COLLOQUIUM MATHEMATICUM

VOL. 95 2003 NO. 2

BOUNDARY POTENTIAL THEORY FOR STABLE LÉVY PROCESSES

BY

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Abstract. We investigate properties of harmonic functions of the symmetric stable Lévy process on $\mathbb{R}^d$ without the assumption that the process is rotation invariant. Our main goal is to prove the boundary Harnack principle for Lipschitz domains. To this end we improve the estimates for the Poisson kernel obtained in a previous work. We also investigate properties of harmonic functions of Feynman–Kac semigroups based on the stable process. In particular, we prove the continuity and the Harnack inequality for such functions.

1. Introduction. For $\alpha \in (0,2)$, a Lévy process X_t on $\mathbb{R}^d$ with characteristic exponent Φ is called *stable* with index α if $\Phi(ku) = k^\alpha \Phi(u)$ for $k > 0$, $u \in \mathbb{R}^d$. The stable processes appear in a natural way in limit theorems and have the *scaling property*: for every $a > 0$ the rescaled process $a^{-1/\alpha} X_{at}$ has the same law as X_t.

Recently, remarkable progress has been made in the potential theory of the rotation invariant α-stable Lévy processes (for definitions see Preliminaries). The results obtained include estimates of the Green function and Poisson kernel ([18], [11], [17]), the boundary Harnack principle for α-harmonic functions ([6], [9], [21]), and similar developments in the potential theory of the α-stable Schrödinger operator ([7], [8]). Many of the results are based on the exact formulae for the Poisson kernel and the Green function for the ball established by M. Riesz ([19], [20], [4]).

In this paper we extend some of these results to α-stable Lévy processes which are symmetric but not necessarily rotation invariant. We focus on the behaviour of such processes near the boundary of a domain $D \subset \mathbb{R}^d$.

The results of the present paper complement the earlier ones contained in [10]. We note that the main results of [10] were restricted to $\alpha \leq 1$. They were based on certain estimates of the harmonic measure, which substitute the exact formula for the Poisson kernel of the ball (see Proposition 4.1

2000 *Mathematics Subject Classification*: Primary 60J45, 31C05; Secondary 60G51.

Key words and phrases: stable processes, boundary Harnack principle, 3G Theorem, q-harmonic functions.

Research supported by KBN grant 2 P03A 041 22.

below). In Sections 3 and 4 below we improve these estimates and we obtain the Carleson estimate, boundary Harnack principle and 3G Theorem for all $\alpha \in (0,2)$. We note that the paper is related to the papers of R. F. Bass and D. A. Levin [1] and Z. Vondraček [24]. In particular, we use in a crucial way the Harnack inequality of [1]. We also extend and strengthen some of the results of [24] (see, e.g., Lemmas 4.1 and 4.2).

The results allow for a study of harmonic functions of the Feynman–Kac perturbation of X_t by a multiplicative functional $\exp(\int_0^{\tau_D} q(X_t)\,dt)$. In Section 5 we prove the continuity and Harnack inequality for such functions.

2. Preliminaries. In what follows $\alpha \in (0,2)$ and $d \ge 2$. We denote by (X_t, P^x) a symmetric α-stable Lévy process in $\mathbb{R}^d$ (i.e. homogeneous, with independent increments), with characteristic function of the form

$$E^0 e^{i\langle u, X_t\rangle} = e^{-t\Phi(u)}, \qquad u \in \mathbb{R}^d,\ t \ge 0,$$

where the characteristic exponent Φ is given by

$$\Phi(u) = \int\limits_{S(0,1)} |\langle u, \xi\rangle|^\alpha \,\mu(d\xi),$$

and μ is a finite, symmetric measure on $S(0,1)$. We assume that μ is absolutely continuous and has a density f_μ with respect to the uniform measure on $S(0,1)$ and there exists a constant $c_1 > 1$ such that

$$c_1^{-1} \le f_\mu(\xi) \le c_1, \qquad \xi \in S(0,1).$$

The Lévy measure ν of such a process has a density f_ν with respect to the Lebesgue measure on $\mathbb{R}^d$ and there exists $M = M(\alpha,\mu) > 1$ such that

$$\frac{M^{-1}}{|x|^{d+\alpha}} \le f_\nu(x) \le \frac{M}{|x|^{d+\alpha}}, \qquad x \in \mathbb{R}^d. \tag{2.1}$$

Note that f_ν is discontinuous whenever f_μ is.

The process X_t has the infinitesimal generator

$$\mathcal{A}\varphi(x) = \int\limits_{\mathbb{R}^d} (\varphi(x+y) - \varphi(x) - \mathbf{1}_{B(0,1)}(y)\langle y, \nabla\varphi(x)\rangle) f_\nu(y)\,dy$$

(see [2]; for connections with pseudo-differential operators see also [16]).

Let $p(t;x,y) = p_t(y-x)$ be the *transition density* of X_t. The function $p_t(x) = p_t(-x)$ is continuous in (t,x) for $t > 0$ (see e.g. [23]), and has the *scaling property*: $p_t(x) = t^{-d/\alpha} p_1(x/t^{1/\alpha})$. From the scaling property and [14] it follows that there exists a constant $c_2 = c_2(\alpha,\mu)$ such that

$$p(t,x) \le c_2 \min(t|x|^{-d-\alpha}, t^{-d/\alpha}), \qquad x \in \mathbb{R}^d,\ t > 0. \tag{2.2}$$

We assume as we may that the sample paths of X_t are right-continuous and have left-hand limits. The process is strong Markov with respect to the so-called "standard filtration" $\{\mathcal{F}_t;\ t \le 0\}$ and quasi left-continuous on

$[0,\infty]$. The *shift operator* is denoted by θ_t: $\theta_t X_s = X_{s+t}$, $s,t \ge 0$. The operator θ_t is also extended to Markov times τ and is denoted by θ_τ.

The potential kernel of X_t is given by

$$K(x) = \int_0^\infty p(t,x)\,dt, \qquad x \in \mathbb{R}^d.$$

By Lemma 3.1 of [10] there exists a constant $C_1 = C_1(\alpha,\mu)$ such that

$$\frac{C_1^{-1}}{|x|^{d-\alpha}} \le K(x) \le \frac{C_1}{|x|^{d-\alpha}}, \qquad x \in \mathbb{R}^d. \tag{2.3}$$

Moreover, by (2.2) and the dominated convergence theorem, it is easy to see that $K(\cdot)$ is continuous in the extended sense on $\mathbb{R}^d$ ([24]).

