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ESTIMATES OF GREEN FUNCTIONS AND THEIR APPLICATIONS FOR PARABOLIC OPERATORS WITH SINGULAR POTENTIALS

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Abstract. We prove global pointwise estimates for the Green function of a parabolic operator with potential in the parabolic Kato class on a $C^{1,1}$ cylindrical domain Ω. We apply these estimates to obtain a new and shorter proof of the Harnack inequality [16], and to study the boundary behavior of nonnegative solutions.

Introduction. The problem of bounding Green functions and its applications to study elliptic and parabolic equations have received much attention by several authors in different situations. In the elliptic setting, it was shown by Hueber [8] that the Green function G_L of an elliptic operator

$$L=\sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2}{\partial x_i\partial x_j}+\sum_{i=1}^{n} b_i(x)\frac{\partial}{\partial x_i}+c(x)$$

with bounded Hölder continuous coefficients, $c\le 0$ and real, symmetric, uniformly elliptic matrix $(a_{ij})_{i,j}$ on a $C^{1,1}$ bounded domain $D\subset\mathbb{R}^n$, $n\ge 3$, satisfies the following pointwise estimates:

$$k^{-1}\,\frac{\varphi(x,y)}{|x-y|^{n-2}}\le G_L(x,y)\le k\,\frac{\varphi(x,y)}{|x-y|^{n-2}}$$

for all $x,y\in D$, where $\varphi(x,y)=\min\bigl(1,\frac{d(x)d(y)}{|x-y|^2}\bigr),\ d(x)=d(x,\partial\Omega)$.

For the Green function G_Δ of the Laplace operator, these estimates are due to Widman [13], Grüter and Widman [7], and Zhao [19]. As a simple consequence, G_L and G_Δ are comparable in the following sense: $k^{-1}G_\Delta\le G_L\le kG_\Delta$ for some constant $k>0$. This comparison result first proved by Hueber and Sieveking [9] enabled them to prove the equivalence of the L-harmonic measure and the Δ-harmonic measure on the boundary of D. These comparisons are very important in the sense that they allow the transfer to L of potential-theoretic results valid for Δ. In [19], Zhao studied

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the Schrödinger operator $L = \frac{1}{2}\Delta + q$, with potential q in the elliptic Kato class $K_n^{\mathrm{loc}}, n \geq 3$, i.e. q satisfies

$$\lim_{\alpha \to 0} \sup_{x \in D} \int_{|x-y| \leq \alpha} \frac{|q(y)|}{|x-y|^{n-2}}\, dy = 0,$$

on a $C^{1,1}$ bounded domain D. Assuming that (D, q) satisfies the gauge condition: $\sup\bigl[\mathrm{spec}\bigl(\frac{1}{2}\Delta + q\bigr)/D\bigr] < 0$, he proved the comparability of the Green functions G_L and G_Δ. This allowed him to prove the existence of the Poisson kernel of the Dirichlet problem corresponding to the Schrödinger operator. These results were later extended by Cranston, Fabes and Zhao [3] to the general Schrödinger operator $L = \mathcal{A} + q$ where $\mathcal{A} = -\mathrm{div}(A(x)\nabla_x)$ and $q \in K_n^{\mathrm{loc}}$, $n \geq 3$, and so L and $\mathcal{A}$ have the same potential theory.

