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NONANALYTICITY OF SOLUTIONS TO $\partial_t u = \partial_x^2 u + u^2$

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Abstract. It is proved that the solution to the initial value problem $\partial_t u = \partial_x^2 u + u^2$, $u(0,x) = 1/(1+x^2)$, does not belong to the Gevrey class G^s in time for $0 \le s < 1$. The proof is based on an estimation of a double sum of products of binomial coefficients.

1. Introduction. We consider the characteristic Cauchy problem for the semilinear heat equation

$$\begin{aligned} \partial_t u &= \partial_x^2 u + u^2, \\ u(0,x) &= \varphi(x). \end{aligned} \tag{1}$$

The equation describes the heat flow with a temperature dependent source and was studied by many authors, mainly in the Sobolev space setting ([F], [P1], [P2], [KST], [W], [Ō], [AH]). Since the nonlinearity, $f(u) = u^2$, is locally Lipschitz in u, it follows from standard results that any nonnegative solution of (1) is in fact classical, but it exists only locally in time ([F], [P1], [P2]). Also the following smoothing effect is known. For any choice $\varphi \in L^2(\mathbb{R})$ there exists $T > 0$ such that the solution u of (1) has an analytic continuation to $\Omega = \{(te^{i\theta}, x+iy) : 0 < t < T,\ |\theta| < \alpha < \pi/2,\ x \in \mathbb{R},\ |y| < \sqrt{t}\}$ ([AH]). The above result does not guarantee the analyticity in time at $t = 0$ even if the initial data is analytic. Indeed, if φ is analytic then (1) has a unique formal solution

$$u(t,x) = \sum_{n=0}^{\infty} \varphi_n(x) t^n, \tag{2}$$

where φ_n are given by the recurrence relations

$$\begin{cases} \varphi_0 = \varphi, \\ \varphi_{n+1} = \dfrac{1}{n+1}\Big(\partial_x^2 \varphi_n + \displaystyle\sum_{i=0}^{n} \varphi_i \varphi_{n-i}\Big), \quad n \in \mathbb{N}_0. \end{cases} \tag{3}$$

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We shall prove that this formal solution belongs to the Gevrey class G^1, but it is divergent if φ does not extend to an entire function of exponential order 2 and has nonnegative Taylor coefficients. Recall here that S. Kovalevskaya already proved in [K] that the condition that φ is entire of exponential order at most 2 is necessary and sufficient for the existence of an analytic solution to the linear heat equation $\partial_t u = \partial_x^2 u, u(0,x) = \varphi(x)$ (see also [LMS]). So it seems that the condition of nonnegativity of Taylor coefficients of φ is not necessary for the divergence of formal solutions to (1). However, when trying to prove the divergence without this condition we encountered difficulties connected with the influence of the nonlinear part of the equation. We only managed to prove the divergence of the formal solution in the "simplest" case of $\varphi(x) = 1/(1+x^2)$. This function is analytic in a strip along $\mathbb{R}$ and it belongs to $H^\infty(\mathbb{R})$ (so it has a finite energy). Our main result reads as follows:

THEOREM 1. *Let* $\varphi(x) = c/(1+x^2)$ *with* $0 < c < 2$. *Then the formal solution* (2) *to the initial value problem* (1) *does not belong to the Gevrey class* G^s *in time for* $0 \le s < 1$. *Thus*, *the solution of* (1) *is not analytic in time at* $t = 0$.

The proof of the main theorem is based on the following lemma which seems to be of independent interest.

THE MAIN LEMMA. *For* $k, n \in \mathbb{N}_0$ *put*

$$C(k,n) = \sum_{i=0}^{n} \sum_{l=0}^{k} \binom{n}{i} \binom{2k}{2l} \Big/ \binom{2k+2n}{2l+2i}. \tag{4}$$

Then

$$C(k,n) \le \begin{cases} 2\frac{2}{5} & \text{for } n \in \mathbb{N}_0 \text{ if } k = 0,1, \\ k+1 & \text{for } n \in \mathbb{N}_0 \text{ if } k \ge 2. \end{cases} \tag{5}$$

Our method of the proof of the main theorem can also be applied to the Cauchy problem for the Korteweg–de Vries equation $\partial_t u = \partial_x^3 u + 2u\partial_x u$, $u(0,x) = 1/(1+x^2)$. However, the proof of the corresponding version of the main lemma is much more involved and will be published elsewhere ([Ł]).