Let D denote an open set in $\mathbb{R}^d$. We set $\tau_D = \inf\{t \ge 0;\, X_t \notin D\}$, the *first exit time* of D. By (P_t^D) we denote the semigroup of the process (X_t) killed on exiting D. The semigroup (P_t^D) is determined by transition densities $p_t^D(x,y)$, which are symmetric: $p_t^D(x,y) = p_t^D(y,x)$, and continuous in (t,x,y) for $t > 0$ and $x,y \in D$ (cf. [12]).

We let

$$G_D(x,y) = \int_0^\infty p_t^D(x,y)\,dt$$

and call $G_D(x,y)$ the *Green function* for D. For $x,y \in D$ we have

$$G_D(x,y) = K(x,y) - E^x K(X_{\tau_D}, y), \tag{2.4}$$

where $K(x,y) = K(y-x)$. The function $G_D(x,y)$ is symmetric: $G_D(x,y) = G_D(y,x)$, $x,y \in D$, and jointly continuous in $x,y \in D$ for $x \ne y$ (cf. [12]).

We say that a domain D in $\mathbb{R}^d$ is *Green-bounded* if $\sup_{x\in\mathbb{R}^d} E^x\tau_D < \infty$. For instance, D is Green-bounded whenever $|D| < \infty$ (see the proof of Theorem 1.17 in [12]).

For $x \in D$, we write ω_D^x to denote the harmonic measure of D:

$$\omega_D^x(A) = P^x(X_{\tau_D} \in A), \qquad x \in D,\ A \subset \mathbb{R}^d.$$

It follows from [15] that, on $(\overline{D})^{\mathrm{c}}$, ω_D^x is absolutely continuous with respect to the Lebesgue measure, with density function $P_D(x,\cdot)$ (the *Poisson kernel*) given by

$$P_D(x,y) = \int_D G_D(x,z) f_\nu(y-z)\,dz, \qquad y \in (\overline{D})^{\mathrm{c}}. \tag{2.5}$$

Moreover, it follows from [22] that the boundary of a Lipschitz domain has zero harmonic measure. By Lemma 3.3 of [10] we have

$$P_D(x,y) \le M\, E^x\tau_D\,(\mathrm{dist}(D,y))^{-d-\alpha}, \quad x \in D,\ y \in (\overline{D})^{\mathrm{c}}, \tag{2.6}$$

and

$$(2.7)\qquad P_D(x,y) \geq M^{-1}\, E^x\tau_D\,(\mathrm{dist}(D,y)+\mathrm{diam}(D))^{-d-\alpha}, \quad x\in D,\ y\in(\overline{D})^{\mathrm{c}}.$$

Let u be a Borel measurable function on $\mathbb{R}^d$. We say that u is *harmonic* in an open set $D\subset\mathbb{R}^d$ if

$$(2.8)\qquad u(x)=E^x u(X_{\tau_U}),\qquad x\in U,$$

for every bounded open set U with $\overline{U}\subset D$. It is called *regular harmonic* in D if (2.8) holds for $U=D$. If D is unbounded then by the usual convention $E^x u(X_{\tau_D}) = E^x[\tau_D<\infty; u(X_{\tau_D})]$. Under (2.8) it is always assumed that the expectation in (2.8) is absolutely convergent; in particular, finite.

EXAMPLE. (a) Let $f_\mu \equiv \Gamma((d+\alpha)/2)/(2\pi^{(d-1)/2}\Gamma((1+\alpha)/2))$. We write $\widehat{X}_t$, $\widehat{\Phi}$, $\widehat{\nu}$, $\widehat{f}_\nu$, $\widehat{K}$ to denote the corresponding process, its characteristic exponent, etc. $\widehat{X}_t$ is the rotation invariant stable process mentioned in the introduction. We have $\widehat{\Phi}(u)=|u|^\alpha$, $\widehat{f}_\nu=\mathcal{A}(d,-\alpha)|x|^{-d-\alpha}$, and $\widehat{K}(x)=\mathcal{A}(d,\alpha)|x|^{-d-\alpha}$, where $\mathcal{A}(d,\gamma)=\Gamma((d-\gamma)/2)/(2^\gamma\pi^{d/2}|\Gamma(\gamma/2)|)$. An explicit formula for the Poisson kernel for a ball is also known:

$$\widehat{P}_{B(0,r)}(x,y)=c_\alpha^d\left[\frac{r^2-|x|^2}{|y|^2-r^2}\right]^{\alpha/2}|x-y|^{-d},\qquad |x|<r,\ |y|>r,$$

where $c_\alpha^d=\Gamma(d/2)\pi^{-d/2-1}\sin(\pi\alpha/2)$ (see [4]).

(b) Let $T:\mathbb{R}^d\to\mathbb{R}^d$ be a linear isomorphism. Define $X_t=T\widehat{X}_t$. For every fixed $a>0$, the measure $\varepsilon^{-1}P^0(X_\varepsilon\in dx)$ converges vaguely on $\{|x|>a\}$ as $\varepsilon\to 0$ to the Lévy measure $\nu(dx)$ (see, e.g., [2]), hence

$$f_\nu(x)=\mathcal{A}(d,-\alpha)|T|^{-1}|T^{-1}x|^{-d-\alpha},\qquad x\neq 0.$$

The process X_t satisfies our assumptions. In particular

$$\frac{c_3^{-1}}{|x|^{d+\alpha}}\leq f_\nu(x)\leq\frac{c_3}{|x|^{d+\alpha}}$$

for some constant $c_3=c_3(T)>1$.

3. Harmonic functions and Harnack inequality. In [10] we proved the Harnack inequality for nonnegative functions that are harmonic with respect to a symmetric α-stable process for $\alpha\in(0,1]$. In a recent work of R. F. Bass and D. A. Levin [1] the Harnack inequality is proved for stable processes with arbitrary $\alpha\in(0,2)$. We note that the authors of [1] use a different definition of harmonicity but their proof is valid in our setting without any important changes (see also [24]).

LEMMA 3.1 (Harnack inequality, [1]). *There exists a constant $C_2 = C_2(\alpha,\mu)$ such that if u is nonnegative and bounded on $\mathbb{R}^d$ and harmonic* *in* $B(x_0,16)$ *then*

$$u(x_1)\le C_2u(x_2),\qquad x_1,x_2\in B(x_0,1).$$

Note that by scaling we can take $B(x_0,16r)$ and $B(x_0,r)$, $r>0$, instead of $B(x_0,16)$ and $B(x_0,1)$ in Lemma 3.1. Moreover, as we prove below, one can release the assumption that u is bounded.

