COLLOQUIUM MATHEMATICUM

BOUNDARY POTENTIAL THEORY FOR
STABLE LÉVY PROCESSES

BY

PAWEŁ SZTONYK (Wrocław)

Abstract. We investigate properties of harmonic functions of the symmetric stable Lévy process on \( \mathbb{R}^d \) without the assumption that the process is rotation invariant. Our main goal is to prove the boundary Harnack principle for Lipschitz domains. To this end we improve the estimates for the Poisson kernel obtained in a previous work. We also investigate properties of harmonic functions of Feynman–Kac semigroups based on the stable process. In particular, we prove the continuity and the Harnack inequality for such functions.

1. Introduction. For \( \alpha \in (0, 2) \), a Lévy process \( X_t \) on \( \mathbb{R}^d \) with characteristic exponent \( \Phi \) is called stable with index \( \alpha \) if \( \Phi(ku) = k^\alpha \Phi(u) \) for \( k > 0 \), \( u \in \mathbb{R}^d \). The stable processes appear in a natural way in limit theorems and have the scaling property: for every \( a > 0 \) the rescaled process \( a^{-1/\alpha} X_t \) has the same law as \( X_t \).

Recently, remarkable progress has been made in the potential theory of the rotation invariant \( \alpha \)-stable Lévy processes (for definitions see Preliminaries). The results obtained include estimates of the Green function and Poisson kernel ([18], [11], [17]), the boundary Harnack principle for \( \alpha \)-harmonic functions ([6], [9], [21]), and similar developments in the potential theory of the \( \alpha \)-stable Schrödinger operator ([7], [8]). Many of the results are based on the exact formulae for the Poisson kernel and the Green function for the ball established by M. Riesz ([19], [20], [4]).

In this paper we extend some of these results to \( \alpha \)-stable Lévy processes which are symmetric but not necessarily rotation invariant. We focus on the behaviour of such processes near the boundary of a domain \( D \subset \mathbb{R}^d \).

The results of the present paper complement the earlier ones contained in [10]. We note that the main results of [10] were restricted to \( \alpha \leq 1 \). They were based on certain estimates of the harmonic measure, which substitute the exact formula for the Poisson kernel of the ball (see Proposition 4.1

2000 Mathematics Subject Classification: Primary 60J45, 31C05; Secondary 60G51.

Key words and phrases: stable processes, boundary Harnack principle, 3G Theorem, \( q \)-harmonic functions.

Research supported by KBN grant 2 P03A 041 22.
below). In Sections 3 and 4 below we improve these estimates and we obtain the Carleson estimate, boundary Harnack principle and 3G Theorem for all \( \alpha \in (0, 2) \). We note that the paper is related to the papers of R. F. Bass and D. A. Levin [1] and Z. Vondraček [24]. In particular, we use in a crucial way the Harnack inequality of [1]. We also extend and strengthen some of the results of [24] (see, e.g., Lemmas 4.1 and 4.2).

The results allow for a study of harmonic functions of the Feynman–Kac perturbation of \( X_t \) by a multiplicative functional \( \exp(\int_0^T q(X_t) \, dt) \). In Section 5 we prove the continuity and Harnack inequality for such functions.

2. Preliminaries. In what follows \( \alpha \in (0, 2) \) and \( d \geq 2 \). We denote by \((X_t, P^x)\) a symmetric \( \alpha \)-stable Lévy process in \( \mathbb{R}^d \) (i.e. homogeneous, with independent increments), with characteristic function of the form

\[
E^0 e^{i \langle u, X_t \rangle} = e^{-t \Phi(u)}, \quad u \in \mathbb{R}^d, \ t \geq 0,
\]

where the characteristic exponent \( \Phi \) is given by

\[
\Phi(u) = \int_{S(0,1)} |\langle u, \xi \rangle|^\alpha \mu(d\xi),
\]

and \( \mu \) is a finite, symmetric measure on \( S(0,1) \). We assume that \( \mu \) is absolutely continuous and has a density \( f_\mu \) with respect to the uniform measure on \( S(0,1) \) and there exists a constant \( c_1 > 1 \) such that

\[
c_1^{-1} \leq f_\mu(\xi) \leq c_1, \quad \xi \in S(0,1).
\]

The Lévy measure \( \nu \) of such a process has a density \( f_\nu \) with respect to the Lebesgue measure on \( \mathbb{R}^d \) and there exists \( M = M(\alpha, \mu) > 1 \) such that

\[
M^{-1/|x|^{d+\alpha}} \leq f_\nu(x) \leq \frac{M}{|x|^{d+\alpha}}, \quad x \in \mathbb{R}^d.
\]

Note that \( f_\nu \) is discontinuous whenever \( f_\mu \) is.

The process \( X_t \) has the infinitesimal generator

\[
\mathcal{A} \varphi(x) = \int_{\mathbb{R}^d} (\varphi(x+y) - \varphi(x) - \mathbf{1}_{B(0,1)}(y) \langle y, \nabla \varphi(x) \rangle)f_\nu(y) \, dy
\]

(see [2]; for connections with pseudo-differential operators see also [16]).

Let \( p(t; x, y) = p_t(y-x) \) be the transition density of \( X_t \). The function \( p_t(x) = p_t(-x) \) is continuous in \((t, x)\) for \( t > 0 \) (see e.g. [23]), and has the scaling property: \( p_t(x) = t^{-d/\alpha} p_1(x/t^{1/\alpha}) \). From the scaling property and [14] it follows that there exists a constant \( c_2 = c_2(\alpha, \mu) \) such that

\[
p(t, x) \leq c_2 \min(t|x|^{-d-\alpha}, t^{-d/\alpha}), \quad x \in \mathbb{R}^d, \ t > 0.
\]

We assume as we may that the sample paths of \( X_t \) are right-continuous and have left-hand limits. The process is strong Markov with respect to the so-called “standard filtration” \( \{F_t; t \leq 0\} \) and quasi left-continuous on
The shift operator is denoted by $\theta_t$: $\theta_t X_s = X_{s+t}$, $s, t \geq 0$. The operator $\theta_t$ is also extended to Markov times $\tau$ and is denoted by $\theta_\tau$.