In the parabolic setting, it is well known that the fundamental solution satisfies the Gaussian estimates in different situations (see Aronson [1, 2], Fabes and Stroock [6], and Zhang [14–17]). In particular, Zhang proved these estimates for parabolic operators with lower order terms in some parabolic Kato classes. The parabolic Kato class is a natural generalization of the elliptic Kato class, and it is considered to be the biggest possible space so that the Gaussian bounds for the fundamental solution hold. The Gaussian estimates are used to study nonnegative solutions of the corresponding parabolic equations. In the half-space the analogous estimates were proved in [11], and used to study parabolic potentials. In [10], Hui studied the heat equation and proved that the Green function of a smooth cylindrical domain satisfies an upper Gaussian estimate involving the distance to the boundary. More importantly, lower and upper estimates were proved in [12] for the Green function of the operator $L = \partial/\partial t - \mathrm{div}(A(x,t)\nabla_x) + B(x,t)\nabla_x$ with B in the parabolic Kato class K_{n+1} on a $C^{1,1}$ cylindrical domain; they were then used to establish the comparability results for Green functions and harmonic measures extending their elliptic counterparts initially proved by Cranston and Zhao [4] for $\frac{1}{2}\Delta + b(x)\nabla_x$. In contrast to the elliptic case, nothing is proved about the boundary behavior of the Green function for parabolic operators with singular potentials, and the existence of such estimates in this case remains unknown.

The main goal of the present paper is to investigate this problem for the parabolic operator

$$\mathcal{L} = \frac{\partial}{\partial t} - \mathrm{div}(A(x,t)\nabla_x) + V(x,t)$$

on $\Omega = D \times]0, T[$, where D is a $C^{1,1}$ bounded domain in $\mathbb{R}^n$, $n \geq 1$, and $0 < T < \infty$. The matrix A is assumed to be real, symmetric, uniformly elliptic, i.e. $(1/\mu)I \leq A(x,t) \leq \mu I$ for some $\mu \geq 1$, with μ-Lipschitz coefficients, and V in the parabolic Kato class K_n as introduced by Zhang [16, 17], i.e.

$V \in L^1_{\mathrm{loc}}(\Omega)$ and $\lim_{h\to 0} N_h^\alpha(V) = 0$, where

$$N_h^\alpha(V) \equiv \sup_{x,t} \int_{t-h}^{t} \int_D |V(z,\tau)| \, \frac{1}{(t-\tau)^{n/2}} \exp\left(-\alpha \, \frac{|x-z|^2}{t-\tau}\right) dz \, d\tau$$
$$+ \sup_{y,s} \int_{s}^{s+h} \int_D |V(z,\tau)| \, \frac{1}{(\tau-s)^{n/2}} \exp\left(-\alpha \, \frac{|z-y|^2}{\tau-s}\right) dz \, d\tau$$

for all $\alpha > 0$. The existence and uniqueness of the $\mathcal{L}$-Green function G on Ω were shown in [16, 17].

Before describing the main body of our paper we recall the following estimates of the Green function G_0 of Ω of the unperturbed operator $\mathcal{L}_0 = \partial/\partial t - \operatorname{div}(A(x,t)\nabla_x)$.

Theorem I ([12]). *There exist constants $k, c, c' > 0$, depending only on n, μ, D and T, such that*

$$k^{-1}\psi(x,y,t-s) \, \frac{\exp\left(-c' \frac{|x-y|^2}{t-s}\right)}{(t-s)^{n/2}} \le G_0(x,t,y,s)$$
$$\le k\psi(x,y,t-s) \, \frac{\exp\left(-c \frac{|x-y|^2}{t-s}\right)}{(t-s)^{n/2}}$$

for all $(x,t), (y,s) \in \Omega$ with $s < t$, where

$$\psi(x,y,u) = \min\left(1, \frac{d(x)}{\sqrt{u}}, \frac{d(y)}{\sqrt{u}}, \frac{d(x)d(y)}{u}\right)$$

and $d(x) = d(x, \partial D)$ denotes the distance from x to the boundary of D.

In Section 1, we prove that the $\mathcal{L}$-Green function G satisfies the estimates of Theorem I with constants depending on V only in terms of the rate of convergence of $N_h^\alpha(V)$ to zero as $h \to 0$. Our idea is based on the resolvent equation $G = G_0 - G * (VG_0)$ and Theorem I. The control of the term $G * (VG_0)$ constitutes a real difficulty in the proof. Apart from being interesting in themselves, these estimates reveal the behavior of the Green function of the perturbed operator $\mathcal{L}$ especially near the boundary. Moreover, they simplify proofs of certain known results which were initially obtained by means of involved analytical calculations or considerations based on boundary Harnack principles (see [16, 17], and [12] for a bibliography).

