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# ON STRONG UNIFORM DIMENSION OF LOCALLY FINITE GROUPS

BY

A. SAKOWICZ (Białystok)

**Abstract.** We give the description of locally finite groups with strongly balanced subgroup lattices and we prove that the strong uniform dimension of such groups exists. Moreover we show how to determine this dimension.

**1. Lattice preliminaries.** All lattices considered in this paper have the least and the greatest element, denoted by 0 and 1 respectively. They do not need to be finite (in contrast to [1, 9]). We will apply also some other notation and terminology about lattices, as in [2, 9]. In particular if $L$ is a lattice we will say that $L$ is *balanced* if for all $x, y, z \in L$ we have

$$x \wedge y = 0 \ \& \ (x \vee y) \wedge z = 0 \ \Rightarrow \ (y \vee z) \wedge x = 0 \ \& \ (z \vee x) \wedge y = 0,$$

and $L$ is *strongly balanced* if all nonempty intervals of $L$ are balanced.

It is easy to show that every distributive and even modular lattice is strongly balanced. Hence strong balancedness can be considered as a generalization of modularity. In [5], with motivation coming from Theorem 6.1.10 of [12], nearly modular lattices were introduced, also as a generalization of modular ones. In the last section of this paper we indicate that there is no inclusion between the class of strongly balanced and the class of nearly modular lattices.

Further properties of balanced and strongly balanced lattices can be found in [9, 10, 13].

If $a, u \in L$ then, as in [6, 10], we will say that $a$ is *essential in* $L$ if $a \wedge x \neq 0$ for every $0 \neq x \in L$, and $u$ is *uniform in* $L$ if $u \neq 0$ and every element from $(0, u]$ is essential in $[0, u]$. For example any atom is a uniform element and 1 is an essential element in every nontrivial lattice.

Let $L$ be a lattice. It will be called *locally uniform* ([10]) if any nontrivial interval $[0, a] \subseteq L$ contains a uniform element, and *strongly locally uniform* if any of its nontrivial intervals is locally uniform. Clearly every (strongly) atomic lattice is (strongly) locally uniform. Notably, any finite lattice is

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strongly locally uniform. However, even in a distributive locally uniform lattice, a sublattice need not be locally uniform.

EXAMPLE 1.1. Let $A$ be an infinite set and $\mathcal{A}$ be the set of all subsets of $A$. Then $\mathcal{A}$ is a Boolean algebra, hence it is a distributive lattice. Moreover, for any proper subsets $B \subset C \subseteq A$ we can construct an atom in the interval $[B, C]$ by adding a one-element set $\{c\}$ to $B$ where $c \in C \setminus B$. Hence $\mathcal{A}$ is a strongly locally uniform lattice.

Now let $A_1, A_2$ be infinite subsets of $A$ such that $A_1 \cap A_2 = \emptyset$ and $A_1 \cup A_2 = A$. Analogously we define infinite subsets $A_{11}$, $A_{12}$, $A_{21}$, $A_{22}$ of $A$ such that $A_{11} \cap A_{12} = \emptyset$, $A_{11} \cup A_{12} = A_1$, $A_{21} \cap A_{22} = \emptyset$, and $A_{21} \cup A_{22} = A_2$. Continuing this process we obtain a family of infinite subsets $\{A_1, A_2, A_{11}, A_{12}, A_{21}, A_{22}, \ldots\}$ such that the Boolean subalgebra $\mathcal{B} = \langle A_1, A_2, A_{11}, A_{12}, A_{21}, A_{22}, \ldots\rangle \subset \mathcal{A}$ has the same $0 = \emptyset$ and has no uniform element ([7]).

Let $L$ be a lattice and let $X \subset L \setminus \{0\}$ be a subset. As in [6, 10], if $X = \{x_1, \ldots, x_n\}$ for some $n < \infty$ then we will say that $X$ is *independent* (*in* $L$) if for every $1 \le i \le n$, $x_i \wedge (\bigvee_{k \ne i} x_k) = 0$. If $X$ is infinite then we will say that $X$ is *independent* if each of its finite subsets is independent in the previous sense. Further a subset $B \subset L$ will be called a *base* of $L$ if any element of $B$ is uniform and $B$ is a maximal independent subset of $L$. If $B$ is a base of $L$ then the cardinality of $B$ will be called the *uniform dimension* of $L$ and will be denoted by $\mathrm{u}(L)$. The following result about bases and the uniform dimension is crucial.