LEMMA 3.2. *If* u *is nonnegative and harmonic in* $B(x_0,16)$ *then*

$$u(x_1)\le C_2u(x_2),\qquad x_1,x_2\in B(x_0,1/2).$$

Proof. Let $B=B(x_0,8)$ and let $x_1,x_2\in B(x_0,1/2)$. The function $\omega_B^x(A)$ is nonnegative, bounded and regular harmonic in B for every set $A\subset B^{\mathrm c}$ so we have

$$\omega_B^{x_1}(A)\le C_2\omega_B^{x_2}(A),$$

and that means

$$\int\limits_A P_B(x_1,y)\,dy\le C_2\int\limits_A P_B(x_2,y)\,dy,$$

for every $A\subset B^{\mathrm c}$. We get

$$P_B(x_1,y)\le C_2P_B(x_2,y)\tag{3.1}$$

for a.e. $y\in B^{\mathrm c}$. Since $u(x)=\int_{B^{\mathrm c}}u(y)P_B(x,y)\,dy$, $x\in B(x_0,1/2)$, the assertion follows. ∎

The proof of the following "chain" Harnack inequality follows from Lemma 3.2 and (2.7) by an easy adaptation of the proof of Lemma 2 in [6] and is therefore omitted.

LEMMA 3.3. *Let* $x_1,x_2\in\mathbb R^d$, $r>0$ *and* $k\in\mathbb N$ *satisfy* $|x_1-x_2|<2^kr$. *Let* $u\ge0$ *be a function which is harmonic in* $B(x_1,r)\cup B(x_2,r)$. *Then*

$$J^{-1}2^{-k(d+\alpha)}u(x_2)\le u(x_1)\le J2^{k(d+\alpha)}u(x_2)\tag{3.2}$$

for a constant $J=J(\alpha,\mu)$.

In what follows, we set $\delta_D(x)=\operatorname{dist}(x,D^{\mathrm c})$. Lemma 3.4 below gives a lower bound for the Green function G_D. By (2.3) and (2.4),

$$G_D(x,y)\le C_1|y-x|^{\alpha-d},\qquad x,y\in D,$$

for every open set $D\subset\mathbb R^d$.

LEMMA 3.4. *Let* D *be an open set in* $\mathbb R^d$ *and* $x,z\in D$. *Suppose that* $a>0$ *is given. There exists a constant* $C_3=C_3(\alpha,\mu,a)$ *such that if* $|z-x|\le a(\delta_D(x)\wedge\delta_D(z))$ *then*

$$G_D(x,z)\ge C_3|z-x|^{\alpha-d}.$$

Proof. Let $x,z\in D$. We may and do assume that $\delta_D(x)\le\delta_D(z)$. Let $b=(2C_1^2)^{1/(\alpha-d)}$. If $|z-x|<b\delta_D(x)$ then by (2.3) and (2.4) we have

$$\begin{aligned} G_D(x,z) &\geq C_1^{-1}|z-x|^{\alpha-d} - C_1 E^x|z-X_{\tau_D}|^{\alpha-d} \\ &\geq C_1^{-1}|z-x|^{\alpha-d} - C_1(\delta_D(z))^{\alpha-d} \\ &\geq C_1^{-1}|z-x|^{\alpha-d} - C_1(|z-x|/b)^{\alpha-d} = \tfrac{1}{2}C_1^{-1}|z-x|^{\alpha-d}. \end{aligned}$$

Now, we assume that

$$b\delta_D(x) \leq |z-x| \leq a\delta_D(x).$$

Let $w \in B(x, b\delta_D(x)) \setminus B\big(x, \frac{1}{2}b\delta_D(x)\big)$. The function $G_D(x,\cdot)$ is harmonic in $B\big(w, \frac{1}{4}b\delta_D(x)\big) \cup B\big(z, \frac{1}{4}b\delta_D(x)\big)$, hence by Lemma 3.3 and the above we obtain

$$G_D(x,z) \geq cG_D(x,w) \geq \tfrac{1}{2}cC_1^{-1}|w-x|^{\alpha-d} \geq \tfrac{1}{2}cC_1^{-1}|z-x|^{\alpha-d},$$

where $c = c(\alpha,\mu,a)$. ■

Corollary 3.1. *Let D be a bounded open set in $\mathbb{R}^d$ and $F \subset D$ such that $\varrho = \operatorname{dist}(F, D^c) > 0$. There exists a constant $C_4 = C_4(\alpha,\mu,\varrho,D)$ such that*

$$G_D(x,z) \geq C_4|z-x|^{\alpha-d}, \quad x,z \in F.$$

Proof. For $x,z \in F$ we have $\delta_D(x) \wedge \delta_D(z) \geq \varrho$ and $|z-x| < \operatorname{diam}(D)$, so we can take $a = \operatorname{diam}(D)/\varrho$ in Lemma 3.4. ■

4. Exit time and Poisson kernel for a ball. In the following lemma we improve [24, Theorem 5.7].

Lemma 4.1. *Let D be a bounded open set. The function $P_D(\cdot,\cdot)$ is jointly continuous in $D \times (\overline{D})^c$.*

Proof. Let $x_0 \in D$, $y_0 \in (\overline{D})^c$, and $x_n \to x_0$, $y_n \to y_0$. Set $\varrho = \delta_D(x_0)/2$, $\eta = \operatorname{dist}(y_0, D)/2$. We may and do assume that $x_n \in B(x_0,\varrho)$ and $y_n \in B(y_0,\eta)$ for all $n \in \mathbb{N}$. We have

$$\begin{aligned} P_D(x,y) &= \int_D G_D(x,z)f_\nu(y-z)\,dz = \int_{\mathbb{R}^d} G_D(x,z)f_\nu(y-z)\,dz \\ &= \int_{\mathbb{R}^d} G_D(x,y-z)f_\nu(z)\,dz, \quad x \in D, y \in (\overline{D})^c. \end{aligned}$$

Moreover, by (2.1) and (2.3) we get for $\varepsilon = \alpha/(2(d-\alpha))$,

$$\begin{aligned} &\int_{\mathbb{R}^d} (G_D(x_n, y_n - z)f_\nu(z))^{1+\varepsilon}\,dz \\ &\qquad \leq (C_1M)^{1+\varepsilon} \int_{y_n - D} |y_n - z - x_n|^{(\alpha-d)(1+\varepsilon)}|z|^{(-d-\alpha)(1+\varepsilon)}\,dz \end{aligned}$$

$$
\begin{aligned}
&\le (C_1M)^{1+\varepsilon}\int_D |z-x_n|^{(\alpha-d)(1+\varepsilon)}|y_n-z|^{(-d-\alpha)(1+\varepsilon)}\,dz\\
&\le (C_1M)^{1+\varepsilon}\eta^{(-d-\alpha)(1+\varepsilon)}\int_{B(x_n,\mathrm{diam}(D))}|z-x_n|^{(\alpha-d)(1+\varepsilon)}\,dz\\
&= c(\mathrm{diam}(D))^{\alpha+\varepsilon(\alpha-d)}<\infty,
\end{aligned}
$$

hence the functions $G_D(x_n, y_n - z)f_\nu(z)$ are uniformly integrable and the assertion follows by the continuity of $G_D(\cdot,\cdot)$, because the set of irregular points of D is of Lebesgue measure 0. ∎

The following lemma is an extension of (3.1).