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2. Gevrey estimates

DEFINITION 1. We say that the formal power series (2) is in the *Gevrey class $G^s(\Omega)$ in time*, $s \geq 0$, $\Omega \subset \mathbb{R}$, if for any compact set $K \subset\subset \Omega$ one can find $L < \infty$ such that

$$\sup_{n\in\mathbb{N}_0} \sup_{x\in K} \frac{|\varphi_n(x)|}{L^n (n!)^s} < \infty. \tag{6}$$

In the proof of Theorem 2 we shall need

LEMMA 1. *Let $\nu, \mu, m \in \mathbb{N}_0$. Then*

$$\sum_{k=0}^{m} \frac{(k+\nu)!}{k!} \frac{(m-k+\mu)!}{(m-k)!} = \frac{\nu!\mu!(m+\nu+\mu+1)!}{(\nu+\mu+1)!\, m!}. \tag{7}$$

Proof. The formula (7) is equivalent to

$$\sum_{k=0}^{m} \binom{k+\nu}{\nu}\binom{m-k+\mu}{\mu} = \binom{m+\nu+\mu+1}{\nu+\mu+1},$$

which can be proved by combinatorial tricks (see [PBM, Form. 4.2.5.36]).

THEOREM 2. *Let φ be analytic in $\Omega \subset \mathbb{R}$. Then the formal solution* (2) *to the initial value problem* (1) *belongs to $G^1(\Omega)$ in time.*

Proof. Let K be compact in Ω. Since $\varphi_0 = \varphi$ is analytic in Ω we can find $1 \leq C < \infty$ such that for $m \in \mathbb{N}_0$,

$$\sup_{x\in K} |\partial^m \varphi_0(x)| \leq C^{m+1} m!, \qquad \sup_{x\in K} |\partial^m \varphi_0^2(x)| \leq C^{m+2}(m+1)!. \tag{8}$$

We shall prove that for $n \in \mathbb{N}, m \in \mathbb{N}_0$,

$$\sup_{x\in K} |\partial^m \varphi_n(x)| \leq 2C^{m+2n+1} \frac{(m+2n)!}{n!}, \tag{9}$$

which implies (6) with $s = 1$ and $L = 4C^2$. For $n = 1$ we have $\varphi_1 = \partial^2\varphi_0 + \varphi_0^2$. Hence by (8) we get $\sup_{x\in K} |\partial^m \varphi_1(x)| \leq 2C^{m+3}(m+2)!$. To prove (9) for $n \geq 2$ note that the recurrence relations (3) imply

$$\varphi_n = \frac{1}{n!}\Big(\partial^{2n}\varphi_0 + \sum_{j=0}^{n-1} j!\partial^{2n-2j-2} \sum_{i=0}^{j} \varphi_i\varphi_{j-i}\Big), \quad n \in \mathbb{N}. \tag{10}$$

Next by the Leibniz rule, the inductive assumption and Lemma 1 we derive for $j \geq 1$,

$$\begin{aligned}\sup_{x\in K} |\partial^m(\varphi_0\varphi_j)(x)| &\leq \sum_{k=0}^{m} \binom{m}{k} C^{k+1}k!\, \frac{2}{j!}\, C^{m-k+2j+1}(m-k+2j)! \\ &\leq \frac{2}{j!(2j+1)}\, C^{m+2j+2}(m+2j+1)!,\end{aligned}$$

and for $j \geq 2, 1 \leq i \leq j-1$,

$$\sup_{x\in K} |\partial^m(\varphi_i\varphi_{j-i})(x)| \leq \sum_{k=0}^{m} \binom{m}{k} \frac{2}{i!}\, C^{k+2i+1}(k+2i)! \times \frac{2}{(j-i)!}\, C^{m-k+2j-2i+1}(m-k+2j-2i)!$$
$$\leq \frac{4}{i!(j-i)!}\, \frac{(2i)!(2j-2i)!}{(2j+1)!}\, C^{m+2j+2}(m+2j+1)!.$$

