T}_\infty$) the tubular family in $\mathrm{mod}\,(\widetilde{\Lambda})$ coming from Γ_0 (respectively Δ_0) as described above. Next,

$$\mathcal{T}_+ \subset \mathrm{mod}(\Gamma) \subset \mathrm{mod}(\widetilde{\Lambda})$$

is given by all indecomposable Γ-modules X with

$$\langle \mathbf{h}_0, [X]\rangle_\Gamma > 0 \quad \text{and} \quad \langle [X], \mathbf{h}_\infty\rangle_\Gamma > 0$$

(this just accounts for all stable tubular families in $\mathrm{mod}(\Gamma)$). Similarly

$$\mathcal{T}_- \subset \mathrm{mod}(\Delta^{(-1)}) \subset \mathrm{mod}(\widetilde{\Lambda})$$

is given by all indecomposable $\Delta^{(-1)}$-modules Y with

$$\langle \mathbf{h}_\infty^{(-1)}, [Y]\rangle_{\Delta^{(-1)}} > 0 \quad \text{and} \quad \langle [Y], \mathbf{h}_0\rangle_{\Delta^{(-1)}} > 0$$

(this just accounts for all stable tubular families in $\mathrm{mod}(\Delta^{(-1)})$). Then let

$$\mathcal{H} = \mathcal{T}_- \vee \mathcal{T}_0 \vee \mathcal{T}_+ \vee \mathcal{T}_\infty.$$

Note that by Lemma 6.1 and tubularity we also have

$$\underline{\mathrm{mod}}(\widetilde{\Lambda}) = \bigvee_{g\in\mathbb{Z}} \underline{\mathcal{H}}^{(g)}.$$

Next, the identification of $\widetilde{\Lambda}$ with the repetitive algebra of Γ (respectively $\Delta^{(-1)}$) gives us canonical isomorphisms

$$K_0(\Gamma) \overset{\pi_\Gamma}{\leftarrow} K_0(\underline{\mathrm{mod}}(\widetilde{\Lambda})) \overset{\pi_\Delta}{\rightarrow} K_0(\Delta^{(-1)}),$$

which in turn yields an isomorphism $K_0(\Delta^{(-1)}) \to K_0(\Gamma)$. Here $K_0(\underline{\mathrm{mod}}(\widetilde{\Lambda}))$ denotes the Grothendieck group of the triangulated category $\underline{\mathrm{mod}}(\widetilde{\Lambda})$. If $X \in \underline{\mathrm{mod}}(\widetilde{\Lambda})$ we denote by $[X]$ the corresponding class in $K_0(\Gamma)$.

Define the set of roots (respectively positive roots) for Γ as

$$R := \{\mathbf{x} \in K_0(\Gamma) \setminus \{0\} \mid q_\Gamma(\mathbf{x}) \le 1\},$$

respectively

$$R^+ := \{\mathbf{x} \in R \mid \langle \mathbf{x}, \mathbf{h}_\infty\rangle_\Gamma > 0 \text{ or } (\langle \mathbf{x}, \mathbf{h}_\infty\rangle_\Gamma = 0 \text{ and } \langle \mathbf{h}_0, \mathbf{x}\rangle_\Gamma > 0)\}.$$

Then $R^+ = \{[X] \mid X \in \underline{\mathcal{H}} \text{ indecomposable}\}$. Similarly to the case of dimension vectors, let R^+_{S} be the set of Schur roots in R^+. Note that these correspond precisely to the classes of objects in $\underline{\mathcal{H}}$ with trivial endomorphism rings. Moreover, for $\mathbf{x} \in R^+$ we denote by $\underline{\mathcal{H}}(\mathbf{x})$ the set of all indecomposable $X \in \underline{\mathcal{H}}$ with $[X] = \mathbf{x}$.

REMARK 6.3. Let $\mathbf{x} \in R^+$ be isotropic, i.e. $\langle \mathbf{x}, \mathbf{x}\rangle_\Gamma = 0$. Then $\mathbf{x} = a\mathbf{h}_0 + b\mathbf{h}_\infty$ with $a > 0$ or ($a = 0$ and $b > 0$). Thus if we define for $\gamma \in \mathbb{Q}_\infty$ the set $R^\gamma = \{\mathbf{y} \in R^+ \mid \langle \mathbf{h}_0, \mathbf{y}\rangle_\Gamma / \langle \mathbf{y}, \mathbf{h}_\infty\rangle_\Gamma = \gamma\}$ we find that $\mathbf{x} \in R^{b/a}$. If $\mathbf{y} \in R^\gamma$ with $\gamma \ne b/a$ we get $0 \ne \langle \mathbf{x}, \mathbf{y}\rangle_\Gamma = -\langle \mathbf{y}, \mathbf{x}\rangle_\Gamma$. Since $\underline{\mathcal{H}}$ is hereditary we conclude that $\mathrm{Ext}^1_{\widetilde{\Lambda}}(X, Y) \ne 0$ or $\mathrm{Ext}^1_{\widetilde{\Lambda}}(Y, X) \ne 0$ if $X \in \underline{\mathcal{H}}(\mathbf{x})$ and $Y \in \underline{\mathcal{H}}(\mathbf{y})$.