LEMMA 4.2. *For every $\varrho \in (0,1)$ there exists a constant $C_5 = C_5(\alpha,\mu,\varrho)$ such that for every $y \in \operatorname{int} B(0,1)^{\mathrm{c}}$ and $x_1, x_2 \in B(0,\varrho)$ we have*

$$P_{B(0,1)}(x_1,y) \le C_5 P_{B(0,1)}(x_2,y).$$

Proof. Let $B = B(0,1)$ and let $x_1, x_2 \in B(x_0,\varrho)$. The function $\omega_B^x(A)$ is nonnegative, bounded and regular harmonic in B for every set $A \subset B^{\mathrm{c}}$, hence by Lemma 3.3 we have

$$\omega_B^{x_1}(A) \le c\,\omega_B^{x_2}(A)$$

where $c = c(\alpha,\mu,\varrho)$, and that means

$$\int_A P_B(x_1,y)\,dy \le c\int_A P_B(x_2,y)\,dy$$

for every $A \subset B^{\mathrm{c}}$. We conclude that $P_B(x_1,y) \le cP_B(x_2,y)$ for almost all, hence all, $y \in \operatorname{int} B^{\mathrm{c}}$. ∎

LEMMA 4.3. *Let $D \subset \mathbb{R}^d$ be an open set and u be a harmonic function on D. Then $u \in C(D)$.*

Proof. Let $x_0 \in D$, $\varrho = \frac{1}{2}\delta_D(x_0)$, and $B = B(x_0,\varrho)$. Let $x_n \to x_0$; then $x_n \in B(x_0,\varrho/2)$ for all $n \ge n_0$. We have $u(x_n) = \int_{B^{\mathrm{c}}} u(y)P_B(x_n,y)\,dy$, and by Lemma 4.2,

$$|u(y)|P_B(x_n,y) \le c|u(y)|P_B(x_0,y), \qquad y \in B^{\mathrm{c}},\ n \ge n_0,$$

where $c = c(\alpha,\mu)$. Moreover, from the definition of harmonic functions

$$\int_{B^{\mathrm{c}}} c|u(y)|P_B(x_0,y)\,dy = cE^{x_0}|u(X_{\tau_B})| < \infty;$$

hence, by Lemma 4.1 and dominated convergence, we get

$$\lim_{n\to\infty} u(x_n) = u(x_0),$$

so u is continuous at x_0 and in D. ∎

The following estimate is given in [10].

Proposition 4.1. *For every* $\varrho \in (0,1)$ *there exist* $C = C(\alpha,\mu,\varrho)$ *and* $\eta = \eta(\alpha,\mu)$ *such that*

$$P_{B(0,1)}(x,y) \le C(|y|-1)^{-\alpha+\eta}, \quad |x|<\varrho,\ |y|>1. \tag{4.1}$$

Our next goal in this section is to strengthen this estimate and prove that $\eta = \alpha/2$ (see Proposition 4.2 below).

The following lemma was communicated to us by M. Lewandowski.

Lemma 4.4. *There exists a constant* $C_6 = C_6(\alpha,\mu)$ *such that*

$$E^x \tau_{B(0,1)} \le C_6(1-|x|^2)^{\alpha/2}, \qquad x \in B(0,1).$$

Proof. For $x \ne 0$ let $Z_t = \langle X_t, x\rangle/|x|$. Then Z_t is a Lévy process on $\mathbb{R}$ and we have for $u \in \mathbb{R}$,

$$E^0 e^{iuZ_t} = \exp\Big(-t|u|^\alpha \int\limits_{S(0,1)} |\langle x/|x|,\xi\rangle|^\alpha\,\mu(d\xi)\Big).$$

We see that Z_t has the same distributions as $c_x^{1/\alpha} Y_t$ where Y_t denotes the symmetric α-stable Lévy process on $\mathbb{R}$ with $E^0 e^{iuY_t} = e^{-t|u|^\alpha}$, and $c_x = \int_{S(0,1)} |\langle x/|x|,\xi\rangle|^\alpha\,\mu(d\xi)$. Moreover, if $X_t \in B(0,1)$ then $|Z_t| \le |X_t|\,|x|/|x| < 1$, hence

$$E^x \tau_{B(0,1)} \le E^{|x|} s,$$

where $s = \inf\{t \ge 0 : |Z_t| > 1\} \overset{D}{=} c_x^{-1} \inf\{t \ge 0 : |Y_t| > 1\}$. Let $s' = \inf\{t \ge 0 : |Y_t| > 1\}$. It is well known (see [13]) that

$$E^{|x|} s' = \frac{(1-|x|^2)^{\alpha/2}}{\Gamma(\alpha+1)}$$

so we obtain

$$E^x \tau_{B(0,1)} \le c_x^{-1} \frac{(1-|x|^2)^{\alpha/2}}{\Gamma(\alpha+1)}.$$

Finally, by our assumptions on μ, we have $c^{-1} < c_x < c$ where $c = c(\alpha,\mu)$. ∎

The following proposition improves (4.1).