In Section 2, we give some applications of the Green function estimates. We first present an alternative and shorter proof of the Harnack inequality recently established by Zhang [16], using the idea of Fabes and Stroock [6]. Our method is new and can be applied to other operators. We next prove a boundary Harnack principle and a comparison theorem for nonnegative $\mathcal{L}$-solutions vanishing on a part of the lateral boundary. These results, first proved for elliptic and less general parabolic operators, are the main regularity properties of nonnegative solutions which are used to study the potential theory of the corresponding operators; for instance, to study the Martin boundary, to prove the so called doubling property of harmonic measures, etc. (we refer the reader to [5], [3] and the references given there). Another important application of the Green function estimates concerns the equivalence of the $\mathcal{L}$-parabolic measure, $\mathcal{L}^*$-parabolic measure and surface measure on the lateral boundary of Ω, which is stated at the end of the paper. The proof of this result follows the idea developed in [12]; therefore it is omitted.

We need to recall some known results. For an open subset Ω of $\mathbb{R}^{n+1}$, we denote by $\partial_{\mathrm{p}}\Omega$ the parabolic boundary of Ω, i.e. $\partial_{\mathrm{p}}\Omega$ is the set of points on the boundary of Ω which can be connected to some interior point of Ω by a closed curve having a strictly increasing t-coordinate. We have

THEOREM II (Minimum principle). *Let Ω be a bounded open set in $\mathbb{R}^{n+1}$ and u an $\mathcal{L}$-superparabolic function in Ω satisfying $\liminf_{z\to z_0} u(z) \ge 0$ for all $z_0 \in \partial_{\mathrm{p}}\Omega$. Then $u \ge 0$.*

The $\mathcal{L}$-Green function G of Ω has the *reproducing property*:

$$G(x,t,y,s) = \int_D G(x,t,\xi,\tau)G(\xi,\tau,y,s)\,d\xi$$

for all $x, y \in D$ and $s < \tau < t$ ([16]).

1. Estimates for the $\mathcal{L}$-Green function. In this section we prove the following main result.

THEOREM 1.1. *Let V be in the parabolic Kato class. Then there exist constants $k, c_1, c_2 > 0$, depending on n, μ, T, D, and on V only in terms of the rate of convergence of $N_h^\alpha(V)$ to zero as $h \to 0$, such that*

$$k^{-1}\psi(x,y,t-s)\,\frac{\exp\left(-c_2\frac{|x-y|^2}{t-s}\right)}{(t-s)^{n/2}} \le G(x,t,y,s) \le k\psi(x,y,t-s)\,\frac{\exp\left(-c_1\frac{|x-y|^2}{t-s}\right)}{(t-s)^{n/2}}$$

for all $(x,t),(y,s) \in \Omega$ with $s < t$, where $\psi(x,y,u) = \min(1, d(x)/\sqrt{u}, d(y)/\sqrt{u}, d(x)d(y)/u)$.

Proof. We can assume that $V \in L^\infty$; the general case is covered by a limiting argument in Lemma 6.3 of [16].

We first prove the upper bound. By the minimum principle, we have

$$G \leq e^{T\|V\|_\infty} G_0.$$

In view of Theorem I and the previous inequality, let k_0 be the least positive number such that

$$G(x,t,y,0) \leq k_0 \min\left(1, \frac{d(x)}{\sqrt{t}}\right) \min\left(1, \frac{d(y)}{\sqrt{t}}\right) \frac{\exp\left(-\frac{c}{2}\frac{|x-y|^2}{t}\right)}{t^{n/2}} \tag{1}$$

for all $x, y \in D$ and $0 < t \leq h$ for some fixed h. Our aim is to prove that k_0 depends on V only in terms of the quantity $N_h^\alpha(V)$.