The potential kernel of $X_t$ is given by
\[
K(x) = \int_0^\infty p(t,x) \, dt, \quad x \in \mathbb{R}^d.
\]
By Lemma 3.1 of [10] there exists a constant $C_1 = C_1(\alpha, \mu)$ such that
\[
(2.3) \quad \frac{C_1^{-1}}{|x|^{d-\alpha}} \leq K(x) \leq \frac{C_1}{|x|^{d-\alpha}}, \quad x \in \mathbb{R}^d.
\]
Moreover, by (2.2) and the dominated convergence theorem, it is easy to see that $K(\cdot)$ is continuous in the extended sense on $\mathbb{R}^d$ ([24]).

Let $D$ denote an open set in $\mathbb{R}^d$. We set $\tau_D = \inf\{t \geq 0; X_t \notin D\}$, the first exit time of $D$. By $(P_t^D)$ we denote the semigroup of the process $(X_t)$ killed on exiting $D$. The semigroup $(P_t^D)$ is determined by transition densities $p_t^D(x,y)$, which are symmetric: $p_t^D(x,y) = p_t^D(y,x)$, and continuous in $(t,x,y)$ for $t > 0$ and $x,y \in D$ (cf. [12]).

We let
\[
G_D(x,y) = \int_0^\infty p_t^D(x,y) \, dt
\]
and call $G_D(x,y)$ the Green function for $D$. For $x,y \in D$ we have
\[
(2.4) \quad G_D(x,y) = K(x,y) - E^x K(X_{\tau_D}, y),
\]
where $K(x,y) = K(y-x)$. The function $G_D(x,y)$ is symmetric: $G_D(x,y) = G_D(y,x)$, $x,y \in D$, and jointly continuous in $x,y \in D$ for $x \neq y$ (cf. [12]).

We say that a domain $D$ in $\mathbb{R}^d$ is Green-bounded if $\sup_{x \in \mathbb{R}^d} E^x \tau_D < \infty$. For instance, $D$ is Green-bounded whenever $|D| < \infty$ (see the proof of Theorem 1.17 in [12]).

For $x \in D$, we write $\omega^x_D$ to denote the harmonic measure of $D$:
\[
\omega^x_D(A) = P^x(X_{\tau_D} \in A), \quad x \in D, \ A \subset \mathbb{R}^d.
\]
It follows from [15] that, on $(\overline{D})^c$, $\omega^x_D$ is absolutely continuous with respect to the Lebesgue measure, with density function $P_D(x, \cdot)$ (the Poisson kernel) given by
\[
(2.5) \quad P_D(x,y) = \int_D G_D(x,z) f_\nu(y-z) \, dz, \quad y \in (\overline{D})^c.
\]
Moreover, it follows from [22] that the boundary of a Lipschitz domain has zero harmonic measure. By Lemma 3.3 of [10] we have
\[
(2.6) \quad P_D(x,y) \leq M E^x \tau_D (\text{dist}(D,y))^{-d-\alpha}, \quad x \in D, \ y \in (\overline{D})^c,
\]
and
(2.7) \[ P_D(x,y) \geq M^{-1} E^x \tau_D (\text{dist}(D,y) + \text{diam}(D))^{-d-\alpha}, \quad x \in D, \quad y \in (\overline{D})^c. \]

Let \( u \) be a Borel measurable function on \( \mathbb{R}^d \). We say that \( u \) is \textit{harmonic} in an open set \( D \subset \mathbb{R}^d \) if
\[
(2.8) \quad u(x) = E^x u(X_{\tau_D}), \quad x \in U,
\]
for every bounded open set \( U \) with \( \overline{U} \subset D \). It is called \textit{regular harmonic} in \( D \) if (2.8) holds for \( U = D \). If \( D \) is unbounded then by the usual convention \( E^x u(X_{\tau_D}) = E^x [u(X_{\tau_D})] \). Under (2.8) it is always assumed that the expectation in (2.8) is absolutely convergent; in particular, finite.

\textbf{Example.} (a) Let \( f_{\mu} \equiv \Gamma((d + \alpha)/2)/(2\pi^{(d-1)/2} \Gamma((1 + \alpha)/2)) \). We write \( \hat{\mathcal{X}}_t, \hat{\Phi}, \hat{\nu}, \hat{f}_\nu, \hat{K} \) to denote the corresponding process, its characteristic exponent, etc. \( \hat{\mathcal{X}}_t \) is the rotation invariant stable process mentioned in the introduction. We have \( \hat{\Phi}(u) = |u|^\alpha \), \( \hat{f}_\nu = \mathcal{A}(d, -\alpha)|x|^{-d-\alpha} \), and \( \hat{K}(x) = \mathcal{A}(d, \alpha)|x|^{-d-\alpha} \), where \( \mathcal{A}(d, \gamma) = \Gamma((d - \gamma)/2)/(2^{\gamma} \pi^{d/2} \Gamma(\gamma/2)) \). An explicit formula for the Poisson kernel for a ball is also known:
\[
\hat{P}_{B(0,r)}(x,y) = c_d \alpha \left[ \frac{r^2 - |x|^2}{|y|^2 - r^2} \right]^{\alpha/2} |x - y|^{-d}, \quad |x| < r, \quad |y| > r,
\]
where \( c_d = \Gamma(d/2)\pi^{-d/2} \sin(\pi \alpha/2) \) (see [4]).

(b) Let \( T : \mathbb{R}^d \to \mathbb{R}^d \) be a linear isomorphism. Define \( X_t = T \hat{\mathcal{X}}_t \). For every fixed \( a > 0 \), the measure \( \varepsilon^{-1} P^0(X_\varepsilon \in dx) \) converges vaguely on \( \{|x| > a\} \) as \( \varepsilon \to 0 \) to the Lévy measure \( \nu(dx) \) (see, e.g., [2]), hence
\[
f_\nu(x) = \mathcal{A}(d, -\alpha)|T|^{-1}|T^{-1}x|^{-d-\alpha}, \quad x \neq 0.
\]
The process \( X_t \) satisfies our assumptions. In particular
\[
\frac{c_3^{-1}}{|x|^{d+\alpha}} \leq f_\nu(x) \leq \frac{c_3}{|x|^{d+\alpha}}
\]
for some constant \( c_3 = c_3(T) > 1 \).