Proposition 4.2. *For every* $\varrho \in (0,1)$ *there exists a constant* $C_7 = C_7(\alpha,\mu,\varrho)$ *such that*

$$P_{B(0,1)}(x,y) \le C_7(|y|-1)^{-\alpha/2}, \qquad |x|<\varrho,\ |y|>1. \tag{4.2}$$

Proof. Let $B = B(0,1)$. By symmetry of the Green function, Fubini's theorem and Lemma 4.4 we have for $y \in \operatorname{int}(B^{\mathrm{c}})$,

$$\begin{aligned}\int\limits_B P_B(w,y)\,dw &= \int\limits_B\int\limits_B G_B(w,z) f_\nu(y-z)\,dz\,dw\\ &= \int\limits_B f_\nu(y-z)\Big(\int\limits_B G_B(z,w)\,dw\Big)\,dz\end{aligned}$$

$$
\begin{aligned}
&= \int_B f_\nu(y-z) E^z \tau_B \, dz \le M C_6 \int_B |y-z|^{-d-\alpha} (1-|z|^2)^{\alpha/2} \, dz \\
&\le M C_6 2^{\alpha/2} \int_{B(y,|y|-1)^{\mathrm{c}}} |y-z|^{-d-\alpha} |y-z|^{\alpha/2} \, dz \\
&= c_1 (|y|-1)^{-\alpha/2},
\end{aligned}
$$

with $c_1 = c_1(\alpha, \mu)$. Moreover, for $x \in B(0, \varrho)$ from Lemma 4.2 we obtain

$$
\int_B P_B(w,y) \, dw \ge \int_{B(0,\varrho)} P_B(w,y) \, dw \ge c_2 P_B(x,y),
$$

where $c_2 = c_2(\alpha, \mu, \varrho)$. ∎

The following results extend the Carleson estimate and boundary Harnack principle given in [10] for $\alpha \le 1$ to all $\alpha \in (0,2)$.

THEOREM 4.1 (Carleson estimate). *Let D be a domain such that $0 \in \partial D$. Let $\kappa > 0$ and $B(A,\kappa) \subset D \cap B(0,1)$. There exists a constant $M_1 = M_1(\alpha,\mu)$ such that for all functions $u \ge 0$, regular harmonic in $D \cap B(0,2)$ and equal to 0 in $D^{\mathrm{c}} \cap B(0,2)$, we have*

$$
u(x) \le M_1 \kappa^{-\alpha} w(A) \le M_1 \kappa^{-\alpha} u(A), \qquad x \in D \cap B(0,3/2),
$$

where w is the regular harmonic function in $D \cap B(0,1)$ defined by

$$
w(x) = \begin{cases} u(x), & x \in B(0,3/2)^{\mathrm{c}} \cup (D^{\mathrm{c}} \cap B(0,1)), \\ 0, & 1 \le |x| < 3/2. \end{cases}
$$

THEOREM 4.2 (Boundary Harnack principle). *Let D be an open set, $Q \in \partial D$, $r > 0$, and suppose that $B(A, \kappa r) \subset D \cap B(Q,r)$. There exists a constant $C_8 = C_8(\alpha,\mu)$ such that for all functions $u, v \ge 0$, regular harmonic in $D \cap B(Q,2r)$ and equal to 0 in $D^{\mathrm{c}} \cap B(Q,2r)$, we have*

$$
C_8^{-1} \kappa^{d+\alpha} \frac{u(A)}{v(A)} \le \frac{u(x)}{v(x)} \le C_8 \kappa^{-d-\alpha} \frac{u(A)}{v(A)}, \qquad x \in D \cap B(Q, r/2). \tag{4.3}
$$

The proofs of Theorems 4.1 and 4.2 are direct adaptations of the proofs given in [10] with (4.1) replaced by (4.2).

REMARK. Consider the process $X_t = T\widehat{X}_t$ described in Example (b) above. By [11] and [18] it is fairly easy to see that if D in Theorem 4.2 is a $C^{1,1}$ domain then $c^{-1} \le u(x)/\delta_D(x)^{\alpha/2} \le c$, $x \in D \cap B(Q, r/2)$, which is the same asymptotics as for $\widehat{X}_t$. However, let D be a cone with vertex angle $\lambda \in (0, \pi/2)$. The asymptotics of harmonic functions of $\widehat{X}_t$ at the vertex of the cone changes when $\lambda \to 0$ ([17]). Therefore the asymptotics varies among different processes of the form $X_t = T\widehat{X}_t$.

The main application of Theorems 4.1 and 4.2 is to the Green function of D. We now state for our α-stable process an important result on the Green function of Lipschitz domains called the "3G Theorem". The proof, which relies on Theorems 4.1 and 4.2, is omitted because it is a direct adaptation of arguments developed for Brownian motion in [5] (see also [12]).

THEOREM 4.3 (3G Theorem). *Let D be a bounded Lipschitz domain in $\mathbb{R}^d$. There exists a constant $C_9 = C_9(\alpha, \mu, D)$ such that for all $x, y, z \in D$ we have*

$$\frac{G_D(x,y)G_D(y,z)}{G_D(x,z)} \le C_9\left(\frac{|z-x|}{|y-x|\,|z-y|}\right)^{d-\alpha} \le 2^{d-\alpha}C_9(|y-x|^{\alpha-d} + |z-y|^{\alpha-d}),$$

unless $x = y = z$. In fact, the constant C_9 above depends on D only through its Lipschitz character and diameter.

5. q-harmonic functions. We say that a Borel function q belongs to the *Kato class* $\mathcal{J}$ if q satisfies either of the two equivalent conditions (see [12])

$$\lim_{r\to 0} \sup_{x\in\mathbb{R}^d} \int_{|x-y|\le r} |q(y)K(y-x)|\,dy = 0, \tag{5.1}$$

$$\lim_{t\to 0} \sup_{x\in\mathbb{R}^d} \int_0^t P_s|q|(x)\,ds = 0. \tag{5.2}$$

We write $q \in \mathcal{J}_{\mathrm{loc}}$ if for every bounded Borel set $U \subset \mathbb{R}^d$ we have $\mathbf{1}_U q \in \mathcal{J}$. Clearly $\mathcal{J}_{\mathrm{loc}} \subset L^1_{\mathrm{loc}}$. If $f \in L^\infty(\mathbb{R}^d)$ and $q \in \mathcal{J}$ then $f, fq \in \mathcal{J}$. Note that by (2.3) we have $\mathcal{J} = \mathcal{J}^\alpha$ where $\mathcal{J}^\alpha$ denotes the Kato class for the rotation invariant α-stable process $\widehat{X}_t$ ([7]).

Let U be a domain in $\mathbb{R}^d$ and let $q \in \mathcal{J}$. We define

$$e_q(\tau_U) = \exp\Big(\int_0^{\tau_U} q(X_t)\,dt\Big). \tag{5.3}$$

By (5.2) we have $\int_0^t |q(X_s)|\,ds < \infty$ a.s., for each $t > 0$. Therefore, if $\tau_U < \infty$, the random variable $e_q(\tau_U)$ is well defined.

Let u be a Borel measurable function on $\mathbb{R}^d$. We say that u is *q-harmonic* in an open set $D \subset \mathbb{R}^d$ if

$$u(x) = E^x[\tau_U < \infty;\, e_q(\tau_U)u(X_{\tau_U})], \qquad x \in U,$$

for every bounded open set U with $\overline{U} \subset D$. It is called *regular q-harmonic* in D if the above equality holds for $U = D$.