From [16] we know the integral equation

$$G(x,t,y,s) = G_0(x,t,y,s) - \int_s^t \int_D G(x,t,z,\tau) V(z,\tau) G_0(z,\tau,y,s)\, dz\, d\tau \tag{2}$$

for all $x, y \in D$ and $0 < t \leq T$. From (1), (2) and Theorem I, it follows that

$$\begin{aligned} |G(x,t,y,0) - G_0(x,t,y,0)| \leq k k_0 \int_0^t \int_D \omega(z,\tau)\, \frac{\exp\left(-\frac{c}{2}\frac{|x-z|^2}{t-\tau}\right)}{(t-\tau)^{n/2}}\, |V(z,\tau)| \\ \times \frac{\exp\left(-c\frac{|z-y|^2}{\tau}\right)}{\tau^{n/2}}\, dz\, d\tau, \end{aligned} \tag{3}$$

where

$$\omega(z,\tau) = \min\left(1, \frac{d(x)}{\sqrt{t-\tau}}\right) \min\left(1, \frac{d(z)}{\sqrt{t-\tau}}\right) \min\left(1, \frac{d(z)}{\sqrt{\tau}}\right) \min\left(1, \frac{d(y)}{\sqrt{\tau}}\right).$$

For simplicity we write

$$J(x,t,y,0) = \int_0^t \int_D \omega(z,\tau) \frac{\exp\left(-\frac{c}{2}\frac{|x-z|^2}{t-\tau}\right)}{(t-\tau)^{n/2}}\, |V(z,\tau)|\, \frac{\exp\left(-c\frac{|z-y|^2}{\tau}\right)}{\tau^{n/2}}\, dz\, d\tau.$$

We will estimate J. For $\varrho \in \,]0,1[$ to be chosen later, we have

$$J(x,t,y,0) = \left(\int_0^{\varrho t} + \int_{\varrho t}^t \right) \int_D \ldots\, dz\, d\tau \equiv J_1 + J_2.$$

We first estimate J_1. To this end let us recall the inequality

$$\frac{|x-z|^2}{t-\tau} + \frac{|z-y|^2}{\tau} \geq \frac{|x-y|^2}{t}, \qquad \forall \tau \in \,]0,t[. \tag{4}$$

Then we have

$$J_1(x,t,y,0) = \int_0^{\varrho t}\int_D \omega(z,\tau)\, \frac{\exp\left(-\frac{c}{2}\frac{|x-z|^2}{t-\tau}\right)}{(t-\tau)^{n/2}}\, |V(z,\tau)|\, \frac{\exp\left(-c\frac{|z-y|^2}{\tau}\right)}{\tau^{n/2}}\, dz\, d\tau \tag{5}$$

$$= \int_0^{\varrho t}\int_D \omega(z,\tau)\, \frac{\exp\left(-\frac{c}{2}\left[\frac{|x-z|^2}{t-\tau} + \frac{|z-y|^2}{\tau}\right]\right)}{(t-\tau)^{n/2}}\, |V(z,\tau)|\, \frac{\exp\left(-\frac{c}{2}\frac{|z-y|^2}{\tau}\right)}{\tau^{n/2}}\, dz\, d\tau$$

$$\leq \frac{\exp\left(-\frac{c}{2}\frac{|x-y|^2}{t}\right)}{((1-\varrho)t)^{n/2}} \int_0^{\varrho t}\int_D \omega(z,\tau)|V(z,\tau)|\, \frac{\exp\left(-\frac{c}{2}\frac{|z-y|^2}{\tau}\right)}{\tau^{n/2}}\, dz\, d\tau.$$

On the other hand,

$$\int_0^{\varrho t}\int_D \omega(z,\tau)|V(z,\tau)| \frac{\exp\left(-\frac{c}{2}\frac{|z-y|^2}{\tau}\right)}{\tau^{n/2}}\, dz\, d\tau \tag{6}$$

$$= \int_0^{\varrho t}\Bigg(\int_{d(z)\leq 2d(y)} + \int_{d(z)\geq 2d(y)} \Bigg) \dots\, dz\, d\tau \equiv J_{11} + J_{12}.$$