\textbf{3. Harmonic functions and Harnack inequality.} In [10] we proved the Harnack inequality for nonnegative functions that are harmonic with respect to a symmetric \( \alpha \)-stable process for \( \alpha \in (0, 1] \). In a recent work of R. F. Bass and D. A. Levin [1] the Harnack inequality is proved for stable processes with arbitrary \( \alpha \in (0, 2) \). We note that the authors of [1] use a different definition of harmonicity but their proof is valid in our setting without any important changes (see also [24]).

\textbf{Lemma 3.1} (Harnack inequality, [1]). \textit{There exists a constant} \( C_2 = C_2(\alpha, \mu) \) \textit{such that if} \( u \) \textit{is nonnegative and bounded on} \( \mathbb{R}^d \) \textit{and harmonic}
in \( B(x_0, 16) \) then
\[
u(x_1) \leq C_2 u(x_2), \quad x_1, x_2 \in B(x_0, 1).
\]

Note that by scaling we can take \( B(x_0, 16r) \) and \( B(x_0, r) \), \( r > 0 \), instead of \( B(x_0, 16) \) and \( B(x_0, 1) \) in Lemma 3.1. Moreover, as we prove below, one can release the assumption that \( u \) is bounded.

**Lemma 3.2.** If \( u \) is nonnegative and harmonic in \( B(x_0, 16) \) then
\[
u(x_1) \leq C_2 u(x_2), \quad x_1, x_2 \in B(x_0, 1/2).
\]

*Proof.* Let \( B = B(x_0, 8) \) and let \( x_1, x_2 \in B(x_0, 1/2) \). The function \( \omega_B^x(A) \) is nonnegative, bounded and regular harmonic in \( B \) for every set \( A \subset B^c \) so we have
\[
u_{x_1}(A) \leq C_2 \nu_{x_2}(A),
\]
and that means
\[
\int_A P_B(x_1, y) \, dy \leq C_2 \int_A P_B(x_2, y) \, dy,
\]
for every \( A \subset B^c \). We get
\[
(3.1) \quad P_B(x_1, y) \leq C_2 P_B(x_2, y)
\]
for a.e. \( y \in B^c \). Since \( u(x) = \int_{B^c} u(y) P_B(x, y) \, dy \), \( x \in B(x_0, 1/2) \), the assertion follows. \( \blacksquare \)

The proof of the following “chain” Harnack inequality follows from Lemma 3.2 and (2.7) by an easy adaptation of the proof of Lemma 2 in [6] and is therefore omitted.

**Lemma 3.3.** Let \( x_1, x_2 \in \mathbb{R}^d \), \( r > 0 \) and \( k \in \mathbb{N} \) satisfy \( |x_1 - x_2| < 2^k r \). Let \( u \geq 0 \) be a function which is harmonic in \( B(x_1, r) \cup B(x_2, r) \). Then
\[
(3.2) \quad J^{-1} 2^{-k(d+\alpha)} u(x_2) \leq u(x_1) \leq J 2^{k(d+\alpha)} u(x_2)
\]
for a constant \( J = J(\alpha, \mu) \).

In what follows, we set \( \delta_D(x) = \text{dist}(x, D^c) \). Lemma 3.4 below gives a lower bound for the Green function \( G_D \). By (2.3) and (2.4),
\[
G_D(x, y) \leq C_1 |y - x|^{\alpha-d}, \quad x, y \in D,
\]
for every open set \( D \subset \mathbb{R}^d \).

**Lemma 3.4.** Let \( D \) be an open set in \( \mathbb{R}^d \) and \( x, z \in D \). Suppose that \( a > 0 \) is given. There exists a constant \( C_3 = C_3(\alpha, \mu, a) \) such that if \( |z-x| \leq a(\delta_D(x) \wedge \delta_D(z)) \) then
\[
G_D(x, z) \geq C_3 |z - x|^{\alpha-d}.
\]

*Proof.* Let \( x, z \in D \). We may and do assume that \( \delta_D(x) \leq \delta_D(z) \). Let \( b = (2C_1^2)^{1/(\alpha-d)} \). If \( |z - x| < b \delta_D(x) \) then by (2.3) and (2.4) we have
\[ G_D(x, z) \geq C_1^{-1}|z - x|^{α - d} - C_1 L^x|z - X_{τ_D}|^{α - d} \]
\[ \geq C_1^{-1}|z - x|^{α - d} - C_1(δ_D(z))^{α - d} \]
\[ \geq C_1^{-1}|z - x|^{α - d} - C_1(|z - x|/b)^{α - d} = \frac{1}{2} C_1^{-1}|z - x|^{α - d}. \]

Now, we assume that
\[ bδ_D(x) \leq |z - x| \leq aδ_D(x). \]

Let \( w \in B(x, bδ_D(x)) \setminus B(x, \frac{1}{2} bδ_D(x)) \). The function \( G_D(x, \cdot) \) is harmonic in \( B(w, \frac{1}{4} bδ_D(x)) \cup B(z, \frac{1}{4} bδ_D(x)) \), hence by Lemma 3.3 and the above we obtain
\[ G_D(x, z) \geq c G_D(x, w) \geq \frac{1}{2} c C_1^{-1}|w - x|^{α - d} \geq \frac{1}{2} c C_1^{-1}|z - x|^{α - d}, \]
where \( c = c(α, μ, a) \).

**Corollary 3.1.** Let \( D \) be a bounded open set in \( \mathbb{R}^d \) and \( F \subset D \) such that \( ρ = \text{dist}(F, D^c) > 0 \). There exists a constant \( C_4 = C_4(α, μ, ρ, D) \) such that
\[ G_D(x, z) \geq C_4|x - z|^{α - d}, \quad x, z \in F. \]

**Proof.** For \( x, z \in F \) we have \( δ_D(x) \land δ_D(z) \geq ρ \) and \( |z - x| < \text{diam}(D) \), so we can take \( a = \text{diam}(D)/ρ \) in Lemma 3.4.

