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GROUPS WITH METAMODULAR SUBGROUP LATTICE

BY

M. DE FALCO (Napoli), F. DE GIOVANNI (Napoli),
C. MUSELLA (Napoli) and R. SCHMIDT (Kiel)

Abstract. A group G is called *metamodular* if for each subgroup H of G either the subgroup lattice $\mathfrak{L}(H)$ is modular or H is a modular element of the lattice $\mathfrak{L}(G)$. Metamodular groups appear as the natural lattice analogues of groups in which every non-abelian subgroup is normal; these latter groups have been studied by Romalis and Sesekin, and here their results are extended to metamodular groups.

1. Introduction. A subgroup of a group G is called *modular* if it is a modular element of the lattice $\mathfrak{L}(G)$ of all subgroups of G. It is clear that every normal subgroup of a group is modular, but arbitrary modular subgroups need not be normal; thus modularity may be considered as a lattice generalization of normality. Lattices with modular elements are also called *modular*, and a group G is said to be an *M-group* if $\mathfrak{L}(G)$ is a modular lattice. Abelian groups and the so-called Tarski groups (i.e. infinite groups all of whose proper non-trivial subgroups have prime order) are obvious examples of M-groups. The structure of periodic M-groups has been completely described by K. Iwasawa [5], [6] and R. Schmidt [12].

A group G is called *metahamiltonian* if all its non-abelian subgroups are normal. The structure of groups with this property has been investigated by G. M. Romalis and N. F. Sesekin in a series of papers ([9], [10], [11]), where they proved that if G is a soluble metahamiltonian group, then the commutator subgroup G' of G is finite of prime power order and G has derived length at most 3.

We shall say that a lattice $\mathfrak{L}$ with 0 and 1 is *metamodular* if for each $a \in \mathfrak{L}$ either the interval $[a/0]$ is a modular lattice or a is a modular element of $\mathfrak{L}$. A group will be called a *metamodular group* if its subgroup lattice is metamodular. If φ is an isomorphism from the subgroup lattice of a group G onto the lattice of all subgroups of a group $\overline{G}$, and N is a normal subgroup of G, then the image N^φ of N is a modular element of the lattice $\mathfrak{L}(\overline{G})$; furthermore, φ maps every abelian subgroup of G to a subgroup of $\overline{G}$ having modular subgroup lattice. Thus every lattice-isomorphic image of a

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metahamiltonian group is a metamodular group, and the aim of this article is to provide a lattice analogue of the above-quoted result of Romalis and Sesekin. Recall that a group G is *locally graded* if every finitely generated non-trivial subgroup of G contains a proper subgroup of finite index. We will prove the following theorems.

THEOREM A. *Let G be a locally graded metamodular group. Then G is soluble with derived length at most* 5.

THEOREM B. *Let G be a periodic locally graded metamodular group. Then G contains a finite normal subgroup N such that the factor group G/N has modular subgroup lattice. Moreover, the subgroup G'' is finite of prime power order.*

In the above statement the assumption that the group is locally graded cannot be omitted, as the following example shows.

EXAMPLE. Let A and B be isomorphic Tarski p-groups; a result of Obraztsov [7] shows that A and B can be embedded in a periodic simple 2-generator group G in such a way that $A \cap B = \{1\}$ and every non-cyclic subgroup of G is contained either in a conjugate of A or in a conjugate of B. It follows that all proper subgroups of G have modular subgroup lattices; in particular, the simple group G is a metamodular group, but the lattice $\mathfrak{L}(G)$ is not modular.

It is not clear whether the bound for the derived length obtained in Theorem A is best possible; however, the symmetric group S_4 of degree 4 shows that finite groups with metamodular subgroup lattice need not be metabelian, and in Section 3 we will also construct a finite metamodular group with derived length 4.

Most of our notation is standard and can be found in [8]. We shall use the monograph [13] as a general reference for results on subgroup lattices.

2. Proof of Theorem A. Let $\mathfrak{L}$ be a metamodular lattice. If a is an element of $\mathfrak{L}$ such that the interval $[a/0]$ is not modular, it follows from the definition that every element of the interval $[1/a]$ is modular in $\mathfrak{L}$, so that in particular $[1/a]$ is a modular lattice. As a special case, we get the following result.