If $d(z) \leq 2d(y)$ and $\tau \in \,]0, \varrho t[$, then

$$\omega(z,\tau) \leq \frac{2}{1-\varrho} \min\left(1, \frac{d(x)}{\sqrt{t}}\right) \min\left(1, \frac{d(y)}{\sqrt{t}}\right).$$

If $d(z) \geq 2d(y)$ and $\tau \in \,]0, \varrho t[$, then

$$\omega(z,\tau) \leq \frac{1}{1-\varrho} \min\left(1, \frac{d(x)}{\sqrt{t}}\right) \min\left(1, \frac{d(y)}{\sqrt{t}}\right) \frac{d(z)}{\sqrt{\tau}}$$

$$\leq \frac{2}{1-\varrho} \min\left(1, \frac{d(x)}{\sqrt{t}}\right) \min\left(1, \frac{d(y)}{\sqrt{t}}\right) \frac{|z-y|}{\sqrt{\tau}}.$$

It follows that

$$J_{11} \leq \frac{2}{1-\varrho} \min\left(1, \frac{d(x)}{\sqrt{t}}\right) \min\left(1, \frac{d(y)}{\sqrt{t}}\right) N_h^{c/2}(V) \tag{7}$$

for $0 < t \leq h$, and

$$J_{12} \leq \frac{2}{1-\varrho} \min\left(1, \frac{d(x)}{\sqrt{t}}\right) \min\left(1, \frac{d(y)}{\sqrt{t}}\right) \tag{8}$$

$$\times \int_0^{\varrho t} \int_{d(z)\geq 2d(y)} \frac{|z-y|}{\sqrt{\tau}}\, |V(z,\tau)|\, \frac{\exp\left(-\frac{c}{2}\frac{|z-y|^2}{\tau}\right)}{\tau^{n/2}}\, dz\, d\tau$$

$$\leq \frac{2}{(1-\varrho)\sqrt{c}} \min\left(1, \frac{d(x)}{\sqrt{t}}\right) \min\left(1, \frac{d(y)}{\sqrt{t}}\right)$$

$$\times \int_0^{\varrho t} \int_{d(z)\geq 2d(y)} |V(z,\tau)|\, \frac{\exp\left(-\frac{c}{4}\frac{|z-y|^2}{\tau}\right)}{\tau^{n/2}}\, dz\, d\tau$$

$$\leq \frac{2}{(1-\varrho)\sqrt{c}} \min\left(1, \frac{d(x)}{\sqrt{t}}\right) \min\left(1, \frac{d(y)}{\sqrt{t}}\right) N_h^{c/4}(V)$$

for $0 < t \le h$. Combining (5)–(8), we obtain

$$(9)\quad J_1(x,t,y,0) \le \frac{4c^{-1/2}}{(1-\varrho)^{n/2+1}} N_h^{c/4}(V) \min\Big(1, \frac{d(x)}{\sqrt{t}}\Big) \min\Big(1, \frac{d(y)}{\sqrt{t}}\Big) \frac{\exp\big(-\frac{c}{2}\frac{|x-y|^2}{t}\big)}{t^{n/2}}$$

for all $x, y \in D$ and $0 < t \le h$.

We next estimate J_2. We have

$$(10)\quad J_2(x,t,y,0) = \int_{\varrho t}^{t}\int_D \omega(z,\tau)\frac{\exp\big(-\frac{c}{2}\frac{|x-z|^2}{t-\tau}\big)}{(t-\tau)^{n/2}} |V(z,\tau)| \frac{\exp\big(-c\frac{|z-y|^2}{\tau}\big)}{\tau^{n/2}}\,dz\,d\tau$$
$$= \int_{\varrho t}^{t}\Big(\int_{|z-y|\ge\frac{1}{\sqrt{2}}|x-y|} + \int_{|z-y|\le\frac{1}{\sqrt{2}}|x-y|}\Big)\ldots\,dz\,d\tau \equiv J_{21}+J_{22}.$$