4. **Exit time and Poisson kernel for a ball.** In the following lemma we improve [24, Theorem 5.7].

**Lemma 4.1.** Let \( D \) be a bounded open set. The function \( P_D(\cdot, \cdot) \) is jointly continuous in \( D \times (\overline{D})^c \).

**Proof.** Let \( x_0 \in D, y_0 \in (\overline{D})^c \), and \( x_n \to x_0, y_n \to y_0 \). Set \( ρ = δ_D(x_0)/2, \eta = \text{dist}(y_0, D)/2 \). We may and do assume that \( x_n \in B(x_0, ρ) \) and \( y_n \in B(y_0, \eta) \) for all \( n \in \mathbb{N} \). We have
\[
P_D(x, y) = \int_{D} G_D(x, z) f_ν(y - z) \, dz = \int_{\mathbb{R}^d} G_D(x, z) f_ν(y - z) \, dz = \int_{\mathbb{R}^d} G_D(x, y - z) f_ν(z) \, dz, \quad x \in D, y \in (\overline{D})^c.
\]

Moreover, by (2.1) and (2.3) we get for \( ε = α/(2(α - d)) \),
\[
\int_{\mathbb{R}^d} (G_D(x_n, y_n - z) f_ν(z))^{1+ε} \, dz 
\leq (C_1 M)^{1+ε} \int_{y_n - D} |y_n - z - x_n|^{(α - d)(1+ε)} |z|^{(-d-α)(1+ε)} \, dz
\]
\[
\leq (C_1 M)^{1+\varepsilon} \int_D |z - x_n|^{(\alpha-d)(1+\varepsilon)} |y_n - z|^{(-d-\alpha)(1+\varepsilon)} \, dz \\
\leq (C_1 M)^{1+\varepsilon} \eta^{(-d-\alpha)(1+\varepsilon)} \int_{B(x_n,\text{diam}(D))} |z - x_n|^{(\alpha-d)(1+\varepsilon)} \, dz \\
= c(\text{diam}(D))^{\alpha+\varepsilon(\alpha-d)} < \infty,
\]
hence the functions \( G_{D}(x_n,y_n - z)f_{\nu}(z) \) are uniformly integrable and the assertion follows by the continuity of \( G_{D}(\cdot,\cdot) \), because the set of irregular points of \( D \) is of Lebesgue measure 0. \( \blacksquare \)

The following lemma is an extension of (3.1).

**Lemma 4.2.** For every \( \varrho \in (0,1) \) there exists a constant \( C_5 = C_5(\alpha, \mu, \varrho) \) such that for every \( y \in \text{int} B(0,1)^c \) and \( x_1, x_2 \in B(0, \varrho) \) we have

\[
P_{B(0,1)}(x_1, y) \leq C_5 P_{B(0,1)}(x_2, y).
\]

**Proof.** Let \( B = B(0,1) \) and let \( x_1, x_2 \in B(x_0, \varrho) \). The function \( \omega_{B}(A) \) is nonnegative, bounded and regular harmonic in \( B \) for every set \( A \subset B^c \), hence by Lemma 3.3 we have

\[
\omega_{B}(A) \leq c \omega_{B}(A)
\]
where \( c = c(\alpha, \mu, \varrho) \), and that means

\[
\int_A P_{B}(x_1, y) \, dy \leq c \int_A P_{B}(x_2, y) \, dy
\]
for every \( A \subset B^c \). We conclude that \( P_{B}(x_1, y) \leq c P_{B}(x_2, y) \) for almost all, hence all, \( y \in \text{int} B^c \). \( \blacksquare \)

**Lemma 4.3.** Let \( D \subset \mathbb{R}^d \) be an open set and \( u \) be a harmonic function on \( D \). Then \( u \in C(D) \).

**Proof.** Let \( x_0 \in D \), \( \varrho = \frac{1}{2}\delta_{D}(x_0) \), and \( B = B(x_0, \varrho) \). Let \( x_n \to x_0 \); then \( x_n \in B(x_0, \varrho/2) \) for all \( n \geq n_0 \). We have \( u(x_n) = \int_{B^c} u(y) P_{B}(x_n, y) \, dy \), and by Lemma 4.2,

\[
|u(y)|P_{B}(x_n, y) \leq c |u(y)|P_{B}(x_0, y), \quad y \in B^c, \quad n \geq n_0,
\]
where \( c = c(\alpha, \mu) \). Moreover, from the definition of harmonic functions

\[
\int_{B^c} c |u(y)|P_{B}(x_0, y) \, dy = c E^{x_0} |u(X_{T_B})| < \infty;
\]
hence, by Lemma 4.1 and dominated convergence, we get

\[
\lim_{n \to \infty} u(x_n) = u(x_0),
\]
so \( u \) is continuous at \( x_0 \) and in \( D \). \( \blacksquare \)

The following estimate is given in [10].
Proposition 4.1. For every \( q \in (0, 1) \) there exist \( C = C(\alpha, \mu, q) \) and \( \eta = \eta(\alpha, \mu) \) such that

\[
P_{B(0,1)}(x,y) \leq C(|y| - 1)^{-\alpha + \eta}, \quad |x| < q, \ |y| > 1.
\]

Our next goal in this section is to strengthen this estimate and prove that \( \eta = \alpha/2 \) (see Proposition 4.2 below).

The following lemma was communicated to us by M. Lewandowski.