Hence by (10) and (8) we get

$$\sup_{x\in K} |\partial^m\varphi_n(x)| \leq \frac{1}{n!}\, C^{m+2n+1}(m+2n)! \times \left\{1 + \frac{1}{C(m+2n)}\left[1 + \sum_{j=1}^{n-1} \frac{4}{2j+1}\left(1 + \sum_{i=1}^{j-1} \binom{j}{i} \Big/ \binom{2j}{2i}\right)\right]\right\}$$
$$\leq \frac{2}{n!}\, C^{m+2n+1}(m+2n)!$$

since $C \geq 1$ and

$$\sum_{j=1}^{n-1} \frac{4}{2j+1}\left(1 + \sum_{i=1}^{j-1} \binom{j}{i} \Big/ \binom{2j}{2i}\right) \leq \sum_{j=1}^{n-1} \frac{4}{2j+1}\,(1+j-1) \leq 2n-2.$$

THEOREM 3. *Fix $\varrho \geq 2$. Let φ be analytic in $\Omega \subset \mathbb{R}$ and assume that at a point $\mathring{x} \in \Omega$ the Taylor coefficients of φ are nonnegative. If φ does not extend to an entire function of exponential order ϱ then the formal solution* (2) *of* (1) *does not belong to $G^s(\Omega)$ in time for any $0 \leq s \leq 1 - 2/\varrho$. In particular, it is divergent.*

Proof. Since φ_n are given by (10) the assumption about nonnegativity of the Taylor coefficients of φ implies that

$$\varphi_n(\mathring{x}) \geq \frac{1}{n!}\, \partial^{2n}\varphi(\mathring{x}). \tag{11}$$

Next the condition that φ is not an entire function of exponential order ϱ is equivalent to (see [B, Sec. 2.2])

$$\overline{\lim}\, \sqrt[n]{\partial^{2n}\varphi(\mathring{x})((2n)!)^{1/\varrho-1}} = \infty,$$

which together with (11) contradicts (6) for $s \leq 1 - 2/\varrho$.

REMARK 1. We conjecture that Theorem 3 remains true without the nonnegativity assumption. Also, if φ is an entire function of exponential order $\varrho < 2$ the solution (2) should be an entire function in t of exponential order $\varrho/(2-\varrho)$.

3. Auxiliary lemmas. In the proof of the Main Lemma we shall need a few lemmas.

LEMMA 2. *For $k, n \in \mathbb{N}_0$ put*

$$D(k,n) = \sum_{i=0}^{n} \binom{n}{i} \Big/ \binom{2k+2n}{k+2i}. \tag{12}$$

Then $D(k, n+1) \le D(k,n)$ for $n \ge 3$ if $k = 0$ and for $n \ge 0$ if $k \ge 1$.

Proof. For $0 \le i \le n$ put

$$D(k,n)(i) = \binom{n}{i} \Big/ \binom{2k+2n}{k+2i}.$$

Then for n even, $n \ge 2$, we have

$$D(k,n) = \sum_{i=0}^{n/2-1} 2D(k,n)(i) + D(k,n)(n/2),$$
$$D(k,n+1) = \sum_{i=0}^{n/2-1} 2D(k,n+1)(i) + 2D(k,n+1)(n/2),$$
$$D(k,n+2) = \sum_{i=0}^{n/2} 2D(k,n+2)(i) + D(k,n+2)(n/2+1).$$

So it is sufficient to show that for n even, $n \ge 2$, the following inequalities hold:

$$D(k,n)(i) \ge D(k,n+1)(i) \ge D(k,n+2)(i) \quad \text{for } 0 \le i \le n/2,\ k \in \mathbb{N}_0; \tag{13}$$

$$2D(k,n)(n/2-1) + D(k,n)(n/2) \ge 2D(k,n+1)(n/2-1) + 2D(k,n+1)(n/2) \quad \text{for } n \ge 2,\ k \in \mathbb{N}_0; \tag{14}$$

$$2D(k,n+1)(n/2) \ge 2D(k,n+2)(n/2) + D(k,n+2)(n/2+1) \quad \text{for } n \ge 2 \text{ if } k = 0 \text{ and for } n \ge 0 \text{ if } k \ge 1; \tag{15}$$

$$D(k,0) \ge D(k,1) \ge D(k,2) \quad \text{for } k \in \mathbb{N}. \tag{16}$$

Proof of (13). The first inequality in (13) means that

$$\binom{n}{i} \Big/ \binom{2k+2n}{k+2i} \ge \binom{n+1}{i} \Big/ \binom{2k+2n+2}{k+2i} \quad \text{for } i = 0, \ldots, n/2.$$