THEOREM 1.2 ([10]). *Let $L$ be a balanced and locally uniform lattice.*

(a) *There exists a base in $L$.*

(b) *Every independent set of uniform elements in $L$ can be extended to a base of $L$.*

(c) *Any two bases of $L$ have the same cardinality. Hence* $\mathrm{u}(L)$ *is well defined.*

It is known from [10] that for nonbalanced lattices, or balanced but not locally uniform ones, the uniform dimension cannot be well defined.

Now let $L$ be a strongly balanced and strongly locally uniform lattice. Then $\mathrm{u}([a, 1])$ for every $a \in L$ is well defined. In this case the smallest cardinal number $\alpha$ such that $\mathrm{u}([a, 1]) \le \alpha$ for all $a \in L$ will be called the *strong uniform dimension* of $L$ and will be denoted by $\mathrm{u_s}(L)$.

Let $\prod_{i \in I} L_i$ be the cartesian product of lattices $L_i, i \in I$. For any $k \in I$, let $\varphi_k : L_k \to \prod_{i \in I} L_i$ be the map such that

$$\varphi_k(x) = \{x_i\}_{i \in I} \quad \text{and} \quad x_i = \begin{cases} x & \text{if } i = k, \\ 0 & \text{if } i \ne k. \end{cases}$$

Then $L_k$ is isomorphic to $\varphi_k(L_k)$ for any $k \in I$ and we will identify these lattices.

It is not difficult to show that if for any $i \in I$, $B_i$ is a base of the lattice $L_i$ then the set $\bigcup_{i\in I} B_i$ is a base of $\prod_{i\in I} L_i$ (see [10, 1.7]). Moreover $u$ is a uniform element in $\prod_{i\in I} L_i$ if and only if $u$ is a uniform element in $L_i$ for some $i \in I$.

From these facts we obtain the following:

PROPOSITION 1.3. *Let* $\{L_i \mid i \in I\}$ *be a set of lattices and let* $L = \prod_{i\in I} L_i$. *All lattices* $L_i$ *are* (*strongly*) *balanced or* (*strongly*) *locally uniform if and only if* $L$ *has the same property.*

PROPOSITION 1.4. *Let* $\{L_i \mid i \in I\}$ *be a set of lattices and let* $L = \prod_{i\in I} L_i$.

(a) *If* $L$ *is balanced and locally uniform then* $\mathrm{u}(L) = \sum_{i\in I} \mathrm{u}(L_i)$.

(b) *If* $L$ *is strongly balanced and strongly locally uniform then* $\mathrm{u_s}(L) = \sum_{i\in I} \mathrm{u_s}(L_i)$.

PROPOSITION 1.5. *Let* $L$ *be a lattice and* $K \subseteq L$ *its sublattice.*

(a) *If* $K$ *and* $L$ *are balanced and locally uniform and* $0 \in K$ *then* $\mathrm{u}(K) \le \mathrm{u}(L)$.

(b) *If* $K$ *and* $L$ *are strongly balanced and strongly locally uniform then* $\mathrm{u_s}(K) \le \mathrm{u_s}(L)$.

(c) *If* $K$ *is an interval in* $L$ *and* $L$ *is strongly balanced and strongly locally uniform then* $\mathrm{u_s}(K) \le \mathrm{u_s}(L)$.

From Example 1.1 we know that, in contrast to the finite case (see [1, 9]), the assumptions on $K$ in Proposition 1.5 are necessary.

**2. Strongly balanced groups.** In this section we will find the description of locally finite groups whose subgroup lattices are strongly balanced and show that these lattices are strongly atomic. As usual (see [2, 12]), we denote by $L(G)$ the lattice of all subgroups of a group $G$. Most of our notation about groups is standard and can be found in [11, 12]. In particular, in [12] groups with modular subgroup lattices are called *modular groups.* Similarly, groups with (strongly) balanced subgroup lattices will be called (*strongly*) *balanced groups* and groups with (strongly) atomic subgroup lattices will be called (*strongly*) *atomic.*

In view of properties of (strongly) balanced lattices, we know that a subgroup of a (strongly) balanced or (strongly) atomic group has the same property. Furthermore, any homomorphic image of a strongly balanced (resp. strongly atomic) group is strongly balanced (resp. strongly atomic). However, from [9] we know that a homomorphic image of a balanced group need not be balanced.