Lemma 4.4. There exists a constant \( C_6 = C_6(\alpha, \mu) \) such that

\[
E^x \tau_{B(0,1)} \leq C_6(1 - |x|^2)^{\alpha/2}, \quad x \in B(0,1).
\]

Proof. For \( x \neq 0 \) let \( Z_t = \langle X_t, x \rangle/|x| \). Then \( Z_t \) is a Lévy process on \( \mathbb{R} \) and we have for \( u \in \mathbb{R} \),

\[
E^0 e^{iuZ_t} = \exp \left( -t |u|^\alpha \sum_{S(0,1)} |\langle x/|x|, \xi \rangle|^{\alpha} \mu(d\xi) \right).
\]

We see that \( Z_t \) has the same distributions as \( c_x^{1/\alpha} Y_t \) where \( Y_t \) denotes the symmetric \( \alpha \)-stable Lévy process on \( \mathbb{R} \) with \( E^0 e^{iuY_t} = e^{-t|u|^\alpha} \), and \( c_x = \int_{S(0,1)} |\langle x/|x|, \xi \rangle|^{\alpha} \mu(d\xi) \). Moreover, if \( X_t \in B(0,1) \) then \( |Z_t| \leq |X_t|/|x| < 1 \), hence

\[
E^x \tau_{B(0,1)} \leq E^{|x|} s,
\]

where \( s = \inf\{t \geq 0 : |Z_t| > 1\} \) \( \overset{D}{=} c_x^{-1} \inf\{t \geq 0 : |Y_t| > 1\} \). Let \( s' = \inf\{t \geq 0 : |Y_t| > 1\} \). It is well known (see [13]) that

\[
E^{|x|} s' = \frac{(1 - |x|^2)^{\alpha/2}}{\Gamma(\alpha + 1)}.
\]

so we obtain

\[
E^x \tau_{B(0,1)} \leq c_x^{-1} \frac{(1 - |x|^2)^{\alpha/2}}{\Gamma(\alpha + 1)}.
\]

Finally, by our assumptions on \( \mu \), we have \( c^{-1} < c_x < c \) where \( c = c(\alpha, \mu) \).

The following proposition improves (4.1).

Proposition 4.2. For every \( q \in (0, 1) \) there exists a constant \( C_7 = C_7(\alpha, \mu, q) \) such that

\[
P_{B(0,1)}(x,y) \leq C_7(|y| - 1)^{-\alpha/2}, \quad |x| < q, \ |y| > 1.
\]

Proof. Let \( B = B(0,1) \). By symmetry of the Green function, Fubini’s theorem and Lemma 4.4 we have for \( y \in \text{int}(B^c) \),

\[
\int_B P_B(w,y) \, dw = \int_B \int_B G_B(w,z) f_\nu(y-z) \, dz \, dw
\]

\[
= \int_B f_\nu(y-z) \left( \int_B G_B(z,w) \, dw \right) \, dz
\]
\[
= \int_B f_\nu(y-z)E^z\tau_B\,dz \leq MC_6\int_B |y-z|^{-d-\alpha}(1-|z|^2)_{\alpha/2}\,dz
\]
\[
\leq MC_6^{\alpha/2}\int_{B(y,|y|-1)c} |y-z|^{-d-\alpha}|y-z|_{\alpha/2}\,dz
\]
\[
= c_1(|y|-1)^{-\alpha/2},
\]
with \( c_1 = c_1(\alpha, \mu) \). Moreover, for \( x \in B(0, \varrho) \) from Lemma 4.2 we obtain
\[
\int_B P_B(w, y)\,dw \geq \int_{B(0, \varrho)} P_B(w, y)\,dw \geq c_2P_B(x,y),
\]
where \( c_2 = c_2(\alpha, \mu, \varrho) \).

The following results extend the Carleson estimate and boundary Harnack principle given in [10] for \( \alpha \leq 1 \) to all \( \alpha \in (0, 2) \).

THEOREM 4.1 (Carleson estimate). Let \( D \) be a domain such that \( 0 \in \partial D \). Let \( \kappa > 0 \) and \( B(A, \kappa) \subset D \cap B(0, 1) \). There exists a constant \( M_1 = M_1(\alpha, \mu) \) such that for all functions \( u \geq 0 \), regular harmonic in \( D \cap B(0, 2) \) and equal to \( 0 \) in \( D^c \cap B(0, 2) \), we have
\[
u(x) \leq M_1 \kappa^{-\alpha}w(A) \leq M_1 \kappa^{-\alpha}u(A), \quad x \in D \cap B(0, 3/2),
\]
where \( w \) is the regular harmonic function in \( D \cap B(0, 1) \) defined by
\[
w(x) = \begin{cases} u(x), & x \in B(0, 3/2)^c \cup (D^c \cap B(0, 1)), \\ 0, & 1 \leq |x| < 3/2. \end{cases}
\]

THEOREM 4.2 (Boundary Harnack principle). Let \( D \) be an open set, \( Q \in \partial D, r > 0, \) and suppose that \( B(A, \kappa r) \subset D \cap B(Q, r) \). There exists a constant \( C_8 = C_8(\alpha, \mu) \) such that for all functions \( u, v \geq 0 \), regular harmonic in \( D \cap B(Q, 2r) \) and equal to \( 0 \) in \( D^c \cap B(Q, 2r) \), we have
\[
C_8^{-1} \kappa^{d+\alpha}u(A)/v(A) \leq u(x)/v(x) \leq C_8 \kappa^{-d-\alpha}u(A)/v(A), \quad x \in D \cap B(Q, r/2).
\]

The proofs of Theorems 4.1 and 4.2 are direct adaptations of the proofs given in [10] with (4.1) replaced by (4.2).

REMARK. Consider the process \( X_t = T\hat{X}_t \) described in Example (b) above. By [11] and [18] it is fairly easy to see that if \( D \) in Theorem 4.2 is a \( C^{1,1} \) domain then \( c^{-1} \leq u(x)/\delta_D(x)_{\alpha/2} \leq c, x \in D \cap B(Q, r/2) \), which is the same asymptotics as for \( \hat{X}_t \). However, let \( D \) be a cone with vertex angle \( \lambda \in (0, \pi/2) \). The asymptotics of harmonic functions of \( \hat{X}_t \) at the vertex of the cone changes when \( \lambda \to 0 \) ([17]). Therefore the asymptotics varies among different processes of the form \( X_t = T\hat{X}_t \).

The main application of Theorems 4.1 and 4.2 is to the Green function of \( D \). We now state for our \( \alpha \)-stable process an important result on the Green
Theorem 4.3 (3G Theorem). Let $D$ be a bounded Lipschitz domain in $\mathbb{R}^d$. There exists a constant $C_9 = C_9(\alpha, \mu, D)$ such that for all $x, y, z \in D$ we have

$$\frac{G_D(x, y)G_D(y, z)}{G_D(x, z)} \leq C_9 \left( \frac{|z - x|}{|y - x||z - y|} \right)^{d-\alpha} \leq 2^{d-\alpha} c_9 (|y - x|^\alpha - d + |z - y|^{\alpha - d}),$$

unless $x = y = z$. In fact, the constant $C_9$ above depends on $D$ only through its Lipschitz character and diameter.