Now we might define $\mathcal{T}_\gamma = \{X \in \underline{\mathcal{H}} \mid X \text{ indecomposable and } [X] \in R^\gamma\}$. In our situation this always gives a tubular family of type $(6, 3, 2)$. These considerations as well as the definition of R^+ are inspired by [15].

It is easy to define a map $\sigma\colon R^+ \to K_0(\widetilde{\Lambda})$ such that for each $\mathbf{x} \in R^+$ there exists an indecomposable object $X \in \underline{\mathcal{H}}$ with $[X] = \mathbf{x}$ and $\sigma(\mathbf{x}) = [X]_{\widetilde{\Lambda}}$. Moreover, we have a linear map $\delta\colon K_0(\widetilde{\Lambda}) \to K_0(\Lambda)$ with the property $\delta([X]_{\widetilde{\Lambda}}) = [F_\lambda X]_\Lambda$ for all $\widetilde{\Lambda}$-modules X.

LEMMA 6.4. *For $X, Y \in \mathcal{H} \subset \mathrm{mod}(\widetilde{\Lambda})$ we have*

$$\mathrm{Ext}^1_\Lambda(F_\lambda X, F_\lambda Y) \cong \mathrm{Ext}^1_{\widetilde{\Lambda}}(X, Y) \oplus \mathrm{Ext}^1_{\widetilde{\Lambda}}(Y, X).$$

Proof. It is easy to see that

$$\mathrm{Ext}^1_\Lambda(F_\lambda X, F_\lambda Y) \cong \bigoplus_{i\in\mathbb{Z}} \mathrm{Ext}^1_{\widetilde{\Lambda}}(X, Y^{(i)}).$$

Thus since $\widetilde{\Lambda}$ is self injective, from Lemma 6.1 we obtain

$$\begin{aligned}\mathrm{Ext}^1_\Lambda(F_\lambda X, F_\lambda Y) &\cong \bigoplus_{i\in\mathbb{Z}} \underline{\mathrm{Hom}}_{\widetilde{\Lambda}}(X[-1], Y^{(i)}) \cong \bigoplus_{i\in\mathbb{Z}} \underline{\mathrm{Hom}}_{\widetilde{\Lambda}}(X, \tau_{\widetilde{\Lambda}}^{-i} Y[i+1])\\ &= \underline{\mathrm{Hom}}_{\widetilde{\Lambda}}(X, \tau_{\widetilde{\Lambda}} Y) \oplus \underline{\mathrm{Hom}}_{\widetilde{\Lambda}}(X, Y[1]);\end{aligned}$$

for the last equality we have used the fact that in $\underline{\mathrm{mod}}(\widetilde{\Lambda})$ there are only maps from $\underline{\mathcal{H}}[0]$ to $\underline{\mathcal{H}}[i]$ for $i \in \{0, 1\}$. Our claim now follows from the Auslander–Reiten formula. ∎

Let $\mathbf{x} \in R^+$. We denote by $\mathrm{ind}_\Lambda(\delta\sigma(\mathbf{x}), \mathbf{x})$ the subset of $\mathrm{mod}_\Lambda(\delta\sigma(\mathbf{x}))$ which corresponds to the modules of the form $F_\lambda X$ for $X \in \underline{\mathcal{H}}(\mathbf{x})$. We write

$C_\Lambda(\mathbf{x})$ for the Zariski closure $\overline{\mathrm{ind}_\Lambda(\delta\sigma(\mathbf{x}),\mathbf{x})}$, and $C_\Lambda(i)$ for the closure of the orbit of the indecomposable projective module corresponding to the vertex i.

Remark 6.5. For our next result we need the following observation which should be well known: Let $F\colon \widetilde{\Lambda} \to \Lambda$ be a Galois covering. For each dimension vector $\mathbf{d}$ of $\widetilde{\Lambda}$ we have an obvious morphism of varieties

$$\Phi_{\mathbf{d}}\colon \mathrm{mod}_{\widetilde{\Lambda}}(\mathbf{d}) \times \mathrm{Gl}(\delta\mathbf{d}) \to \mathrm{mod}_\Lambda(\delta\mathbf{d}).$$

If $U \subset \mathrm{mod}_{\widetilde{\Lambda}}(\mathbf{d})$ we write $F_\lambda(U) = \Phi_{\mathbf{d}}(U \times \mathrm{Gl}(\delta\mathbf{d}))$. Now if $\widetilde{\Lambda}$ is locally support-finite, each irreducible component $C \subset \mathrm{mod}_\Lambda(\mathbf{h})$ is of the form $\overline{F_\lambda(C')}$ for some irreducible component $C' \subset \mathrm{mod}_{\widetilde{\Lambda}}(\mathbf{d})$ and some $\mathbf{d}$ with $\delta\mathbf{d} = \mathbf{h}$. In particular C is indecomposable if and only if C' is indecomposable.