For brevity, the group $G$ will be called an *exceptional strongly balanced group* (*ESB-group*) if $G$ is the semidirect product of an elementary abelian $p$-group $P$ by a group $Q = \langle y \mid y^{q^m} = e\rangle$ with $y^{-1}xy = x^k$ for all $x \in P$, where $k$ is an integer such that $k^{q^m} \equiv 1 \pmod{p}$ and $k \not\equiv 1 \pmod{p}$.

All strongly balanced finite groups are described in [1] as follows:

THEOREM 2.1. *Let $G$ be a finite group. Then $G$ is strongly balanced if and only if it is one of the following groups*:

(a) *a modular $p$-group*;
(b) *an $ESB$-group*;
(c) *a direct product of groups given in* (a) *and* (b), *with pairwise coprime orders.*

The next lemma is helpful in the study of strongly balanced and strongly atomic groups.

LEMMA 2.2. *Let $G$ be a group, $P$ be a normal subgroup of $G$ and $H \le K$ be subgroups of $G$.*

(a) *If $HX$ is a subgroup of $G$ for any subgroup $X$ of $P$ then the intervals $[H, HP]$ and $[H \wedge P, P]$ are isomorphic.*

(b) *If $K \cap HP = H$ then the intervals $[H, K]$ and $[HP, KP]$ are isomorphic.*

*Proof.* (a) We consider the mapping $\varphi : [H \wedge P, P] \to [H, HP]$ defined by $\varphi(X) = X \vee H$. By assumption $\varphi(X) = XH$ is a subgroup of $G$ for every $X \in [H \wedge P, P]$. Hence by elementary coset calculation one can see that $\varphi$ is an isomorphism of $[H \wedge P, P]$ and $[H, HP]$.

(b) Now we consider the mapping $\psi : [H, K] \to [HP, KP]$ defined by $\psi(X) = X \vee P$. Since $P$ is a normal subgroup of $G$, $\psi(X) = XP$ is a subgroup for every $X \in [H, K]$. By assumption $HP \cap K = H$. Hence by the isomorphism theorem, $\psi$ is an isomorphism of $[H, K]$ and $[HP, KP]$. ∎

It is obvious that any torsion group is atomic. Hence any abelian torsion group is strongly atomic. Below we extend this observation.

LEMMA 2.3. *Let $G$ be a torsion group. If there exists an abelian normal subgroup $P$ and an element $g$ of $G$ such that $G = P\langle g\rangle$ and $g$ induces a power automorphism on $P$ then $G$ is strongly atomic.*

*Proof.* Let $H \le K$ be subgroups of $G$. Suppose first that $HP \cap K = H$. Then, from Lemma 2.2, $[H, K]$ is isomorphic to $[HP, KP]$. Since $[HP, KP]$ is isomorphic to an interval of $L(G/P)$ and $G/P$ is abelian and torsion, $[H, K]$ has an atom.

Now let $HP \cap K = J > H$. Since $g$ induces a power automorphism on $P$, $XH$ is a subgroup for any subgroup $X$ of $P$. Then the intervals $[H, HP]$ and $[H \wedge P, P]$ are isomorphic, by Lemma 2.2. Hence $[H, J] \subset [H, HP]$ is isomorphic to an interval of $L(P)$ and $P$ is strongly atomic by assumption. So $[H, J]$ has an atom and hence $[H, K]$ also has an atom. ∎

In [1, 2.9] it was proved that every finite ESB-group is strongly balanced. The criterion for being a strongly balanced lattice which was used in this proof can be extended from finite to strongly atomic lattices (see [13]). Using this criterion and Lemma 2.3 we obtain:

THEOREM 2.4. *If $G$ is an ESB-group then $G$ is strongly balanced and strongly atomic.*

It was observed in [1] that even in the finite case, ESB-groups need not be modular. It turns out that the situation is different in the case of locally finite and strongly balanced $p$-groups.