LEMMA 2.1. *Let G be a metamodular group, and let N be a normal subgroup of G. Then at least one of the groups N and G/N has modular subgroup lattice.*

A group G is called a *P^*-group* if it is the semidirect product of an abelian normal subgroup A of prime exponent by a cyclic group $\langle x\rangle$ of prime power order such that x induces on A a power automorphism of prime order (recall here that a *power automorphism* of a group G is an automorphism mapping every subgroup of G onto itself). It is easy to see that the subgroup lattice of any P^*-group is modular, and Iwasawa ([5], [6]) proved that a locally finite group has modular subgroup lattice if and only if it is a direct product

$$G = \operatorname*{Dr}_{i\in I} G_i,$$

where each G_i is either a P^*-group or a primary locally finite group with modular subgroup lattice, and elements of different factors have coprime orders; this direct decomposition will be called the *Iwasawa decomposition* of the M-group G. Recall also that a group G is said to be a *P-group* if either it is abelian of prime exponent or $G = \langle x\rangle \ltimes A$ is a P^*-group with the subgroup $\langle x\rangle$ of prime order.

It is well known that a special role among modular subgroups is played by permutable subgroups; here a subgroup H of a group G is said to be *permutable* if $HK = KH$ for every subgroup K of G. Recall also that a subgroup H of a periodic group G is said to be *P-embedded* in G if the following conditions are satisfied:

• $G/H_G = (\mathrm{Dr}_{i\in I}(S_i/H_G)) \times L/H_G$, where each S_i/H_G is a non-abelian P-group;

• in the above direct decomposition, elements from different factors have coprime orders;

• $H/H_G = (\mathrm{Dr}_{i\in I}(Q_i/H_G)) \times ((H \cap L)/H_G)$, where each Q_i/H_G is a non-normal Sylow subgroup of S_i/H_G;

• $H \cap L$ is a permutable subgroup of G.

All P-embedded subgroups are modular, and it can be proved that every modular subgroup of a locally finite group is either permutable or P-embedded (see [15, Theorem 3.2 and Theorem E]).

Lemma 2.2. *Let G be a finite metamodular group. Then G is soluble.*

Proof. It is clearly enough to prove the statement when G is a finite simple metamodular group. In this case G does not contain proper nontrivial modular subgroups (see [13, Theorem 5.3.1]), and hence every proper subgroup of G has modular subgroup lattice. It follows that all proper subgroups of G are supersoluble, so that G is a soluble group. ■

In order to prove that arbitrary locally graded metamodular groups are soluble with derived length at most 5 we need the following information on modular subgroups of non-periodic groups.

Lemma 2.3. *Let G be a group, and let H be a modular subgroup of G such that the factor group G/H^G is generated by elements of infinite order. Then H is normal in G.*

Proof. Let g be any element of G such that the coset gH^G has infinite order. Then

$$\langle H, g\rangle \cap H^G = \langle H, \langle g\rangle \cap H^G\rangle = H,$$

and so H is normal in $\langle H, g\rangle$. Since the group G is generated by its elements with infinite order modulo H^G, it follows that H is a normal subgroup of G. ■

Proof of Theorem A. Assume by contradiction that the statement is false, so that there exists a finitely generated locally graded metamodular group G with $G^{(5)} \neq \{1\}$. As locally graded M-groups are metabelian (see [13, Theorem 2.4.21]), the lattice $\mathfrak{L}(G^{(3)})$ is not modular, and so $G/G^{(3)}$ is an M-group by Lemma 2.1. It follows that $G'' = G^{(3)}$, so that G'' is a perfect group, and hence it has no finite non-trivial homomorphic images by Lemma 2.2. In particular, G'' is not finitely generated and G/G'' is not periodic, so that G can be generated by its elements of infinite order modulo G'' (see [13, Lemma 2.4.8]). Let H be any subgroup of G'' such that the lattice $\mathfrak{L}(H)$ is not modular. Then by Lemma 2.3 every subgroup of G'' containing H is normal in G. In particular, G''/H is a Dedekind group, so that $H = G''$ and hence every proper subgroup of G'' has modular subgroup lattice. On the other hand, the property of having modular subgroup lattice is local (see [3, Lemma 5.1]), so that G'' is an M-group and $G^{(4)} = \{1\}$. This contradiction completes the proof. ■

3. Proof of Theorem B. The first three lemmas of this section deal with the structure of the factor group G/H_G when H is a modular subgroup of a finite group G.