Expanding the binomial coefficients and cancelling similar factors we get

$$1 \ge \frac{(n+1)(2n+k-2i+1)(2n+k-2i+2)}{(n-i+1)(2n+2k+1)(2n+2k+2)}$$

or equivalently

$$\left(1-\frac{k+2i}{2n+2k+1}\right)\left(1-\frac{k+2i}{2n+2k+2}\right)\le 1-\frac{i}{n+1}\qquad\text{for } i=0,\dots,n/2.$$

Since the first factor is less than the second one, putting $n=2i+m$ it is sufficient to show that for $i,k,m\ge 0$,

$$\left(1-\frac{k+2i}{2m+2k+4i+2}\right)^2\le 1-\frac{i}{m+2i+1}$$

or equivalently

$$\frac{k+2i}{m+k+2i+1}-\frac{1}{4}\left(\frac{k+2i}{m+k+2i+1}\right)^2\ge\frac{i}{m+2i+1}.$$

Put

$$x=\frac{k+2i}{m+k+2i+1}.$$

Then $0\le x<1$ and $x-\frac{1}{4}x^2\ge x/2$. Since

$$x/2=\frac{k/2+i}{m+k+2i+1}\ge\frac{i}{m+2i+1}\qquad\text{for } i,k,m\ge 0$$

we are done.

The second inequality in (13) is proved in the same way.

Proof of (14). We have to prove that for $k\in\mathbb{N}_0$ and n even, $n\ge 2$,

$$\frac{2\binom{n}{n/2-1}}{\binom{2k+2n}{k+n-2}}+\frac{\binom{n}{n/2}}{\binom{2k+2n}{k+n}}\ge\frac{2\binom{n+1}{n/2-1}}{\binom{2k+2n+2}{k+n-2}}+\frac{2\binom{n+1}{n/2}}{\binom{2k+2n+2}{k+n}}.$$

Expanding the binomial coefficients and cancelling similar factors we get

$$\begin{aligned}\frac{(k+n+1)(k+n+2)}{n/2+1}&+\frac{(k+n-1)(k+n)}{n/2}\\&\ge\frac{(n+1)(k+n+2)(k+n+3)(k+n+4)}{(n/2+1)(n/2+2)(2k+2n+1)}\\&\quad+\frac{(n+1)(k+n-1)(k+n)(k+n+2)}{(n/2)(n/2+1)(2k+2n+1)},\end{aligned}$$

which is equivalent to

$$\begin{aligned}2n^5+(6k+9)n^4+(6k^2+34k-5)n^3+(2k^3+41k^2+33k-42)n^2\\+(16k^3+46k^2-14k)n+8k^3-16k^2-24k\ge 0.\end{aligned}$$

Clearly, the last inequality holds for $k\in\mathbb{N}_0$ if $n\ge 2$.

Proof of (15). We have to prove that

$$\frac{2\binom{n+1}{n/2}}{\binom{2k+2n+2}{k+n}}\ge\frac{2\binom{n+2}{n/2}}{\binom{2k+2n+4}{k+n}}+\frac{2\binom{n+2}{n/2+1}}{\binom{2k+2n+4}{k+n+2}}.$$

Expanding the binomial coefficients and cancelling similar factors we have to show

$$1 \geq \frac{(n+2)(k+n+3)(k+n+4)}{(n/2+2)(2k+2n+3)(2k+2n+4)} + \frac{(k+n+1)(k+n+2)}{(2k+2n+3)(2k+2n+4)},$$

which is equivalent to

$$n^3 + (2k+5)n^2 + (k^2+13k+2)n + 8k^2 + 16k - 8 \geq 0.$$

The last inequality holds for $n \geq 1$ if $k = 0$ and for $n \geq 0$ if $k \geq 1$.

Proof of (16). We have

$$D(k,0) = \frac{k!k!}{(2k)!}, \qquad D(k,1) = 2\,\frac{k!(k+2)!}{(2k+2)!},$$
$$D(k,2) = 2\,\frac{k!(k+4) + (k+2)!(k+2)!}{(2k+4)!}.$$

So the first inequality of (16) is equivalent to $\frac{k+2}{2k+1} \leq 1$, which holds for $k \geq 1$. The second inequality is equivalent to

$$\frac{(k+3)(k+4) + (k+1)(k+2)}{(2k+3)(2k+4)} \leq 1,$$

which is also true if $k \geq 1$.