If $|z-y| \ge \frac{1}{\sqrt{2}}|x-y|$ and $\tau \in\,]\varrho t, t[$, then

$$\frac{\exp\big(-c\frac{|z-y|^2}{\tau}\big)}{\tau^{n/2}} \le \frac{\exp\big(-\frac{c}{2}\frac{|x-y|^2}{t}\big)}{(\varrho t)^{n/2}}.$$

Therefore

$$(11)\quad J_{21} \le \frac{\exp\big(-\frac{c}{2}\frac{|x-y|^2}{t}\big)}{(\varrho t)^{n/2}}\int_{\varrho t}^{t}\int_D \omega(z,\tau)\frac{\exp\big(-\frac{c}{2}\frac{|x-z|^2}{t-\tau}\big)}{(t-\tau)^{n/2}}|V(z,\tau)|\,dz\,d\tau.$$

Now we estimate J_{22}. We have

$$(12)\quad J_{22} = \int_{\varrho t}^{t}\int_{|z-y|\le\frac{1}{\sqrt{2}}|x-y|} \omega(z,\tau)\frac{\exp\big(-\frac{c}{2}\frac{|x-z|^2}{t-\tau}\big)}{(t-\tau)^{n/2}} |V(z,\tau)| \frac{\exp\big(-c\frac{|z-y|^2}{\tau}\big)}{\tau^{n/2}}\,dz\,d\tau$$
$$\le (\varrho t)^{-n/2}\int_{\varrho t}^{t}\int_{|z-y|\le\frac{1}{\sqrt{2}}|x-y|} \omega(z,\tau)\frac{\exp\big(-\frac{c}{2}\frac{|x-z|^2}{t-\tau}\big)}{(t-\tau)^{n/2}}|V(z,\tau)|\,dz\,d\tau.$$

If $|z-y| \le \frac{1}{\sqrt{2}}|x-y|$, then $|x-z| \ge |x-y| - |z-y| \ge \big(1-\frac{1}{\sqrt{2}}\big)|x-y|$. Hence

$$\exp\Big(-\frac{c}{2}\frac{|x-z|^2}{t-\tau}\Big) \le \exp\Big(-\frac{c}{4}\frac{|x-z|^2}{t-\tau}\Big)\exp\Big(-\frac{c}{4}\frac{|x-y|^2}{t-\tau}\Big(1-\frac{1}{\sqrt{2}}\Big)^2\Big)$$
$$\le \exp\Big(-\frac{c}{4}\frac{|x-z|^2}{t-\tau}\Big)\exp\Big(-\frac{c}{4}\frac{|x-y|^2}{(1-\varrho)t}\Big(1-\frac{1}{\sqrt{2}}\Big)^2\Big).$$

Now taking ϱ so that $(1-1/\sqrt{2})^2/(2(1-\varrho)) = 1$, we obtain

$$(13)\quad \exp\Big(-\frac{c}{2}\frac{|x-z|^2}{t-\tau}\Big) \le \exp\Big(-\frac{c}{4}\frac{|x-z|^2}{t-\tau}\Big)\exp\Big(-\frac{c}{2}\frac{|x-y|^2}{t}\Big).$$

Combining (12) and (13) yields

$$J_{22} \leq \frac{\exp\left(-\frac{c}{2}\frac{|x-y|^2}{t}\right)}{(\varrho t)^{n/2}} \int_{\varrho t}^{t}\int_{D} \omega(z,\tau)\,\frac{\exp\left(-\frac{c}{4}\frac{|x-z|^2}{t-\tau}\right)}{(t-\tau)^{n/2}}\,|V(z,\tau)|\,dz\,d\tau. \tag{14}$$

From (10), (11) and (14), we have

$$\begin{aligned} &J_2(x,t,y,0) \\ &\quad\leq 2\,\frac{\exp\left(-\frac{c}{2}\frac{|x-y|^2}{t}\right)}{(\varrho t)^{n/2}} \int_{\varrho t}^{t}\int_{D} \omega(z,\tau)\,\frac{\exp\left(-\frac{c}{4}\frac{|x-z|^2}{t-\tau}\right)}{(t-\tau)^{n/2}}\,|V(z,\tau)|\,dz\,d\tau. \end{aligned} \tag{15}$$