LEMMA 2.5. *Let $G$ be a locally finite $p$-group. The following conditions are equivalent*:

(a) *$G$ is strongly balanced.*
(b) *$G$ is modular.*
(c) *$G$ is one of the following groups*:

  (i) *abelian or*
  (ii) *a hamiltonian 2-group or*
  (iii) *$G$ contains an abelian normal subgroup $A$ of exponent $p^k$ with cyclic factor group $G/A$ of order $p^m$ $(k, m \in \mathbb{N})$ and there exists $b \in G$ with $G = A\langle b \rangle$ and an integer $s$ which is at least 2 in case $p = 2$ such that $s < k \leq s + m$ and $b^{-1}ab = a^{1+p^s}$ for all $a \in A$.*

*Proof.* (a) follows from (b) even on the lattice level, and (b) is equivalent to (c) in view of the result of Iwasawa [12, 2.4.14], because $G$ is locally finite.

Hence, it is enough to prove that (a) implies (b). Let $G$ be a strongly balanced $p$-group. Let $H, K$ be any subgroups of $G$ and $h \in H$, $k \in K$. Then $\langle h, k \rangle$ is a finite strongly balanced $p$-group. Hence $\langle h, k \rangle$ is modular, by Theorem 2.1. In view of [12, 2.3.2], any two subgroups of $\langle h, k \rangle$ commute. In particular $\langle h \rangle \langle k \rangle = \langle k \rangle \langle h \rangle$. Thus $HK = KH$ and $HK = H \vee K$ is a subgroup of $G$. Hence, from [11, 1.3.14], $G$ is a modular group. ∎

We will say that $G$ is *decomposable* if $G$ is a direct product of its nontrivial subgroups. Otherwise we will say that $G$ is *indecomposable*. Moreover, as in [12] we will say that the groups $G_i$, $i \in I$, are *coprime* if $(o(g_i), o(g_j)) = 1$ for all $g_i \in G_i, g_j \in G_j$ with $i \neq j$. $\mathrm{Dr}_{i \in I} G_i$ will denote the direct product of groups $G_i$, $i \in I$. In [12] it was proved that if $G = \mathrm{Dr}_{i \in I} G_i$ and $G_i$, $i \in I$, are coprime then $L(G) \simeq \prod_{i \in I} L(G_i)$. We will denote by $\Pi(G)$ the set of all primes dividing the orders of elements from $G$.

Lemma 2.6. *Let $G$ be a locally finite strongly balanced group, $q \in \Pi(G)$, and let $Q$ be the subgroup generated by all $q$-elements of $G$. If $Q$ is not a $q$-group then there exists a prime $p > q$ such that*:

(i) *$Q$ contains the subgroup $P$ generated by all $p$-elements of $G$.*
(ii) *If $g \in G$ and $(o(g), pq) = 1$ then $g \in C_G(Q) \setminus Q$.*
(iii) *$Q$ is an indecomposable group.*
(iv) *$Q$ is an ESB-group.*

*Proof.* Assume that $Q$ is not a $q$-group. Then there exist $q$-elements $y_1, y_2 \in Q$ such that $\langle y_1, y_2 \rangle$ is not a $q$-group. Since $\langle y_1, y_2 \rangle$ is finite and strongly balanced, there exists $p \in \Pi(G)$ with $p > q$ such that $\langle y_1, y_2 \rangle$ is an ESB-group of order $p^i q^j$, $i, j \geq 1$, by Theorem 2.1.

(i) Let $x \in G$ be any $p$-element. Then the subgroup $H = \langle x, y_1, y_2 \rangle$ is also a finite ESB-group of order $p^k q^l$, where $i \leq k$, $j \leq l$. Since $x$ was arbitrarily chosen, we see from Theorem 2.1 that any $p$-element of $G$ is contained in a subgroup generated by $q$-elements. Hence the subgroup $P$ generated by all $p$-elements of $G$ is a subgroup of $Q$.

(ii) Let $y_3, y_4 \in Q$ be any $q$-elements. The subgroup $K = \langle x, y_1, y_2, y_3, y_4 \rangle$ is finite and $H \leq K$ so that $K$ is an ESB-group of order $p^s q^t$, where $k \leq s$, $l \leq t$, by Theorem 2.1. Hence, by the choice of $y_3, y_4$, only $p, q \in \Pi(Q)$.