We always understand that the expectation in the above condition is absolutely convergent. For $q \equiv 0$ we obtain the previous definition of harmonicity. By the strong Markov property of X_t a regular q-harmonic function u is q-harmonic.

An important technical tool in further considerations is the conditional α-stable Lévy motion. For the definition and properties of the conditional process for $\widehat{X}_t$ we refer to [7]; here the definitions are analogous. We recall that for a domain D by the α-stable y-Lévy motion we mean the process conditioned by the Green function $G(\cdot, y)$ of D. If D is a bounded Lipschitz domain then we obtain, by routine arguments (see, e.g., [12]), for $\Phi \geq 0$ measurable with respect to $\mathcal{F}_{\tau_D -}$, and any Borel $f \geq 0$, the following formula:

$$E^x[f(X_{\tau_D})\Phi] = E^x[f(X_{\tau_D})E^x_{X_{\tau_D -}}[\Phi]], \qquad x \in D. \tag{5.4}$$

In what follows E^x_v denotes the expectation for the α-stable v-Lévy process conditioned by $G_B(\cdot, v)$ where B is a given ball in $\mathbb{R}^d$ and $x, v \in B$. The process approaches v in a finite time and is killed then.

LEMMA 5.1. *Let $q \in \mathcal{J}$ and $\varepsilon > 0$. There is $r_0 = r_0(\alpha, \mu, q, \varepsilon)$ such that*

$$\int_B \frac{G_B(x,y)G_B(y,v)}{G_B(x,v)}\,|q(y)|\,dy \leq \varepsilon \tag{5.5}$$

for every ball $B \subset \mathbb{R}^d$ of radius $r \leq r_0$.

Proof. Let $x_0 \in \mathbb{R}^d$, $r > 0$ and $B = B(x_0, r)$. By scaling we have

$$G_B(z,w) = r^{\alpha-d} G\left(\frac{z-x_0}{r}, \frac{w-x_0}{r}\right), \qquad z, w \in B,$$

where G is the Green function for the ball $B(0,1)$.

By the 3G Theorem we have for $v, x, y \in B$,

$$\begin{aligned}\frac{G_B(x,y)G_B(y,v)}{G_B(x,v)} &\leq 2^{d-\alpha} C_9 r^{\alpha-d}\left(\left(\frac{|y-x|}{r}\right)^{\alpha-d} + \left(\frac{|v-y|}{r}\right)^{\alpha-d}\right) \\ &= 2^{d-\alpha} C_9 (|y-x|^{\alpha-d} + |v-y|^{\alpha-d}).\end{aligned}$$

By (5.1) there is $r_0 = r_0(\alpha, \mu, q, \varepsilon)$ such that (5.5) holds if $0 < r \leq r_0$. ∎

We recall the following important fact known as Khasminski's lemma: For every nonnegative q and Markov time τ such that $\tau \leq t + \tau \circ \theta_t$ on $\{t < \tau\}$ for each $t \geq 0$, we have:

$$\text{If} \quad \sup_{x \in \mathbb{R}^d} E^x \int_0^\tau q(X_s)\,ds = \varepsilon < 1 \quad \text{then} \quad \sup_{x \in \mathbb{R}^d} E^x e_q(\tau) < (1-\varepsilon)^{-1}.$$

LEMMA 5.2. *Let $q \in \mathcal{J}$ and $\varepsilon > 0$. Let $r_0 = r_0(\alpha, \mu, q, \varepsilon) > 0$ be the constant of Lemma 5.1. Then for every ball $B \subset \mathbb{R}^d$ of radius $r \leq r_0$ we have*

$$\exp(-\varepsilon) \leq E^x_v e_q(\tau_B) \leq (1-\varepsilon)^{-1}, \qquad x, v \in B. \tag{5.6}$$

Proof. We have

$$E_v^x\Big[\int_0^{\tau_B} q(X_t)\,dt\Big] = G_B(x,v)^{-1}E^x\Big[\int_0^{\tau_B} q(X_t)G_B(X_t,v)\,dt\Big]$$
$$= \int_B \frac{G_B(x,y)G_B(y,v)}{G_B(x,v)}\,q(y)\,dy.$$

The upper bound in (5.6) follows by Lemma 5.1 and Khasminski's lemma applied to the expectations $\{E_v^x : x \in \mathbb{R}^d\}$, and the lower bound follows from Jensen's inequality. ■

The proofs of the next two lemmas are standard (see, e.g., [12] or [7]); we provide them only for the reader's convenience.

LEMMA 5.3. *Let D be a Green-bounded domain in $\mathbb{R}^d$ and $q \in \mathcal{J}$.*

(i) *For every $b > 0$ there exists $a = a(\alpha, q, b)$ such that*

$$G_D|q| \le aG_D\mathbf{1} + b. \tag{5.7}$$

Consequently, for a fixed $q \in \mathcal{J}$ and variable D, we have $\|G_D q\|_\infty \to 0$ as $\|G_D\mathbf{1}\|_\infty \to 0$.

(ii) $G_D q \in L^\infty(\mathbb{R}^d) \cap C(D)$, *and for any $z \in \partial D$ regular for D, we have*

$$\lim_{x\to z} G_D q(x) = 0. \tag{5.8}$$

(iii) *Under the additional hypothesis*: (a) $q \in L^1(D)$, *or* (b) $|D| < \infty$, *we have*

$$\lim_{|x|\to\infty} G_D q(x) = 0. \tag{5.9}$$

Proof. In the proof of (i) we may suppose that $q \ge 0$. For any $s > 0$ let Λ_n denote the indicator of the set $\{\tau_D > ns\}$, $n \ge 0$. The following inequality holds:

$$\int_0^{\tau_D} q(X_t)\,dt = \int_0^{\infty} q(X_t)\mathbf{1}_{\{\tau_D>t\}}\,dt \le \sum_{n=0}^{\infty}\int_{ns}^{(n+1)s} q(X_t)\mathbf{1}_{\{\tau_D>ns\}}\,dt$$
$$= \sum_{n=0}^{\infty}\Lambda_n \int_{ns}^{(n+1)s} q(X_t)\,dt.$$

Taking expectations of both sides, using Fubini's theorem and the Markov property we obtain

$$G_D q(x) \le \sum_{n=0}^{\infty} E^x\Big[\tau_D > ns;\ E^{X_{ns}}\Big[\int_0^s q(X_t)\,dt\Big]\Big].$$

Since $q \in \mathcal{J}$, we can choose $s > 0$ so that

$$\sup_{x\in\mathbb{R}^d} E^x\Big[\int_0^s q(X_t)\,dt\Big] \le b.$$

We then obtain

$$G_D q(x) \le b \sum_{n=0}^{\infty} P^x(\tau_D > ns) \le b(1 + E^x[\tau_D/s]),$$

which gives (5.7) with $a = b/s$.