Note that (15) is similar to the inequality (5) for J_1. Then by the same method used to prove (9), we obtain

$$\begin{aligned} &J_2(x,t,y,0) \\ &\quad\leq \frac{8c^{-1/2}}{\varrho^{n/2+1}} N_h^{c/8}(V)\min\left(1,\frac{d(x)}{\sqrt{t}}\right)\min\left(1,\frac{d(y)}{\sqrt{t}}\right)\frac{\exp\left(-\frac{c}{2}\frac{|x-y|^2}{t}\right)}{t^{n/2}} \end{aligned} \tag{16}$$

for all $x, y \in D$ and $0 < t \leq h$.

Combining (9), (16) and the fact that $(1-1/\sqrt{2})^2/(2(1-\varrho)) = 1$, we get

$$J(x,t,y,0) \leq k' N_h^{c/8}(V)\min\left(1,\frac{d(x)}{\sqrt{t}}\right)\min\left(1,\frac{d(y)}{\sqrt{t}}\right)\frac{\exp\left(-\frac{c}{2}\frac{|x-y|^2}{t}\right)}{t^{n/2}} \tag{17}$$

for all $x, y \in D$ and $0 < t \leq h$. Substituting (17) to (3) gives

$$\begin{aligned} G(x,t,y,0) &\leq k(1+k_0 k' N_h^{c/8}(V))\min\left(1,\frac{d(x)}{\sqrt{t}}\right)\min\left(1,\frac{d(y)}{\sqrt{t}}\right) \\ &\qquad\times \frac{\exp\left(-\frac{c}{2}\frac{|x-y|^2}{t}\right)}{t^{n/2}} \end{aligned}$$

for all $x, y \in D$ and $0 < t \leq h$. Hence, by definition of k_0, we obtain

$$k_0 \leq k + k k_0 k' N_h^{c/8}(V).$$

Choosing h sufficiently small so that $kk'N_h^{c/8}(V) < 1/2$, we have $k_0 \leq 2k$. This completes the proof of the upper bound.

We next prove the lower bound. From (3) and (17), we have

$$\begin{aligned} &|G(x,t,y,0) - G_0(x,t,y,0)| \\ &\quad\leq 2k^2 k' N_h^{c/8}(V)\min\left(1,\frac{d(x)}{\sqrt{t}}\right)\min\left(1,\frac{d(y)}{\sqrt{t}}\right)\frac{\exp\left(-\frac{c}{2}\frac{|x-y|^2}{t}\right)}{t^{n/2}} \end{aligned}$$

for all $x, y \in D$ and $0 < t \le h$. Hence, by Theorem I, we deduce that

$$G(x,t,y,0) \ge \min\Big(1, \frac{d(x)}{\sqrt{t}}\Big) \min\Big(1, \frac{d(y)}{\sqrt{t}}\Big) \times \bigg[\frac{1}{k}\,\frac{\exp\big(-c'\frac{|x-y|^2}{t}\big)}{t^{n/2}} - 2k^2k'N_h^{c/8}(V)\,\frac{1}{t^{n/2}}\bigg]$$

for all $x, y \in D$ and $0 < t \le h$. Then, for h so small that $2k^2k'N_h^{c/8}(V) \le e^{-c'}/(2k)$, we obtain

$$G(x,t,y,0) \ge \frac{e^{-c'}}{2k} \min\Big(1, \frac{d(x)}{\sqrt{t}}\Big) \min\Big(1, \frac{d(y)}{\sqrt{t}}\Big) \frac{1}{t^{n/2}} \tag{18}$$

for all $x, y \in \overline{D}$ and $0 < t \le h$ with $|x-y|^2/t \le 1$.

An inequality like (18) together with the reproducing property of the Green function, and a geometric property of $C^{1,1}$ domains, imply the required lower estimate. For all details we refer the reader to [12]. ∎

By Theorem 1.1 and a simple argument given in [12], we prove the following comparison result for Green functions.