LEMMA 3.1. *Let G be a finite group, and let H be a Hall subgroup of G. If H is modular in G, then $G/H_G = H^G/H_G \times L/H_G$, where the factors have coprime orders and H^G/H_G is an M-group. Moreover, either H is normal in G or H^G/H_G is the direct product of non-abelian P-groups with pairwise coprime orders.*

Proof. Obviously we may suppose that the subgroup H is not normal in G. Then

$$G/H_G = S_1/H_G \times \ldots \times S_t/H_G \times L/H_G,$$

where the factors have pairwise coprime orders, every S_i/H_G is a non-abelian P-group, $H \cap L$ is a permutable subgroup of G,

$$H/H_G = Q_1/H_G \times \ldots \times Q_t/H_G \times (H \cap L)/H_G$$

and Q_i/H_G is a non-normal Sylow subgroup of S_i/H_G for each $i \leq t$ (see [13, Theorem 5.1.14]). As H is a Hall subgroup of G, it follows that the intersection $H \cap L$ is characteristic in L, so that it is normal in G and hence $H \cap L = H_G$. Therefore $H^G = \langle S_1, \ldots, S_t\rangle$ and the lemma is proved. ■

LEMMA 3.2. *Let G be a finite metamodular group, and let H be a Hall subgroup of G. If H is not an M-group, then the factor group G/H_G has modular subgroup lattice.*

Proof. The subgroup H is modular in G, and hence by Lemma 3.1 we have

$$G/H_G = H^G/H_G \times L/H_G,$$

where the factors have coprime orders and H^G/H_G has modular subgroup lattice. On the other hand, $L/H_G \simeq G/H^G$ is an M-group by Lemma 2.1, so that also the lattice $\mathfrak{L}(G/H_G)$ is modular. ∎

LEMMA 3.3. *Let G be a finite group, and let H be a modular subgroup of G such that the index $|G : H|$ is a power p^n of a prime number p. If H is not permutable in G, then the factor group G/H_G is a non-abelian P-group of order $p^n q$, where q is a prime number and $q < p$.*

Proof. Since H is not permutable in G, we have

$$G/H_G = S_1/H_G \times \ldots \times S_t/H_G \times L/H_G,$$

where the factors have pairwise coprime orders, every S_i/H_G is a non-abelian P-group and $H \cap L$ is permutable in G (see [13, Theorem 5.1.14]). On the other hand, the index $|G : H|$ is a power of a prime number, and hence in the above decomposition there is only one non-trivial factor. Therefore G/H_G is a P-group and $|G/H_G| = p^n q$ for some prime number $q < p$. ∎

We will now prove a series of lemmas, which will be used in order to show that the second commutator subgroup of a finite metamodular group has prime power order.

LEMMA 3.4. *Let G be a finite group of order $p^m q^n$, where p and q are prime numbers, and let P be a Sylow p-subgroup of G such that the normalizer $N_G(P)$ is a P^*-group. Then G'' is a q-group.*

Proof. Let Q be a Sylow q-subgroup of G, so that $G = PQ$ and $N_G(P) = P\langle b\rangle$, where P is elementary abelian and b is an element of Q inducing on P a power automorphism of order q. Then

$$[P \cap N_G(Q), b] \leq P \cap Q = \{1\},$$

so that

$$P \cap N_G(Q) \leq C_P(b) = \{1\}$$

and hence

$$N_G(Q) = Q(P \cap N_G(Q)) = Q.$$

It follows that the soluble group G has no normal subgroups of index p, and so it must contain a normal subgroup K such that G/K has order q. Clearly

$N_G(P)$ is not contained in K and

$$N_K(P) = N_G(P) \cap K = P(\langle b \rangle \cap K) = P \times \langle b^q \rangle,$$

so that K is p-nilpotent by Burnside's theorem. Then $Q \cap K$ is a normal subgroup of K and $K/Q \cap K$ is abelian, so that $G'' \leq K' \leq Q$ and G'' is a q-group. ∎

LEMMA 3.5. *Let G be a finite metamodular group, and let P be a Sylow subgroup of G. If P is normal in G, then either G'' is contained in P or $P \cap G'' = \{1\}$.*

Proof. Since P is normal in G, there exists a subgroup H of G such that $G = PH$ and $P \cap H = \{1\}$. If H has modular subgroup lattice, then $G/P \simeq H$ is metabelian and so $G'' \leq P$. Suppose now that H is not an M-group, so that H is a modular subgroup of G and Lemma 3.2 yields that the lattice $\mathfrak{L}(G/H_G)$ is modular. In this case G'' is contained in H, and hence $P \cap G'' = \{1\}$. ∎