5. $q$-harmonic functions. We say that a Borel function $q$ belongs to the Kato class $\mathcal{J}$ if $q$ satisfies either of the two equivalent conditions (see [12])

$$(5.1) \quad \lim_{r \to 0} \sup_{x \in \mathbb{R}^d} \int_{|x - y| \leq r} |q(y)K(y - x)| \, dy = 0,$$

$$(5.2) \quad \lim_{t \to 0} \sup_{x \in \mathbb{R}^d} \int_0^t |P_s|q|(x)\, ds = 0.$$

We write $q \in \mathcal{J}_{\text{loc}}$ if for every bounded Borel set $U \subset \mathbb{R}^d$ we have $1_U q \in \mathcal{J}$. Clearly $\mathcal{J}_{\text{loc}} \subset L_1^{\text{loc}}$. If $f \in L^\infty(\mathbb{R}^d)$ and $q \in \mathcal{J}$ then $f,fq \in \mathcal{J}$. Note that by (2.3) we have $\mathcal{J} = \mathcal{J}^\alpha$ where $\mathcal{J}^\alpha$ denotes the Kato class for the rotation invariant $\alpha$-stable process $\hat{X}_t$ ([7]).

Let $U$ be a domain in $\mathbb{R}^d$ and let $q \in \mathcal{J}$. We define

$$(5.3) \quad e_q(\tau_U) = \exp \left( \int_0^{\tau_U} q(X_t) \, dt \right).$$

By (5.2) we have $\int_0^t |q(X_s)| \, ds < \infty$ a.s., for each $t > 0$. Therefore, if $\tau_U < \infty$, the random variable $e_q(\tau_U)$ is well defined.

Let $u$ be a Borel measurable function on $\mathbb{R}^d$. We say that $u$ is $q$-harmonic in an open set $D \subset \mathbb{R}^d$ if

$$u(x) = E^x[\tau_U < \infty; e_q(\tau_U)u(X_{\tau_U})], \quad x \in U,$$

for every bounded open set $U$ with $\overline{U} \subset D$. It is called regular $q$-harmonic in $D$ if the above equality holds for $U = D$.

We always understand that the expectation in the above condition is absolutely convergent. For $q \equiv 0$ we obtain the previous definition of harmonicity. By the strong Markov property of $X_t$ a regular $q$-harmonic function $u$ is $q$-harmonic.
An important technical tool in further considerations is the conditional \( \alpha \)-stable Lévy motion. For the definition and properties of the conditional process for \( \hat{X}_t \) we refer to [7]; here the definitions are analogous. We recall that for a domain \( D \) by the \( \alpha \)-stable \( y \)-Lévy motion we mean the process conditioned by the Green function \( G(\cdot, y) \) of \( D \). If \( D \) is a bounded Lipschitz domain then we obtain, by routine arguments (see, e.g., [12]), for \( \Phi \geq 0 \) measurable with respect to \( \mathcal{F}_{\tau_D} \), and any Borel \( f \geq 0 \), the following formula:

\[
E^x[f(X_{\tau_D})\Phi] = E^x[f(X_{\tau_D})E^{X_{\tau_D}}_X[\Phi]], \quad x \in D.
\]

In what follows \( E^v \) denotes the expectation for the \( \alpha \)-stable \( v \)-Lévy process conditioned by \( G_B(\cdot, v) \) where \( B \) is a given ball in \( \mathbb{R}^d \) and \( x, v \in B \). The process approaches \( v \) in a finite time and is killed then.

**Lemma 5.1.** Let \( q \in \mathcal{J} \) and \( \varepsilon > 0 \). There is \( r_0 = r_0(\alpha, \mu, q, \varepsilon) \) such that

\[
\left| \int_B G_B(x, y)G_B(y, v) \frac{|q(y)|}{G_B(x, v)} \, dy \leq \varepsilon \right.
\]

for every ball \( B \subset \mathbb{R}^d \) of radius \( r \leq r_0 \).

**Proof.** Let \( x_0 \in \mathbb{R}^d, r > 0 \) and \( B = B(x_0, r) \). By scaling we have

\[
G_B(z, w) = r^{\alpha-d} G\left( \frac{z - x_0}{r}, \frac{w - x_0}{r} \right), \quad z, w \in B,
\]

where \( G \) is the Green function for the ball \( B(0, 1) \).

By the 3G Theorem we have for \( v, x, y \in B \),

\[
\frac{G_B(x, y)G_B(y, v)}{G_B(x, v)} \leq 2^{d-\alpha} C_9 r^{\alpha-d} \left( \left( \frac{|y - x|}{r} \right)^{\alpha-d} + \left( \frac{|v - y|}{r} \right)^{\alpha-d} \right).
\]

By (5.1) there is \( r_0 = r_0(\alpha, \mu, q, \varepsilon) \) such that (5.5) holds if \( 0 < r \leq r_0 \).

We recall the following important fact known as Khasminski’s lemma: For every nonnegative \( q \) and Markov time \( \tau \) such that \( \tau \leq t + \tau \circ \theta_t \) on \( \{t < \tau\} \) for each \( t \geq 0 \), we have:

\[
\text{If } \sup_{x \in \mathbb{R}^d} E^x \left[ \int_0^\tau q(X_s) \, ds \right] < 1 \text{ then } \sup_{x \in \mathbb{R}^d} E^x e_q(\tau) < (1 - \varepsilon)^{-1}.
\]

**Lemma 5.2.** Let \( q \in \mathcal{J} \) and \( \varepsilon > 0 \). Let \( r_0 = r_0(\alpha, \mu, q, \varepsilon) > 0 \) be the constant of Lemma 5.1. Then for every ball \( B \subset \mathbb{R}^d \) of radius \( r \leq r_0 \) we have

\[
\exp(-\varepsilon) \leq E^v e_q(\tau_B) \leq (1 - \varepsilon)^{-1}, \quad x, v \in B.
\]
Proof. We have
\[ E^x_v \left[ \int_0^{\tau_B} q(X_t) \, dt \right] = G_B(x, v)^{-1} E^x \left[ \int_0^{\tau_B} q(X_t) G_B(X_t, v) \, dt \right] \]
\[ = \int_B \frac{G_B(x, y) G_B(y, v)}{G_B(x, v)} q(y) \, dy. \]
The upper bound in (5.6) follows by Lemma 5.1 and Khasminski’s lemma applied to the expectations \( E^x_v : x \in \mathbb{R}^d \), and the lower bound follows from Jensen’s inequality.