Let $r \in \Pi(G), r \notin \{p, q\}$, and $g \in G$ be any $r$-element. Now we consider the subgroup $\langle K, g \rangle$. Since $\langle K, g \rangle$ is finite and strongly balanced, $g \in C_G(K)$, from Theorem 2.1. Again by the choice of $x, y_3, y_4$ we obtain $g \notin Q$ and $g \in C_G(Q)$.

(iii) Assume that $Q$ is decomposable, i.e. there exist subgroups $Q_1$, $Q_2$ such that $Q = Q_1 \times Q_2$. Let $\overline{Q}_1$ be any finite subgroup of $Q_1$ and $\overline{Q}_2$ be any finite subgroup of $Q_2$. We consider the subgroup $\overline{Q} = \overline{Q}_1 \times \overline{Q}_2$. Since $\overline{Q}$ is finite, $\overline{Q}$ is an abelian group or a $q$-group or $\overline{Q}_1, \overline{Q}_2$ are coprime, by Theorem 2.1. Hence any set of $q$-elements of $G$ generates a $q$-group so $Q$ is a $q$-group. This contradicts our assumption and therefore $Q$ is indecomposable.

(iv) Let $x_1 \in Q$ be any $p$-element. Then $M = \langle x, x_1, y_1, y_2, y_3, y_4 \rangle$ is a finite ESB-group of order $p^u q^w$, where $u$, $w$ are positive integers such that $i \leq u$, $j \leq w$. Since $x$, $x_1$, $y_3$, $y_4$ were arbitrarily chosen, any two $p$-elements of $Q$ commute, have prime orders and every subgroup of $P$ is normalized by all $q$-elements of $Q$. So $P$ is a normal, elementary abelian subgroup of $Q$.

From the proof of (i) we know that $[x, y_1] \neq 1$ for any $p$-element $x$. Hence $P \cap Z(Q) = \{e\}$. Let $H$ be an arbitrary finite subgroup of $Q$. Then $PH/P \simeq H/H \cap P$ is a finite subgroup of $Q/P$. Now $P \cap H$ is a Sylow $p$-subgroup of $H$ and thus, by Theorem 2.1, $H/H \cap P$ is a cyclic $q$-group; this implies that $Q/P$ is a locally cyclic $q$-group. Since $Q$ is not abelian, $PZ(Q)/P$ is a proper subgroup of $Q/P$. Hence $PZ(Q)/P$ is a finite cyclic $q$-group and further $Z(Q)$ is a finite cyclic $q$-group, as $Z(Q) \simeq PZ(Q)/P$.

Now we take a $q$-element $y \in M$ with maximal order and let $o(y) = q^m$. Then $M = (M \cap P)\langle y \rangle$ and there exist positive integers $k, r$ such that $m \geq k$ and $q^r \mid p-1$, $k^{q^r} \equiv 1 \pmod{p}$, because $M$ is an ESB-group (see [1]). Moreover for all $x \in M \cap P$, $x^y = x^k$ and $g^{q^r} \in Z(M)$ for any $q$-element $g \in M$. Let $s$ be the maximal integer such that $q^s \mid p-1$. Then, by the choice of $M$, for any $q$-element $g \in Q$ we have $g^{q^s} \in Z(Q)$. If we take $|Z(Q)| = q^t$ then the order of any $q$-element $g$ of $Q$ equals at most $q^{s+t}$.

The above considerations show that if $y$ is a $q$-element of $Q$ of maximal order then $\langle M, y \rangle = (\langle M, y \rangle \cap P)\langle y \rangle$, as $\langle M, y \rangle$ is an ESB-group. Thus $Q = P\langle y \rangle$. Since $y$ acts on $P$ as a power automorphism, $Q$ is an ESB-group. ∎

Now we are in a position to prove the main results of this section.

THEOREM 2.7. *Let $G$ be a locally finite group. Then $G$ is strongly balanced if and only if it is one of the following groups*:

(i) *a modular $p$-group*;
(ii) *an ESB-group*;
(iii) *a direct product of coprime groups given in* (i) *and* (ii).