LEMMA 3.6. *Let G be a finite metamodular group of order $p^m q^n$, where p and q are prime numbers with $p > q$, and let P be a Sylow p-subgroup of G. Then either P is normal in G or the normalizer $N_G(P)$ is a maximal subgroup of G.*

Proof. Assume by contradiction that P is not normal in G and there exists a subgroup H of G such that $N_G(P) < H < G$. Clearly H is not subnormal in G and the index $|G : H|$ is a power of q. As P is not normal in H, the lattice $\mathfrak{L}(H)$ is not modular, so that H is modular in G and G/H_G is a P-group by Lemma 3.3. As $p > q$, it follows that PH_G/H_G is the unique Sylow p-subgroup of G/H_G, a contradiction since H/H_G is core-free. ∎

LEMMA 3.7. *Let G be a finite metamodular group, and let p be the largest prime divisor of the order of G. If P is a Sylow p-subgroup of G and $|\pi(P^G)| = 2$, then the normalizer $N_G(P)$ has modular subgroup lattice.*

Proof. Assume by contradiction that $N_G(P)$ is not an M-group, so that it is a modular subgroup of G. Put $\pi(P^G) = \{p, q\}$, and let H be a q-complement of G containing P. Then $P = P^G \cap H$ is a normal subgroup of H, and so the index $|G : N_G(P)|$ is a power of q. Let $K = (N_G(P))_G$ be the core of $N_G(P)$ in G. Since $N_G(P)$ is a proper self-normalizing subgroup of G, it follows from Lemma 3.3 that G/K is a P-group of order $q^n r$, where r is a prime number and $r < q$. In particular, $P^G/P^G \cap K$ is a q-group and $P^G \cap K \leq N_G(P)$, so that $N_{P^G}(P)$ is subnormal in P^G. Therefore P is subnormal, and so even normal in G. This contradiction proves the lemma. ∎

LEMMA 3.8. *Let G be a finite metamodular group. Then the subgroup G'' has prime power order.*

Proof. Assume by contradiction that the lemma is false, and choose a counterexample G with minimal order. Since all finite M-groups are metabelian, it follows from Lemma 3.2 that every Sylow subgroup of G has modular subgroup lattice. Suppose that G contains a non-trivial normal Sylow subgroup S; then $S \cap G'' = \{1\}$ by Lemma 3.5, and so $G'' \simeq G''S/S$ has prime power order, a contradiction. Therefore G does not contain normal non-trivial Sylow subgroups. Let p be the largest prime divisor of the order of G, and let P be a Sylow p-subgroup of G. As G is soluble by Lemma 2.2, we may consider a Sylow basis Σ of G containing P, and there exists a Sylow q-subgroup Q of G such that $Q \in \Sigma$ and P is not normal in the subgroup $H = PQ$. Clearly the lattice $\mathfrak{L}(H)$ is not modular, and so H is a modular subgroup of G. It follows from Lemma 3.2 that G/H_G is an M-group, so that G'' is contained in H and hence it is a $\{p, q\}$-group. Moreover, PH_G/H_G is a normal subgroup of G/H_G, so that $P \leq H_G$ and H/H_G is a q-group. Thus either H is normal in G or H^G/H_G is a P-group of order $r^m q$, where r is a prime number and $p > r > q$.

Let U and V be the p-complement and the q-complement, respectively, in the Sylow system Σ^* of G associated to Σ. Assume that the lattice $\mathfrak{L}(U)$ is not modular. Then G/U_G is an M-group by Lemma 3.2 and G'' is contained in U, so that G'' is a q-group. This contradiction shows that U has modular subgroup lattice, and a similar argument proves that also V is an M-group. Put $W = U \cap V$, and let

$$W = S_1 \times \ldots \times S_t$$

be the Iwasawa decomposition of the M-group W. Suppose that S_i is a P^*-group for some $i \leq t$. Then S_i must occur as factor also in the Iwasawa decompositions of the groups U and V, so that S_i is a normal subgroup of $G = \langle U, V \rangle$; on the other hand, S_i contains a normal non-trivial Sylow subgroup P_i, and P_i is normal in G. This contradiction proves that every S_i has prime power order, and hence W is a nilpotent group. As $G = QV$, we have $U = QW$ and at most one of the Sylow subgroups of W can generate together with Q a P^*-group in the Iwasawa decomposition of U. Thus either U is nilpotent or $U = (QS_i) \times E_i$, where S_i is a Sylow subgroup of W, QS_i is a P^*-group and $W = S_i \times E_i$. Similarly we find that either V is nilpotent or $V = PS_j \times E_j$, where S_j is a Sylow subgroup of W, PS_j is a P^*-group and $W = S_j \times E_j$.