From now on we no longer assume that $q \ge 0$. Let us recall that P_t^D has the strong Feller property. Since, by (5.7), $G_D q$ is bounded in $\mathbb{R}^d$, $P_t^D(G_D q)$ is continuous in D. From the semigroup property of P_t^D we obtain

$$G_D q - P_t^D(G_D q) = \int_0^t P_s^D q\, ds. \tag{5.10}$$

Since $|P_s^D q| \le P_s|q|$, and $q \in \mathcal{J}$, the right hand side above converges to zero uniformly in $\mathbb{R}^d$ as $t \to 0$ and so $G_D q$ is continuous in D. Next, if $z \in \partial D$ is regular for D, then

$$\begin{aligned}\limsup_{x\to z} |P_t^D G_D q(x)| &\le \limsup_{x\to z} (P_t^D \mathbf{1}(x)) \|G_D q\|_\infty \\ &= \limsup_{x\to z} P^x(\tau_D > t) \|G_D q\|_\infty \\ &\le P^z(\tau_D > t)\|G_D q\|_\infty = 0,\end{aligned}$$

because the function $x \mapsto P^x(\tau_D > t)$ is upper-semicontinuous at z (cf. [12]). Hence (5.8) is also true by the uniform convergence in (5.10) and the above argument.

Under the hypothesis (a) in (iii) we have $G_D q \in L^1(D)$, because $\|G_D q\|_1 \le \|G_D \mathbf{1}\|_\infty \|q\|_1 < \infty$. Since $\lim_{|x|\to\infty} p(t,x,y) = 0$ for each $t > 0$ and $y \in \mathbb{R}^d$, it follows by dominated convergence that $\lim_{|x|\to\infty} P_t^D G_D q(x) = 0$. We obtain (5.9) once again by the uniform convergence in (5.10). Under the hypothesis (b) in (iii) we have $G_D q \in L^\infty(D) \subset L^1(D)$, so the same argument is valid. ■

LEMMA 5.4. *Let $q \in \mathcal{J}_{\mathrm{loc}}$ and u be a nonnegative function which is locally bounded and q-harmonic on an open set $D \subset \mathbb{R}^d$. Then for every open bounded set $U \subset D$ such that $\overline{U} \subset D$ we have*

$$u(x) = E^x u(X_{\tau_U}) + G_U(qu)(x), \qquad x \in U. \tag{5.11}$$

In particular, u is continuous in D.

Proof. By the assumptions $qu\mathbf{1}_U \in \mathcal{J}$ so $\sup_{x\in U} G_U(u|q|)(x) < \infty$ and $G_U(qu)$ is continuous on U by Lemma 5.3. We put

$$\Phi(t) = \mathbf{1}_{\{t<\tau_U\}} q(X_t) u(X_{\tau_U}) \exp \int_t^{\tau_U} q(X_s)\, ds,$$

$$\Psi(t) = \mathbf{1}_{\{t<\tau_U\}} |q(X_t)| u(X_{\tau_U}) \exp \int_t^{\tau_U} q(X_s)\, ds,$$

Using the Markov property along with the Fubini–Tonelli theorem we obtain

$$\begin{aligned}\int_0^\infty E^x[\Psi(t)]\,dt &= E^x\Big[\int_0^{\tau_U} |q(X_t)|(e_q(\tau_U)u(X_{\tau_U}))\circ\theta_t\,dt\Big]\\ &= E^x\Big[\int_0^{\tau_U} |q(X_t)|E^{X_t}[e_q(\tau_U)u(X_{\tau_U})]\,dt\Big]\\ &= E^x\Big[\int_0^{\tau_U} |q(X_t)|u(X_t)\,dt\Big] = G_U(u|q|)(x)<\infty.\end{aligned}$$

Thus, by Fubini's theorem we obtain

$$\int_0^\infty E^x[\Phi(t)]\,dt = G_U(qu)(x),$$

while, at the same time,

$$\begin{aligned}\int_0^\infty E^x[\Phi(t)]\,dt &= E^x[\{e_q(\tau_U)-1\}u(X_{\tau_U})] = E^x[e_q(\tau_U)u(X_{\tau_U})] - E^x u(X_{\tau_U})\\ &= u(x) - E^x u(X_{\tau_U}).\end{aligned}$$

Indeed, $\int_0^{\tau_U}|q(X_s)|\,ds<\infty$ and so the function $[0,\tau_U]\ni t\mapsto \exp\int_t^{\tau_U} q(X_s)\,ds$ is absolutely continuous (a.s.). Its derivative equals $-q(X_t)\exp\int_t^{\tau_U} q(X_s)ds$ a.s. This shows the formula (5.11). The function $x\mapsto E^x u(X_{\tau_U})$ is harmonic, hence continuous on U (Lemma 4.3). ∎

THEOREM 5.1. *Let $q\in\mathcal{J}_{\mathrm{loc}}$, u be a nonnegative q-harmonic function in an open set D, and $F\subset D$ be compact. There exists a constant $C_{10}=C_{10}(\alpha,\mu,q,F,D)$ such that*

$$u(x)\le C_{10}u(y),\qquad x,y\in F. \tag{5.12}$$

If $u(x)=0$ for some $x\in D$ then $u=0$ on D and $u=0$ a.e. on D^{c}.

Proof. Let $F\subset\subset D$ and $\delta_F=\mathrm{dist}(F,D^{\mathrm{c}})$. Put $A=\{x\in D:\mathrm{dist}(x,F)\le \delta_F/2\}$. Then A is a compact subset of D and we have $q\mathbf{1}_A\in\mathcal{J}$. Let $\varrho_0=r_0\wedge(\delta_F/2)$, where $r_0=r_0(\alpha,\mu,q\mathbf{1}_A)$ is the constant from Lemma 5.1 for $\varepsilon=1/2$. Let $x\in F$, $0<r\le\varrho_0$ and $B=B(x,r)$. We have $B\subset A$ and by (5.4),

$$u(y)=E^y[e_q(\tau_B)u(X_{\tau_B})] = E^y[u(X_{\tau_B})E^y_{X_{\tau_B-}}(\tau_B)],\qquad y\in B.$$

Lemma 5.2 yields

$$\tfrac12 E^y u(X_{\tau_B})\le u(y)\le 2E^y u(X_{\tau_B}),\qquad y\in B. \tag{5.13}$$