LEMMA 3. *For $k, n \in \mathbb{N}_0$ let $C(k,n)$ be given by (4). Then*

$$C(k,n) = \sum_{i=0}^{\lfloor k/2 \rfloor} \alpha(k,i) D(2i, k+n-2i), \tag{17}$$

where the coefficients $\alpha(k,i)$ satisfy the recurrence relations

$$\alpha(k,i) = \begin{cases} 1 & \text{for } k \in \mathbb{N}_0,\ i = 0, \\ \binom{2k}{2i} - \sum_{j=0}^{i-1} \binom{k-2j}{i-j} \alpha(k,j) & \text{for } k \in \mathbb{N},\ 1 \leq i \leq \lfloor k/2 \rfloor. \end{cases} \tag{18}$$

Proof. Assuming that $\binom{n}{i} = 0$ if $i < 0$ or $i > n$ we shall prove inductively that for any $m \in \{0, 1, \ldots, \lfloor k/2 \rfloor, \lfloor k/2 \rfloor + 1\}$,

$$\begin{aligned} C(k,n) &= \sum_{i=0}^{m-1} \alpha(k,i) D(2i, k+n-2i) \\ &+ \sum_{j=m}^{k+n-m} \sum_{l=m}^{k-m} \left\{ \binom{2k}{2l} - \sum_{i=0}^{m-1} \alpha(k,i) \binom{k-2i}{l-i} \right\} \binom{n}{j-l} \Big/ \binom{2k+2n}{2j}. \end{aligned} \tag{19}$$

Note that if $m = \lfloor k/2 \rfloor + 1$ then the second summand reduces to zero and we get (17).

Clearly (19) holds trivially for $m = 0$. Now assume (19) for a fixed $m \in \{0, 1, \ldots, \lfloor k/2 \rfloor\}$. Since

$$\sum_{l=m}^{k-m} \binom{k-2m}{l-m}\binom{n}{j-l} = \sum_{l=0}^{k-2m} \binom{k-2m}{l}\binom{n}{j-l-m} = \binom{k+n-2m}{j-m}$$

and

$$\sum_{j=m}^{k+n-m} \frac{\binom{k+n-2m}{j-m}}{\binom{2k+2n}{2j}} = \sum_{j=0}^{k+n-2m} \frac{\binom{k+n-2m}{j}}{\binom{2k+2n}{2j+2m}} = D(2m, k+n-2m)$$

we derive

$$\sum_{j=m}^{k+n-m}\sum_{l=m}^{k-m}\left\{\binom{2k}{2l} - \sum_{i=0}^{m-1}\alpha(k,i)\binom{k-2i}{l-i}\right\}\binom{n}{j-l}\Big/\binom{2k+2n}{2j}$$

$$= \sum_{j=m}^{k+n-m}\sum_{l=m}^{k-m}\alpha(k,m)\binom{k-2m}{l-m}\binom{n}{j-l}\Big/\binom{2k+2n}{2j}$$

$$+ \sum_{j=m}^{k+n-m}\sum_{l=m}^{k-m}\left\{\binom{2k}{2l} - \sum_{i=0}^{m}\alpha(k,i)\binom{k-2i}{l-i}\right\}\binom{n}{j-l}\Big/\binom{2k+2n}{2j}$$

$$= \alpha(k,m)D(2m, k+n-2m)$$

$$+ \sum_{j=m+1}^{k+n-m-1}\sum_{l=m+1}^{k-m-1}\left\{\binom{2k}{2l} - \sum_{i=0}^{m}\alpha(k,i)\binom{k-2i}{l-i}\right\}\binom{n}{j-l}\Big/\binom{2k+2n}{2j}.$$

Clearly, this implies (19) with $m+1$ in place of m.

LEMMA 4. *Let $\alpha(k,i)$, $k, i \in \mathbb{N}_0$, satisfy the recurrence relations* (18). *Then*

$$\alpha(k,i) = 4^i\binom{k}{2i} \quad \textit{for } k \in \mathbb{N}_0,\ 0 \le i \le \lfloor k/2 \rfloor. \tag{20}$$