*Proof.* The groups in (i) are certainly strongly balanced and the groups from (ii) are strongly balanced by Theorem 2.4. Hence, by Proposition 1.3, also the groups from (iii) are strongly balanced.

Conversely, let $G$ be a strongly balanced group. If $p \in \Pi(G)$ then $G_p$ will denote the subgroup generated by all $p$-elements of $G$. Since $G_p$ is a normal subgroup in $G$ for any $p \in \Pi(G)$, $G$ is the algebraic product of subgroups $G_p$.

Assume that for some $q \in \Pi(G)$, $G_q$ is not a $q$-group. Then by Lemma 2.6, $G_q$ contains an abelian subgroup $G_p$ for some $p \in \Pi(G)$ with $q < p$, and furthermore, $G_q$ is an ESB-group. Moreover if an element $e \neq g \in G$ is such that $(o(g), pq) = 1$ then $g \in C_G(G_q) \setminus G_q$. Thus $G_p$ is not a subgroup of $G_r$ for $r \in \Pi(G) \setminus \{p, q\}$.

Hence for any $p \in \Pi(G)$, $G_p$ is a $p$-group or an ESB-group and if $G_p \cap G_q = \{e\}$ then they are coprime. This means that $G$ is the direct product of some groups $G_p$ which are coprime. If $G_p$ is a $p$-group then it is modular, by Theorem 2.5, and if $G_p$ is not modular, then $G_p$ is an ESB-group by arguments used earlier in this proof. ∎

The next corollaries are consequences of Theorem 2.7 and Lemmas 2.5 and 2.6.

COROLLARY 2.8. *Let $G$ be a locally finite group. Then $G$ is strongly balanced if and only if every finite subgroup of $G$ is strongly balanced.*

COROLLARY 2.9. *All locally finite strongly balanced groups are metabelian.*

Theorem 2.10. *If $G$ is a locally finite and strongly balanced group then $G$ is strongly atomic.*

*Proof.* We noticed earlier that any periodic abelian group is strongly atomic. By Theorem 2.4 also any ESB-group is strongly atomic. Hence, we assume that $G$ is a nonabelian strongly balanced $p$-group. In this case from Lemma 2.5 one can easily deduce that $G$ satisfies the assumption of Lemma 2.3. Hence $G$ is strongly atomic.

Since the direct product of strongly atomic lattices is strongly atomic, the proof is completed by applying Theorem 2.7. ∎

The above result cannot be extended to all locally finite groups, even in a weaker form, because there exists a locally finite (hence atomic) group which is not strongly locally uniform ([8]).

**3. Strong uniform dimension of groups.** If $G$ is a group then, as in [9], the uniform dimension (resp. strong uniform dimension) of $L(G)$ will be called the *uniform dimension* (*resp. strong uniform dimension*) of $G$ and will be denoted by $\mathrm{u}(G)$ (resp. $\mathrm{u_s}(G)$).

In [9], both uniform dimensions were determined for all strongly balanced finite groups. From Theorem 2.10, every locally finite strongly balanced group is strongly locally uniform. Hence, by Theorem 1.2, the (strong) uniform dimension of such groups exists. Now we determine these dimensions for such groups.

As a consequence of Propositions 1.4, 1.5 and properties of strongly balanced groups we obtain the following facts:

Proposition 3.1. *Let $G$ be a locally finite and strongly balanced group.*

(a) *If $H \le G$ is a subgroup then* $\mathrm{u}(H) \le \mathrm{u}(G)$ *and* $\mathrm{u_s}(H) \le \mathrm{u_s}(G)$.
(b) *If $H \le G$ is a normal subgroup then* $\mathrm{u_s}(G/H) \le \mathrm{u_s}(G)$.

Proposition 3.2. *Let $G_i$, $i \in I$, be locally finite coprime groups and* $G = \mathrm{Dr}_{i\in I}\, G_i$.

(a) *If $G$ is balanced then* $\mathrm{u}(G) = \sum_{i\in I} \mathrm{u}(G_i)$.
(b) *If $G$ is strongly balanced then* $\mathrm{u_s}(G) = \sum_{i\in I} \mathrm{u_s}(G_i)$.

In view of Theorem 2.7 and Proposition 3.2 we only need to determine the uniform dimensions of modular $p$-groups and ESB-groups. We start with the case of abelian $p$-groups.