The function $h(y)=E^y u(X_{\tau_B})$, $y\in\mathbb{R}^d$, is regular harmonic in B, so by Lemma 3.3 we obtain

$$c_1^{-1}E^y u(X_{\tau_B})\le E^x u(X_{\tau_B})\le c_1 E^y u(X_{\tau_B}),\qquad y\in B(x,r/2),$$

where $c_1 = c_1(\alpha,\mu)$. By (5.13) and the above we get

$$(4c_1)^{-1}u(x) \le u(y) \le 4c_1 u(x), \quad y \in B(x,r/2). \tag{5.14}$$

We now consider $z \in F$ such that $|z-x| \ge \varrho_0/2$. Let $B_1 = B(z,\varrho_0/4)$. Note that $B_1 \subset A$ and $B_1 \cap B(x,\varrho_0/4) = \emptyset$. By (2.7), (5.13) and (5.14) with $r = \varrho_0$ we obtain

$$\begin{aligned}
u(z) &\ge \frac{1}{2}\,E^z u(X_{\tau_{B_1}}) \ge \frac{1}{2}\int\limits_{B(x,\varrho_0/4)} P_{B_1}(z,w)\,dw \\
&\ge \frac{1}{2}\,M^{-1}E^z\tau_{B_1}\int\limits_{B(x,\varrho_0/4)} (|w-z|+\varrho_0/4)^{-d-\alpha}u(w)\,dw \\
&\ge \frac{1}{2}\,M^{-1}(\varrho_0/4)^{\alpha}E^0\tau_{B(0,1)}(4c_1)^{-1}u(x)(2|x-z|)^{-d-\alpha}|B(x,\varrho_0/4)| \\
&\ge c_2 u(x),
\end{aligned}$$

where $c_2 = M^{-1}8^{-d-\alpha-1}(\omega_d/d)E^0\tau_{B(0,1)}c_1^{-1}(\varrho_0/\operatorname{diam}(F))^{d+\alpha}$. From this and (5.14) with $r = \varrho_0$, (5.12) follows.

We now assume that $x \in D$ and $u(x) = 0$. By the first part of the proof, for every $B = B(x,r)$ with $r > 0$ small enough we have

$$0 = u(x) \ge \frac{1}{2}\,E^x u(X_{\tau_B}) \ge \frac{1}{2}\int\limits_{D^c} P_B(x,y)u(y)\,dy.$$

It follows that $u = 0$ a.e. on D^c. The pointwise equality $u = 0$ on D is a consequence of (5.12). ∎

From (5.12) it follows that every nonnegative q-harmonic function is locally bounded on D, hence by Lemma 5.4 we get the following

COROLLARY 5.1. *Every function q-harmonic in D is continuous in D.*

Acknowledgments. The author is grateful to Drs. K. Bogdan and M. Ryznar for discussions and suggestions, and to Dr. M. Lewandowski for communicating Lemma 4.4.

REFERENCES

[1] R. F. Bass and D. A. Levin, *Harnack inequalities for jump processes*, Potential Anal. 17 (2002), 375–388.

[2] J. Bertoin, *Lévy Processes*, Cambridge Univ. Press, Cambridge, 1996.

[3] R. M. Blumenthal and R. K. Getoor, *Markov Processes and Potential Theory*, Pure Appl. Math., Academic Press, New York, 1968.

[4] R. M. Blumenthal, R. K. Getoor and D. B. Ray, *On the distribution of first hits for the symmetric stable processes*, Trans. Amer. Math. Soc. 99 (1961), 540–554.

[5] K. Bogdan, *Sharp estimates for the Green function in Lipschitz domains*, J. Math. Anal. Appl. 243 (2000), 326–337.

[6] K. Bogdan, *The boundary Harnack principle for the fractional Laplacian*, Studia Math. 123 (1997), 43–80.

[7] K. Bogdan and T. Byczkowski, *Potential theory for the α-stable Schrödinger operator on bounded Lipschitz domains*, ibid. 133 (1999), 53–92.

[8] —, —, *Potential theory of the Schrödinger operator based on the fractional Laplacian*, Probab. Math. Statist. 20 (2000), 293–335.

[9] —, —, *Probabilistic proof of boundary Harnack principle for α-harmonic functions*, Potential Anal. 11 (1999), 135–156.

[10] K. Bogdan, A. Stós and P. Sztonyk, *Potential theory for Lévy stable processes*, Bull. Polish Acad. Sci. Math. 50 (2002), 361–372.

[11] Z.-Q. Chen and R. Song, *Estimates on Green functions and Poisson kernels for symmetric stable processes*, Math. Ann. 312 (1998), 465–501.

[12] K. L. Chung and Z. Zhao, *From Brownian Motion to Schrödinger's Equation*, Springer, New York, 1995.

[13] R. K. Getoor, *First passage times for symmetric stable processes in space*, Trans. Amer. Math. Soc. 101 (1961), 75–90.

[14] P. Głowacki and W. Hebisch, *Pointwise estimates for densities of stable semigroups of measures*, Studia Math. 104 (1993), 243–258.

[15] N. Ikeda and S. Watanabe, *On some relations between the harmonic measure and the Lévy measure for a certain class of Markov processes*, J. Math. Kyoto Univ. 2 (1962), 79–95.

[16] N. Jacob, *Pseudo-Differential Operators and Markov Processes*, *Vol. 1*: *Fourier Analysis and Semigroups*, Imperial College Press, London, 2002.

[17] T. Kulczycki, *Exit time and Green function of cone for symmetric stable processes*, Probab. Math. Statist. 19 (1999), 337–374.

[18] —, *Properties of Green function of symmetric stable processes*, ibid. 17 (1997), 339–364.

[19] M. Riesz, *Intégrales de Riemann–Liouville et potentiels*, Acta Sci. Math. (Szeged) 9 (1938), 1–42.

[20] —, *Rectification au travail "Intégrales de Riemann–Liouville et potentiels"*, ibid., 116–118.

[21] R. Song and J.-M. Wu, *Boundary Harnack principle for symmetric stable processes*, J. Funct. Anal. 168 (1999), 403–427.

[22] P. Sztonyk, *On harmonic measures for Lévy processes*, Probab. Math. Statist. 20 (2000), 383–390.

[23] S. J. Taylor, *Sample path properties of a transient stable process*, J. Math. Mech. 16 (1967), 1229–1246.

[24] Z. Vondraček, *Basic potential theory of certain nonsymmetric strictly α-stable processes*, Glas. Mat. Ser. III 37 (57) (2002), 211–233.

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Received 3 June 2002;
revised 17 June 2002 (4232)