Proof. By (18) and (20) it is sufficient to show that for $i \ge 1$,

$$\binom{2k}{2i} = \sum_{j=0}^{i} 4^j \binom{k-2j}{i-j}\binom{k}{2j}.$$

To this end fix $i \in \mathbb{N}$ and observe that

$$\binom{2k}{2i} = \frac{k(k-1)\cdot\ldots\cdot(k-i+1)}{i!}\cdot\frac{(2k-1)(2k-3)\cdot\ldots\cdot(2k-2i+1)}{1\cdot 3\cdot\ldots\cdot(2i-1)}$$

is a polynomial in k of degree $2i$ with leading coefficient $2^{2i}/(2i)! = 2^i/(i!(2i-1)!!)$, vanishing at $k = 0, 1/2, 1, \ldots, i-1, i-1/2$. Next note that for $j = 1, \ldots, i$,

$$2^j \binom{k-2j}{i-j}\binom{k}{2j} = 2^j \cdot \frac{(k-i-j+1)\cdot \ldots \cdot (k-2j)}{(i-j)!} \cdot \frac{(k-2j+1)\cdot \ldots \cdot k}{(2j)!}$$
$$= \frac{(k-i+1)\cdot \ldots \cdot k}{i!} \cdot \frac{(i-j+1)\cdot \ldots \cdot i}{j!} \cdot \frac{(k-i-j+1)\cdot \ldots \cdot (k-i)}{(2j-1)!!}$$

vanishes at $k = 0, 1, \ldots, i-1$. So it is sufficient to show that

$$W(k) := 1 + \sum_{j=1}^{i} 2^j \cdot \frac{(i-j+1)\cdot \ldots \cdot i}{j!} \cdot \frac{(k-i-j+1)\cdot \ldots \cdot (k-i)}{(2j-1)!!}$$

is a polynomial of degree i with leading coefficient $2^i/(2i-1)!!$, vanishing for $k = 1/2, 3/2, \ldots, i-1/2$. Since the first two statements are clear we shall prove the third one. To this end put $k = i - 1/2 - m$ with $m = 0, \ldots, i-1$. Then

$$W(k) = 1 + \sum_{j=1}^{i} (-1)^j \binom{i}{j} \frac{(2m+1)(2m+3)\cdot \ldots \cdot (2m+2j-1)}{(2j-1)!!}$$
$$= 1 + \sum_{j=1}^{i} (-1)^j \binom{i}{j} \frac{(2j+1)(2j+3)\cdot \ldots \cdot (2j+2m-1)}{(2m-1)!!} = 0$$

since $\sum_{j=0}^{i} (-1)^i \binom{i}{j} P(j) = 0$ for any polynomial P of degree $\le i-1$.

4. Proof of the Main Lemma. Let $k = 0$. Then $C(0,n) = D(0,n)$. Since by Lemma 2, $D(0,n+1) \le D(0,n)$ for $n \ge 3$ and $D(0,0) = 1$, $D(0,1) = 2$, $D(0,2) = 2\frac{1}{3}$, $D(0,3) = 2\frac{2}{5}$ we get the conclusion. If $k = 1$ then

$$C(1,n) = 2 + \sum_{i=1}^{n} \binom{n}{i} \Big/ \binom{2n+2}{2i} + \sum_{i=1}^{n} \binom{n}{i-1} \Big/ \binom{2n+2}{2i} = C(0, n+1)$$

and we are reduced to the previous case. Now let $k \ge 2$. Then by Lemmas 3 and 4,

$$C(k,n) = D(0, k+n) + \sum_{i=1}^{\lfloor k/2 \rfloor} 4^i \binom{k}{2i} D(2i, k+n-2i).$$

Since by Lemma 2, $D(2i,n) \ge D(2i,n+1)$ for $n \ge 0$ if $i \ge 1$ and for $n \ge 3$ if $i = 0$, this implies that $C(k,n) \ge C(k,n+1)$ for $n \ge 0$ (for $k = 2, n = 0$ we have $C(2,0) = 3 > C(2,1) = 2\frac{14}{15}$). The final computation $C(k,0) = k+1$ completes the proof.

5. Proof of Theorem 1. Assuming that (2) is a formal power series solution of (1) we easily get the recurrence relations (3) for φ_n. Next we observe that φ_n can be written in the form

$$\varphi_n(x) = \frac{1}{n!}\sum_{k=0}^{\infty}(-1)^{n+k}A(n,2k)x^{2k}, \quad n \in \mathbb{N}_0, \tag{21}$$

where the coefficients $A(n,2k)$ satisfy

$$\begin{aligned} A(0,2k) &= c, \\ A(n+1,2k) &= (2k+1)(2k+2)A(n,2k+2) \\ &\qquad - \sum_{i=0}^{n}\binom{n}{i}\sum_{l=0}^{k}A(i,2l)A(n-i,2k-2l). \end{aligned} \tag{22}$$

Indeed, $\varphi_0(x) = \sum_{k=0}^{\infty}(-1)^k cx^{2k}$ and assuming (21) for a fixed $n \in \mathbb{N}_0$, by (3) we easily get the second formula in (22).