The proofs of the next two lemmas are standard (see, e.g., [12] or [7]); we provide them only for the reader’s convenience.

**Lemma 5.3.** Let \( D \) be a Green-bounded domain in \( \mathbb{R}^d \) and \( q \in \mathcal{J} \).

(i) For every \( b > 0 \) there exists \( a = a(\alpha, q, b) \) such that
\begin{equation}
(5.7) \quad G_D |q| \leq a G_D 1 + b.
\end{equation}
Consequently, for a fixed \( q \in \mathcal{J} \) and variable \( D \), we have \( \|G_D q\|_{\infty} \to 0 \) as \( \|G_D 1\|_{\infty} \to 0 \).

(ii) \( G_D q \in L^\infty(\mathbb{R}^d) \cap C(D) \), and for any \( z \in \partial D \) regular for \( D \), we have
\begin{equation}
(5.8) \quad \lim_{x \to z} G_D q(x) = 0.
\end{equation}

(iii) Under the additional hypothesis: (a) \( q \in L^1(D) \), or (b) \( |D| < \infty \), we have
\begin{equation}
(5.9) \quad \lim_{|x| \to \infty} G_D q(x) = 0.
\end{equation}

Proof. In the proof of (i) we may suppose that \( q \geq 0 \). For any \( s > 0 \) let \( A_n \) denote the indicator of the set \( \{\tau_D > ns\} \), \( n \geq 0 \). The following inequality holds:
\[ \int_0^{\tau_D} q(X_t) \, dt = \int_0^\infty q(X_t) 1_{\{\tau_D > t\}} \, dt \leq \sum_{n=0}^{\infty} \int_{ns}^{(n+1)s} q(X_t) 1_{\{\tau_D > ns\}} \, dt \]
\[ = \sum_{n=0}^{\infty} A_n \int_{ns}^{(n+1)s} q(X_t) \, dt. \]
Taking expectations of both sides, using Fubini’s theorem and the Markov property we obtain
\[ G_D q(x) \leq \sum_{n=0}^{\infty} E^x \left[ \int_{ns}^{(n+1)s} q(X_t) \, dt \right] \leq \sum_{n=0}^{\infty} E^{X_{ns}} \left[ \int_0^{s} q(X_t) \, dt \right]. \]
Since \( q \in \mathcal{J} \), we can choose \( s > 0 \) so that
\[ \sup_{x \in \mathbb{R}^d} E^x \left[ \int_0^{s} q(X_t) \, dt \right] \leq b. \]
We then obtain
\[ G_D q(x) \leq b \sum_{n=0}^{\infty} P^x(\tau_D > ns) \leq b(1 + E^x[\tau_D/s]), \]
which gives (5.7) with \( a = b/s \).

From now on we no longer assume that \( q \geq 0 \). Let us recall that \( P^D_t \) has the strong Feller property. Since, by (5.7), \( G_D q \) is bounded in \( \mathbb{R}^d \), \( P^D_t(G_D q) \) is continuous in \( D \). From the semigroup property of \( P^D_t \) we obtain
\[
G_D q - P^D_t(G_D q) = \int_0^t P^D_s q \, ds.
\]
(5.10)

Since \( |P^D_s q| \leq P_s |q| \), and \( q \in \mathcal{J} \), the right hand side above converges to zero uniformly in \( \mathbb{R}^d \) as \( t \to 0 \) and so \( G_D q \) is continuous in \( D \). Next, if \( z \in \partial D \) is regular for \( D \), then
\[
\limsup_{x \to z} |P^D_t G_D q(x)| \leq \limsup_{x \to z} (P^D_t 1(x)) \|G_D q\|_{\infty} = \|G_D q\|_{\infty} = 0,
\]

because the function \( x \mapsto P^x(\tau_D > t) \) is upper-semicontinuous at \( z \) (cf. [12]). Hence (5.8) is also true by the uniform convergence in (5.10) and the above argument.

Under the hypothesis (a) in (iii) we have \( G_D q \in L^1(D) \), because \( \|G_D q\|_1 \leq \|G_D 1\|_{\infty} \|q\|_1 < \infty \). Since \( \lim_{|x| \to \infty} p(t,x,y) = 0 \) for each \( t > 0 \) and \( y \in \mathbb{R}^d \), it follows by dominated convergence that \( \lim_{|x| \to \infty} P^D_t G_D q(x) = 0 \). We obtain (5.9) once again by the uniform convergence in (5.10). Under the hypothesis (b) in (iii) we have \( G_D q \in L^\infty(D) \subset L^1(D) \), so the same argument is valid. \( \blacksquare \)

**Lemma 5.4.** Let \( q \in \mathcal{J}_{\text{loc}} \) and \( u \) be a nonnegative function which is locally bounded and \( q \)-harmonic on an open set \( D \subset \mathbb{R}^d \). Then for every open bounded set \( U \subset D \) such that \( \overline{U} \subset D \) we have
\[
(5.11) \quad u(x) = E^x u(X_{\tau_U}) + G_U(q u)(x), \quad x \in U.
\]
In particular, \( u \) is continuous in \( D \).