Assume by contradiction that the subgroup H is not normal in G, so that H^G/H_G is a P-group of order $r^m q$ with $r > q$. Let R be the Sylow r-subgroup of G in Σ. Clearly RQ is not nilpotent, and hence $RQ = QS_i$ is a P^*-group. It follows that Q is cyclic and $R = S_i$ is a normal subgroup of U. Thus R cannot be normalized by P, so that PR is not nilpotent and so $PR = PS_j$ is a P^*-group. Therefore $R = S_i = S_j$ is the unique Sylow subgroup of W which is not normal in G, so that $W = R$ and G is a $\{p,q,r\}$-group. As Q is cyclic and $p > r > q$, we deduce that G contains a normal q-complement (see [8, 10.1.9]), so that V is normal in G, and hence also P is a normal subgroup of G. This contradiction shows that H is normal in G, so that the normal closure P^G of P is a $\{p,q\}$-group and the normalizer $N_G(P)$ has modular subgroup lattice by Lemma 3.7.

Suppose now that V is not nilpotent, so that $V = PS_j \times E$ and PS_j is a P^*-group; in particular, P is normal in PS_j and PS_j is a direct factor in the Iwasawa decomposition of $N_G(P)$. Thus $N_H(P) = P \times Q_0$ with $Q_0 \leq Q$, and hence H has a normal p-complement by Burnside's theorem. It follows that Q is normal in H, and so even in G, a contradiction. Therefore the subgroup V is nilpotent. Since $G = UV$, every non-trivial Sylow subgroup of W is not normal in U. Assume that H is properly contained in G, i.e. that G is not a $\{p,q\}$-group. Thus $W \neq \{1\}$, and in particular the subgroup U is not nilpotent, so that

$$U = QS_i \times E_i = QS_i$$

is a P^*-group. Moreover, $Q = H \cap U$ is a normal subgroup of U, so that Q is abelian of exponent q and S_i is cyclic of order r^n, where r is a prime number. Suppose that the normalizer $N_G(Q)$ is an M-group; then

$$N_H(Q) = Q \times (P \cap N_H(Q)),$$

and it follows from Burnside's theorem that H is q-nilpotent, so that P is normal in H and hence even in G, which is not the case. Therefore $N_G(Q)$ is not an M-group, and so it is a modular subgroup of G. Since Q is normal in U, the index $|G : N_G(Q)|$ is a power of p, and Lemma 3.3 implies that $G/(N_G(Q))_G$ is a non-abelian P-group of order $p^k s$, where s is a prime number and $s < p$. On the other hand, G is a $\{p,q,r\}$-group and its $\{p,r\}$-subgroups are nilpotent, so that $s = q$ and G contains a normal subgroup N of index q. Since $G/H \simeq S_i$, it follows that $G/H \cap N$ is cyclic of order qr^n; but $U = QS_i$ is a Hall $\{q,r\}$-subgroup of G and it has no elements of order qr^n. This contradiction proves that $G = H = PQ$ is a $\{p,q\}$-group.

As the lattice $\mathfrak{L}(N_G(P))$ is modular, it follows now from Lemma 3.4 that $N_G(P)$ is not a P^*-group, so that $N_G(P)$ is nilpotent and hence $N_G(P) = P \times Q_0$, where Q_0 is a proper subgroup of Q. Let Q_1 be a subgroup of Q such that Q_0 is a maximal subgroup of Q_1. Since $N_G(P)$ is a maximal subgroup of G by Lemma 3.6, we see that Q_0 is normal in $G = \langle N_G(P), Q_1\rangle$. Let $N = (N_G(P))_G$ be the core of $N_G(P)$ in G, and let M/N be the unique minimal normal subgroup of the primitive soluble group G/N. Then $G = MN_G(P)$ and $M \cap N_G(P) = N$, and in particular