CLAIM. *We have*

$$\begin{aligned} c(2k+1)\cdot\ldots\cdot(2k+2n)[1-\varepsilon(1,k+n-1)-\ldots-\varepsilon(n,k)] \\ \le A(n,2k) \le c(2k+1)\cdot\ldots\cdot(2k+2n) \end{aligned} \tag{23}$$

with $\varepsilon(n,k)$, $n \in \mathbb{N}$, $k \in \mathbb{N}_0$, *defined by*

$$(2k+2n-1)(2k+2n)\varepsilon(n,k) = \begin{cases} 2.4c & \text{if } k = 0,1 \text{ and } n \ge 2, \\ c(k+1) & \text{if } k \ge 2 \text{ or } n = 1. \end{cases} \tag{24}$$

Furthermore, *the assumption* $0 < c < 2$ *implies*

$$\varepsilon(1,k+n-1) + \ldots + \varepsilon(n,k) < 1. \tag{25}$$

Proof. Note that to prove (25) it is sufficient to show that for $l \in \mathbb{N}$ and $c < 2$,

$$\varepsilon(l) := \varepsilon(1,l-1) + \ldots + \varepsilon(l,0) < 1.$$

We compute $\varepsilon(1) = c/2 < 1$ if $c < 2$; $\varepsilon(2) = c\frac{2+2.4}{3\cdot 4} < 1$ if $c < \frac{30}{11}$. Finally, for $l \ge 3$ we get

$$\varepsilon(l) = c\,\frac{l+(l-1)+\ldots+3+2.4+2.4}{(2l-1)(2l)} = c\,\frac{l(l+1)+3.6}{4l(2l-1)} < 1 \quad \text{if } c < \frac{50}{13}.$$

To prove (23) observe that for $n = 1$ we have

$$A(1,2k) = c(2k+1)(2k+2) - c^2(k+1) = c(2k+1)(2k+2)[1-\varepsilon(1,k)].$$

Now assume (23) for a fixed $n \in \mathbb{N}_0$. Since in (22) we subtract a positive term (by (25)) we get

$$A(n+1,2k) \le (2k+1)(2k+2)A(n,2k+2) \le c(2k+1)\cdot\ldots\cdot(2k+2n+2).$$

To estimate $A(n+1,2k)$ from below observe that by the inductive assumption, the Main Lemma and the definition of $\varepsilon(n+1,k)$ we have

$$\begin{aligned}\sum_{i=0}^{n}\binom{n}{i}\sum_{l=0}^{k}&A(i,2l)A(n-i,2k-2l)\\ &\le c^2\sum_{i=0}^{n}\sum_{l=0}^{k}\binom{n}{i}\frac{(2l+2i)!}{(2l)!}\,\frac{(2k+2n-2l-2i)!}{(2k-2l)!}\\ &= c^2\,\frac{(2k+2n)!}{(2k)!}\sum_{i=0}^{n}\sum_{l=0}^{k}\binom{n}{i}\binom{2k}{2l}\Big/\binom{2k+2n}{2l+2i}\\ &\le c(2k+1)\cdot\ldots\cdot(2k+2n+2)\varepsilon(n+1,k).\end{aligned}$$

So

$$A(n+1,2k)\ge c(2k+1)\cdot\ldots\cdot(2k+2n+2)[1-\varepsilon(1,k+n)-\ldots-\varepsilon(n+1,k)].$$

Now, to finish the proof of Theorem 1 take $K=\{0\}$ in Definition 1. Then

$$|\varphi_n(0)|=\frac{1}{n!}\,A(n,0)\ge c\,\frac{(2n)!}{n!}\,[1-\varepsilon(1,n-1)-\ldots-\varepsilon(n,0)],$$

which by (25) contradicts (6) for $0\le s<1$ if $0<c<2$.