In fact, since $\widetilde{\Lambda}$ is locally support-finite we conclude from [9] that there exists a finite collection of dimension vectors $\mathbf{d}_1,\dots,\mathbf{d}_m$ such that

$$\mathrm{mod}_\Lambda(\mathbf{d}) = \bigcup_{i=1}^{m} F_\lambda(\mathrm{mod}_{\widetilde{\Lambda}}(\mathbf{d}_i)).$$

If $C'_1,\dots,C'_n$ denotes the collection of the irreducible components of the varieties $\mathrm{mod}_{\widetilde{\Lambda}}(\mathbf{d}_i)$, $1 \le i \le m$, we trivially have

$$\mathrm{mod}_\Lambda(\mathbf{d}) = \bigcup_{i=1}^{n} \overline{F_\lambda(C'_i)};$$

but the $\overline{F_\lambda(C'_i)}$ are irreducible closed subsets of $\mathrm{mod}_\Lambda(\mathbf{h})$ and our claim follows.

Theorem 6.6. *The map $\mathbf{x} \mapsto C_\Lambda(\mathbf{x})$ induces a bijection between the set R_{S}^+ of Schur roots and the indecomposable irreducible components of varieties of Λ-modules which contain no projective-injective module. For $\mathbf{x} \in R_{\mathrm{S}}^+$ the component $C_\Lambda(\mathbf{x})$ contains an open orbit if $\mathbf{x}$ is real, otherwise $C_\Lambda(\mathbf{x})$ is not generically reduced.*

Proof. By Theorem 1.1 and Remark 6.5 our $C_\Lambda(\mathbf{x})$ with $\mathbf{x} \in R_{\mathrm{S}}^+$ are the only candidates for indecomposable components, besides the $C_\Lambda(i)$.

Assume first that $\mathbf{x}$ is real. Then there exists a unique indecomposable $X \in \underline{\mathcal{H}}$ with $[X] = \mathbf{x}$, moreover $\mathrm{Ext}^1_{\widetilde{\Lambda}}(X,X) = 0$, thus by Lemma 6.4 we also have $\mathrm{Ext}^1_\Lambda(F_\lambda X, F_\lambda X) = 0$. By Voigt's Lemma $F_\lambda X$ represents an open orbit in $\mathrm{mod}_\Lambda(\delta\sigma(\mathbf{x}))$.

If $\mathbf{x}$ is not real, we have a one-parameter family $(X_t)_{t\in k} \subset \underline{\mathcal{H}}$ of $\widetilde{\Lambda}$-modules with $[X_t] = \mathbf{x}$ for all t. From Lemma 6.2 (use the notation from there) we have

$$\delta\sigma(\mathbf{x}) = [F_\lambda X_t]_\Lambda = (2a+b, 3a+2b, 4a+2b, 3a+2b, 2a+b)$$

and

$$\dim \mathrm{End}_\Lambda(F_\lambda X_t) = 1 + 6a^2 + 2b^2 + 6ab$$

for some $a, b \in \mathbb{N}_0$. Thus

$$\dim \mathrm{End}_{\Lambda}(F_{\lambda} X_t) - q_{\mathbb{A}_5}([F_{\lambda} X_t]_{\Lambda}) = 1.$$

Now $(F_{\lambda} X_t)_{t \in k}$ is a family of non-isomorphic modules, and they are dense in some component since for example by [16, 12] each irreducible component of $\mathrm{mod}_{\Lambda}(\mathbf{d})$ has dimension $\dim \mathrm{Gl}(\mathbf{d}) - q_{\mathbb{A}_5}(\mathbf{d})$. Finally $\dim \mathrm{Ext}^1_{\Lambda}(F_{\lambda} X_t, F_{\lambda} X_t) = 2$ by Lemma 6.4. Thus $C_{\Lambda}(\mathbf{x})$ is not generically reduced by Voigt's Lemma. Note that for a non-real root $\mathbf{x} \in R^+_{\mathrm{S}}$, we have $\mathrm{ext}^1_{\Lambda}(C_{\Lambda}(\mathbf{x}), C_{\Lambda}(\mathbf{x})) = 0$, but $\dim \mathrm{Ext}^1_{\Lambda}(X, X) = 2$ for all X in a dense subset U of $C_{\Lambda}(\mathbf{x})$. ∎

COROLLARY 6.7. *Let* $\mathbf{x}, \mathbf{y} \in R^+_{\mathrm{S}}$. *Then* $\mathrm{ext}^1_{\Lambda}(C_{\Lambda}(\mathbf{x}), C_{\Lambda}(\mathbf{y})) = 0$ *if and only if there exist* $X \in \underline{\mathcal{H}}(\mathbf{x})$ *and* $Y \in \underline{\mathcal{H}}(\mathbf{y})$ *with* $\mathrm{Ext}^1_{\widetilde{\Lambda}}(X, Y) = 0 = \mathrm{Ext}^1_{\widetilde{\Lambda}}(Y, X)$.