$$|M/N| = |G : N_G(P)| = q^k,$$

with $k \geq 2$ because P is not normal in G and $q < p$. Moreover, Q_0 lies in N, so that $N_G(P)/N$ is a p-group and Q is a proper subgroup of M; it follows that Q_0 is properly contained in N, and so $N = P_0 \times Q_0$, where $P_0 = P \cap N$ is a non-trivial normal subgroup of G. Since Q is not normal in G, the subgroup M is not nilpotent, and hence there exist (at least) q maximal subgroups $X_1, \dots, X_q$ of M containing N which are not nilpotent. Assume that X_i is a modular subgroup of G for some $i \leq q$; as $N < X_i < M$, we obtain

$$N_G(P) < \langle N_G(P), X_i \rangle < G,$$

which contradicts the maximality of $N_G(P)$ in G. This shows that the subgroups $X_1, \dots, X_q$ are not modular in G, so that they have modular subgroup lattices, and hence $X_1, \dots, X_q$ are P^*-groups. Therefore P_0 is elementary abelian and X_i induces on P_0 a group of power automorphisms for each $i \leq q$. It follows that also M induces on P_0 a group of power automorphisms, so that $M/C_M(P_0)$ is a cyclic non-trivial group. Thus $C_M(P_0)$ is a normal subgroup of G such that

$$N < C_M(P_0) < M,$$

which is impossible. ∎

We can now prove the main result of the paper. In our argument we need information on groups in which every subgroup has finite index in a modular subgroup; the structure of such groups has recently been investigated in [1] and [2].

Proof of Theorem B. The group G is soluble with derived length at most 5 by Theorem A. In order to prove that G contains a finite normal subgroup N such that G/N is an M-group, it can obviously be assumed that G is not an M-group, so that G contains a finite subgroup E such that the lattice $\mathfrak{L}(E)$ is not modular (see [3, Lemma 5.1]), and E is modular in G. Since every modular subgroup of a locally finite group is either permutable or P-embedded (see [13, Theorem 6.2.17]), we have

$$G/E_G = S/E_G \times L/E_G,$$

where S/E_G is an M-group, $L \cap E$ is a permutable subgroup of G and the set $\pi(S/E_G) \cap \pi(L/E_G)$ is empty. Let H be any subgroup of L containing E_G. Then $\langle H, E \rangle$ is a modular subgroup of G, and hence $\langle H, E \rangle \cap L$ is modular in L. On the other hand,

$$\langle H, E \rangle \cap L = \langle H, E \cap L \rangle = H(E \cap L)$$

and so the index $|\langle H, E \rangle \cap L : H|$ is finite. Therefore each subgroup of L/E_G has finite index in a modular element of $\mathfrak{L}(L/E_G)$ and there exists a finite normal subgroup N of L such that $E_G \leq N$ and the lattice $\mathfrak{L}(L/N)$ is modular (see [1]). Clearly N is a normal subgroup of G, and G/N is an M-group. In particular, G/N is metabelian, so that G'' is finite and there exists a finite subgroup G_0 of G such that $G_0'' = G''$. Therefore G'' has prime power order by Lemma 3.8, and the theorem is proved. ∎

Finally, it will now be proved that there exist finite metamodular groups with derived length 4. It is well known that the symmetric group S_4 of degree 4 has precisely two non-isomorphic representation groups (see [14]); one of them is $\mathrm{GL}(2,3)$ and the other is a group G of order 48 with just one subgroup Z of order 2. Then $G/Z \simeq S_4$ and G has derived length 4. Moreover, since G has only one subgroup of order 2, every subgroup of order 8 or 12 of G has modular subgroup lattice. Let M/Z be the normal subgroup of order 4 of G/Z, and let X be any subgroup of G such that the lattice $\mathfrak{L}(X)$ is not modular; then X contains M and so it is a modular subgroup of G because $G/M \simeq S_3$. Therefore the group G has metamodular subgroup lattice.

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Dipartimento di Matematica e Applicazioni
Università di Napoli
via Cintia
I-80126 Napoli, Italy
E-mail: defalco@matna2.dma.unina.it
degiova@matna2.dma.unina.it
musella@matna2.dma.unina.it

Mathematisches Seminar
Universität Kiel
Ludewig-Meyn-Str. 4
D-24098 Kiel, Germany
E-mail: schmidt@math.uni-kiel.de

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