Lemma 3.3. *If $G$ is an abelian $p$-group then* $\mathrm{u}(G) = \mathrm{u_s}(G)$ *and both dimensions are equal to the rank of $G$.*

*Proof.* Let $G$ be an abelian $p$-group. Then the rank of $G$ is equal to the $p$-rank of $G$. On the other hand, cyclic $p$-groups are uniform. Hence the result follows by the arguments from [4, §16]. ∎

In the case of nonabelian $p$-groups or ESB-groups we can use the following lemma, which is an easy extension of the results from [9].

LEMMA 3.4. *Let $G$ be a torsion group. If there exists an abelian normal subgroup $P$ and an element $g$ of $G$ such that $G = P\langle g\rangle$ and $g$ induces a power automorphism on $P$ then* $\mathrm{u_s}(G) = \mathrm{u_s}(P) + 1$.

It can be checked, as in [9], that for the quaternion group $Q_8$ of order 8 we have $\mathrm{u_s}(Q_8) = 2$ and $\mathrm{u}(Q_8) = 1$. Hence Lemma 3.4 is not true for the uniform dimension. Moreover, $Q_8$ is the only indecomposable group $G$ such that $\mathrm{u_s}(G) \neq \mathrm{u}(G)$.

Summarizing the results of this section we obtain:

THEOREM 3.5. *Let $G$ be a locally finite and strongly balanced group. Then* $\mathrm{u_s}(G) = \mathrm{u}(G) + 1$ *if and only if $G$ has a direct factor which is the quaternion group of order* 8. *In any other case* $\mathrm{u_s}(G) = \mathrm{u}(G)$.

The theorem below shows that the finiteness of the strong uniform dimension gives some other finiteness conditions for strongly balanced locally finite groups.

THEOREM 3.6. *Let $G$ be a locally finite and strongly balanced group. Then the* (*strong*) *uniform dimension of $G$ is finite if and only if $G$ is a direct product of finitely many finite modular $p$-groups*, *finite ESB-groups and quasicyclic groups.*

*Proof.* Let $G$ be a strongly balanced group. If $G$ is a nonabelian infinite indecomposable group then $G$ is a nonabelian modular $p$-group or an ESB-group. Hence $G$ contains an infinite abelian $p$-subgroup $N$ of finite exponent. So, from Lemma 3.3, $\mathrm{u}(N) = \mathrm{u_s}(N) = r_p(N) = \infty$.

If $G$ is indecomposable and abelian, then $G$ is a $p$-group and the rank of $G$ is finite if and only if $G$ is a cyclic $p$-group or a quasicyclic group $C_{p^\infty}$, by [4].

In the general case, $\mathrm{u}(G)$ is finite if and only if $G$ has finitely many direct factors and the rank of each factor is finite. Thus $G$ is a direct product of finitely many finite modular $p$-groups, finite ESB-groups and quasicyclic groups. ■

REMARKS. The examples of torsion but not locally finite groups which have modular subgroup lattices are provided in [12]. They are Tarski groups, that is, infinite groups all of whose nontrivial subgroups have prime order. Such groups are strongly atomic and both their uniform dimensions are equal to 2.

In [3] it was proved that for every odd prime $p$ there exists an infinite simple group whose proper subgroups are cyclic $p$-groups. Moreover, it was shown that for every odd prime $p$ there exist continuum many nonisomorphic such groups with isomorphic subgroup lattices. All these groups are strongly balanced and strongly atomic. Among these groups there are ones with exponent greater than $p$. Such groups are nonmodular, but strongly atomic. The uniform dimensions of all these groups are 2.

The above examples show that investigation of strongly balanced but not locally finite groups requires methods which are completely different from the ones used in this paper.7

From [5] we know that all finite groups are nearly modular, but from [1, 9] we know that there exist finite groups which are not strongly balanced. On the other hand any nonmodular example of Deryabina mentioned above is strongly balanced but not nearly modular. Hence there is no inclusion between the class of strongly balanced and the class of nearly modular groups.

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Institute of Mathematics
University of Białystok
Akademicka 2
15-267 Białystok, Poland
E-mail: sakowicz@math.uwb.edu.pl

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