6. Final remarks

REMARK 2. By the method presented in the paper one can also prove that the solution to the Cauchy problem

$$\partial_t u=\Delta u+u^2,$$
$$u(0,x)=\frac{c}{1+x^2},\qquad x\in\mathbb{R}^d,\ d\ge 2,$$

does not belong to $G^s(\mathbb{R}^d)$ in time for any $0\le s<1$ and c small enough. In this case the coefficient $C(k,n)$ in the Main Lemma takes the form

$$C(k,n)=\begin{cases}\displaystyle\sum_{i=0}^{n}\sum_{l=0}^{k}\binom{n}{i}\binom{k}{l}^2\Big/\binom{k+n}{l+i}^2 & \text{if } d=2,\\ \displaystyle\sum_{i=0}^{n}\sum_{l=0}^{k}\binom{n}{i}\binom{2k+2}{2l+1}\Big/\binom{2k+2n+2}{2l+2i+1} & \text{if } d=3,\end{cases}$$

and one can prove that $C(k,n)$ is bounded by $k+3$.

REMARK 3. Our method can also be applied to the Cauchy problem for the KdV equation

$$\partial_t u=\partial_x^3 u+2u\cdot\partial_x u,\qquad u(0,x)=\frac{c}{1+x^2}. \tag{26}$$

Namely, we can prove ([Ł]) that if $0<c<5$ the solution to (26) does not belong to $G^s(\mathbb{R})$ in time for $0\le s<2$. The counterpart of $C(k,n)$ is

$$C(k,n) = \begin{cases} \displaystyle\sum_{i=0}^{n} \sum_{l=0}^{k+1-i \bmod 2} \binom{n}{i} \binom{2k+2}{2l+i \bmod 2} \Big/ \binom{2k+3n+2}{2l+3i+i \bmod 2} & \text{if } n \text{ is even,} \\ \displaystyle\sum_{i=0}^{n} \sum_{l=0}^{k} \binom{n}{i} \binom{2k+1}{2l+i \bmod 2} \Big/ \binom{2k+3n+1}{2l+3i+i \bmod 2} & \text{if } n \text{ is odd,} \end{cases}$$

and is bounded by 3 if $k=0$ and by $k+2$ if $k \ge 1$.

Added in proof. In summer 2002 P. Byers and A. Himonas constructed, by another method, a nonanalytic solution to the KdV equation for a globally analytic initial data.

REFERENCES

[AH] H. Aikawa and N. Hayashi, *Holomorphic solutions of semilinear heat equations*, Complex Variables 16 (1991), 115–125.

[B] R. P. Boas, *Entire Functions*, Academic Press, New York, 1954.

[F] H. Fujita, *On the blowing up of solutions of the Cauchy problem for* $u_t = \Delta u + u^{1+\alpha}$, J. Fac. Sci. Univ. Tokyo Sect. 1 13 (1966), 109–124.

[KST] K. Kobayashi, T. Sirao and H. Tanaka, *On the growing up problem for semilinear heat equations*, J. Math. Soc. Japan 29 (1977), 407–424.

[K] S. Kowalevski, *Zur Theorie der partiellen Differentialgleichungen*, J. Reine Angew. Math. 80 (1875), 1–32.

[LMS] D. A. Lutz, M. Miyake and R. Schäfke, *On the Borel summability of divergent solutions of the heat equation*, Nagoya Math. J. 154 (1999), 1–29.

[Ł] G. Łysik, *Nonanalyticity in time of solutions to the Korteweg–de Vries equation*, in preparation.

[Ō] S. Ōuchi, *On the analyticity in time of solutions of initial boundary value problems for semi-linear parabolic differential equations with monotone nonlinearity*, J. Fac. Sci. Univ. Tokyo Ser. 1A 21 (1974), 19–41.

[P1] R. G. Pinsky, *Existence and nonexistence of global solutions for* $u_t = \Delta u + a(x)u^p$ *in* $\mathbb{R}^d$, J. Differential Equations 133 (1997), 152–177.

[P2] —, *The behavior of the life span for solutions to* $u_t = \Delta u + a(x)u^p$ *in* $\mathbb{R}^d$, ibid. 147 (1998), 30–57.

[PBM] A. P. Prudnikov, Yu. A. Brychkov and O. I. Marichev, *Integrals and Series. Elementary Functions*, Nauka, Moscow, 1981 (in Russian); English transl.: Gordon and Breach, 1986.

[W] F. B. Weissler, *Existence and nonexistence of global solutions for a semilinear heat equation*, Israel J. Math. 38 (1981), 29–40.

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