**Proof.** By the assumptions \( qu \mathbf{1}_U \in \mathcal{J} \) so \( \sup_{x \in U} G_U(u|q|)(x) < \infty \) and \( G_U(q u) \) is continuous on \( U \) by Lemma 5.3. We put
\[
\Phi(t) = 1_{\{t < \tau_U\}} q(X_t) u(X_{\tau_U}) \exp \int_t^{\tau_U} q(X_s) \, ds,
\]
\[
\Psi(t) = 1_{\{t < \tau_U\}} q(X_t) u(X_{\tau_U}) \exp \int_t^{\tau_U} q(X_s) \, ds,
\]
Using the Markov property along with the Fubini–Tonelli theorem we obtain
\[
\int_0^\infty E^x[\Psi(t)] \, dt = E^x \left[ \int_0^{\tau_U} |q(X_t)| (e_q(\tau_U) u(X_{\tau_U})) \circ \theta_t \, dt \right]
\]
\[
= E^x \left[ \int_0^{\tau_U} |q(X_t)| E^{X_t} [e_q(\tau_U) u(X_{\tau_U})] \, dt \right]
\]
\[
= E^x \left[ \int_0^{\tau_U} |q(X_t)| u(X_t) \, dt \right] = G_U(u|q)|(x) < \infty.
\]
Thus, by Fubini’s theorem we obtain
\[
\int_0^\infty E^x[\Phi(t)] \, dt = G_U(qu)(x),
\]
while, at the same time,
\[
\int_0^{\tau_U} |q(X_s)| \, ds < \infty \text{ and so the function } [0, \tau_U) \ni \tau \mapsto \exp \int_0^\tau q(X_s) \, ds \text{ is absolutely continuous (a.s.). Its derivative equals } -q(X_\tau) \exp \int_0^\tau q(X_s)\, ds \text{ a.s. This shows the formula (5.11). The function } x \mapsto E^x u(X_{\tau_U}) \text{ is harmonic, hence continuous on } U \text{ (Lemma 4.3).}
\]

**Theorem 5.1.** Let \( q \in \mathcal{J}_{\text{loc}}, u \) be a nonnegative \( q \)-harmonic function in an open set \( D \), and \( F \subset D \) be compact. There exists a constant \( C_{10} = C_{10}(\alpha, \mu, q, F, D) \) such that
\[
(5.12) \quad u(x) \leq C_{10} u(y), \quad x, y \in F.
\]
If \( u(x) = 0 \) for some \( x \in D \) then \( u = 0 \) on \( D \) and \( u = 0 \) a.e. on \( D^c \).

**Proof.** Let \( F \subset D \) and \( \delta_F = \text{dist}(F, D^c) \). Put \( A = \{x \in D : \text{dist}(x, F) \leq \delta_F/2\} \). Then \( A \) is a compact subset of \( D \) and we have \( q 1_A \in \mathcal{J} \). Let \( \varrho_0 = r_0 \wedge (\delta_F/2) \), where \( r_0 = r_0(\alpha, \mu, q, 1_A) \) is the constant from Lemma 5.1 for \( \varepsilon = 1/2 \). Let \( x \in F, 0 < r \leq \varrho_0 \) and \( B = B(x, r) \). We have \( B \subset A \) and by (5.4),
\[
u(y) = E^y [e_q(\tau_B) u(X_{\tau_B})] = E^y [u(X_{\tau_B}) E^y_{X_{\tau_B}}(\tau_B)], \quad y \in B.
\]
Lemma 5.2 yields
\[
u(y) = E^y u(X_{\tau_B}) \leq 2 E^y u(X_{\tau_B}), \quad y \in B.
\]
The function \( h(y) = E^y u(X_{\tau_B}), y \in \mathbb{R}^d, \) is regular harmonic in \( B \), so by Lemma 3.3 we obtain
\[
c_1^{-1} E^y u(X_{\tau_B}) \leq E^x u(X_{\tau_B}) \leq c_1 E^y u(X_{\tau_B}), \quad y \in B(x, r/2),
\]
where \( c_1 = c_1(\alpha, \mu) \). By (5.13) and the above we get

\[
(4c_1)^{-1} u(x) \leq u(y) \leq 4c_1 u(x), \quad y \in B(x, r/2).
\]

We now consider \( z \in F \) such that \( |z - x| \geq \varrho_0/2 \). Let \( B_1 = B(z, \varrho_0/4) \). Note that \( B_1 \subset A \) and \( B_1 \cap B(x, \varrho_0/4) = \emptyset \). By (2.7), (5.13) and (5.14) with \( r = \varrho_0 \) we obtain

\[
u(z) \geq \frac{1}{2} E^z u(X_{\tau_{B_1}}) \geq \frac{1}{2} \int_{B(x, \varrho_0/4)} P_{B_1}(z, w) \, dw
\]

\[
\geq \frac{1}{2} M^{-1} E^z \tau_{B_1} \int_{B(x, \varrho_0/4)} (|w - z| + \varrho_0/4)^{-d-\alpha} u(w) \, dw
\]

\[
\geq \frac{1}{2} M^{-1} (\varrho_0/4)^{\alpha} E^{0}_{\tau B(0,1)} (4c_1)^{-1} u(x)(2|x - z|)^{-d-\alpha} |B(x, \varrho_0/4)|
\]

\[
\geq c_2 u(x),
\]

where \( c_2 = M^{-1} 8^{-d-\alpha-1}(\omega_d/d)E^{0}_{\tau B(0,1)} c_1^{-1}(\varrho_0/\text{diam}(F))^{d+\alpha} \). From this and (5.14) with \( r = \varrho_0 \), (5.12) follows.

We now assume that \( x \in D \) and \( u(x) = 0 \). By the first part of the proof, for every \( B = B(x, r) \) with \( r > 0 \) small enough we have

\[
od = u(x) \geq \frac{1}{2} E^x u(X_{\tau_B}) \geq \frac{1}{2} \int_{D^c} P_B(x, y) u(y) \, dy.
\]

It follows that \( u = 0 \) a.e. on \( D^c \). The pointwise equality \( u = 0 \) on \( D \) is a consequence of (5.12).

**Acknowledgments.** The author is grateful to Drs. K. Bogdan and M. Ryznar for discussions and suggestions, and to Dr. M. Lewandowski for communicating Lemma 4.4.

**REFERENCES**


Institute of Mathematics
Wrocław University of Technology
Wybrzeże Wyspiańskiego 27
50-370 Wrocław, Poland
E-mail: sztonyk@im.pwr.wroc.pl

*Received 3 June 2002;
revised 17 June 2002*