Proof. This follows directly from Theorem 6.6 and Lemma 6.4. ∎

We conclude that each irreducible component C of $\mathrm{mod}_{\Lambda}(\mathbf{d})$ admits a unique decomposition into indecomposable components

$$\bigoplus_{i=1}^{l} C_{\Lambda}(\mathbf{x}_i)^{n(i)} \oplus \bigoplus_{i=1}^{5} C_{\Lambda}(i)^{m(i)}$$

with the $\mathbf{x}_i \in R^+_{\mathrm{S}}$ pairwise different. If say $\mathbf{x}_1$ is isotropic, we have

$$\langle \mathbf{h}_0, \mathbf{x}_i \rangle / \langle \mathbf{x}_i, \mathbf{h}_{\infty} \rangle = \langle \mathbf{h}_0, \mathbf{x}_1 \rangle / \langle \mathbf{x}_1, \mathbf{h}_{\infty} \rangle$$

for $i = 2, \ldots, l$ by Remark 6.3, i.e. the $\mathbf{x}_i$ are dimension vectors of indecomposable objects which all belong to the same tubular family, and we conclude from the last corollary that $l \leq 9 = (6-1)+(3-1)+(2-1)+1$. If all the $\mathbf{x}_i$ are real, we get $l \leq 10$. In fact, in this case there exist $X_i \in \underline{\mathcal{H}}(\mathbf{x}_i)$ with $\mathrm{Ext}^1_{\widetilde{\Lambda}}(X_i, X_j) = 0$ for all $1 \leq i, j \leq l$ (see the proof of Theorem 6.6 and Corollary 6.7). Thus $\bigoplus_{i=1}^{l} X_i$ is a partial tilting object in the hereditary category $\underline{\mathcal{H}}$, which has Grothendieck group isomorphic to $K_0(\Gamma) \cong \mathbb{Z}^{10}$. Thus in any case the number of non-isomorphic summands is bounded by 15, which is the number of positive roots for $\mathbb{A}_5$.

Let A be any finitely generated k-algebra, and $\{C_i \mid i \in I\}$ a set of pairwise different indecomposable irreducible components of varieties of A-modules such that $\mathrm{ext}^1_A(C_i, C_j) = 0$ for all $i \neq j$. If the set is maximal with this property, then we call it a *maximal set of indecomposable components* for A.

For $A = \Lambda$ it follows from our considerations that a maximal set of indecomposable components for Λ contains either 14 or 15 components. At most one of them does not contain a dense orbit, and in this case, the maximal set contains 14 components.

Note also that $\mathrm{ext}^1_{\Lambda}(C, C) = 0$ for any indecomposable irreducible component C (there is no contradiction to the fact that $\dim \mathrm{Ext}^1_{\Lambda}(M, M) \geq 2$ for all $M \in C$ if C corresponds to an isotropic root as noted in the proof of

Theorem 6.6). This immediately implies $\mathrm{ext}^1_\Lambda(C,C)=0$ for any irreducible component C.

Finally, note that the preprojective algebra $P(Q)$ for Q a quiver of type $\mathbb{D}_4$ admits a Galois covering, which is the repetitive algebra of a tubular algebra of type $(3,3,3)$. One checks easily that Lemma 6.1 holds also in this situation. Thus we can repeat the above program. In particular, we find that a maximal set of indecomposable components for $P(Q)$ contains either $11=(3-1)+(3-1)+(3-1)+1+4$ or $12=8+4$ elements. Observe that a quiver of type $\mathbb{D}_4$ admits 12 positive roots.

One might ask the following question: Let Q be any Dynkin quiver. Assume that the number of positive roots of Q is n. Let $\{C_i \mid i \in I\}$ be a maximal set of indecomposable components for $P(Q)$. Assume that C_i contains a dense n_i-parameter family of indecomposable $P(Q)$-modules for all i. Is I a finite set and does the formula

$$|I| = n - \sum_{i\in I} n_i$$

hold? This was verified before for Q of type $\mathbb{A}_i$ with $i \leq 4$. By our above considerations this holds also for Q of type $\mathbb{A}_5$ or $\mathbb{D}_4$.

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