COROLLARY 1.2. *Let V be in the parabolic Kato class. Then there exist positive constants k, c_3 and c_4, depending only on n, μ, T, D, and on V in terms of the rate of convergence of $N_h^\alpha(V)$ to zero as $h \to 0$, such that*

$$k^{-1}G_{c_3} \le G \le kG_{c_4}$$

on Ω, where G_{c_i} is the Green function of $\partial/\partial t - c_i\Delta_x$ on Ω.

REMARKS 1.3. 1. The constants c_3 and c_4 are independent of T, in view of the comparison on $D \times]0,1[$ and the reproducing property.

2. In general the estimates in Theorem 1.1 are not global in time. This is clear from the following simple example. Consider $L = \partial/\partial t - \Delta_x + c$, where c is a positive constant, and $\Omega = B(0,1) \times]0,\infty[$. Let G denote the L-Green function of Ω. The L-fundamental solution is given by

$$\Gamma(x,t,y,0) = \frac{1}{(4\pi t)^{n/2}} \exp\Big(-\frac{|x-y|^2}{4t}\Big) \exp(-ct)$$

for all x, $y \in \mathbb{R}^n$ and $t > 0$. Suppose there is a global lower bound. Then

$$k^{-1} \min\Big(1, \frac{d(x)}{\sqrt{t}}, \frac{d(y)}{\sqrt{t}}, \frac{d(x)d(y)}{t}\Big) \frac{\exp\big(-c_2\frac{|x-y|^2}{t}\big)}{t^{n/2}} \le \Gamma(x,t,y,0)$$

for all $t > 0$ and $x, y \in B(0,1)$. If we choose $x = y$, then we get

$$\min(1, d(x)/\sqrt{t}, d^2(x)/t) \le k\exp(-ct)$$

for all $t > 0$, which is a contradiction.

3. The estimates in Theorem 1.1 may fail to hold when the domain D is only Lipschitz. The following example illustrates this point. Let $D = \{x = (x_1, x_2) \in \mathbb{R}^2 : x_1 > 0, x_2 > 0\}$ and fix $y \in D$ with $|y| > 1$. Put $U = D \cap B(0,1)$. Consider the parabolic operators

$$L_1 = \frac{\partial}{\partial t} - \Delta_x, \quad L_2 = \frac{\partial}{\partial t} - \left(\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_1 \partial x_2} + \frac{\partial^2}{\partial x_2^2}\right)$$

and let $u_1(x) = x_1 x_2$, $u_2(x) = x_1^2 x_2 + x_1 x_2^2$. The function u_i is a positive L_i-solution on $\Omega = D \times]0,T[$ for $i = 1,2$. Let G_i denote the L_i-Green function of Ω for $i = 1,2$. From the comparison theorem (Theorem 1.6 in [5]), it follows that there exist two positive constants k_1 and k_2 such that, for $t \in]0,T[$ fixed,

$$k_1^{-1} \le \frac{u_1(x)}{G_1(x,t,y,0)} \le k_1 \quad \text{for all } x \in U,$$
$$k_2^{-1} \le \frac{u_2(x)}{G_2(x,t,y,0)} \le k_2 \quad \text{for all } x \in U.$$

This shows that

$$k^{-1} \le \frac{u_1(x) G_2(x,t,y,0)}{u_2(x) G_1(x,t,y,0)} \le k \quad \text{for all } x \in U.$$

Suppose that the estimates of Theorem 1.1 are true. Then

$$k_3^{-1} \exp\left(-c\,\frac{|x-y|^2}{t}\right) \le \frac{G_1(x,t,y,0)}{G_2(x,t,y,0)} \le k_3 \exp\left(c\,\frac{|x-y|^2}{t}\right) \quad \text{for all } x \in D.$$

The previous two-sided inequalities now imply that u_1/u_2 is bounded near zero, which is not true.

2. Applications. In this section we give some applications of the Green function bounds established in Section 1. A new and shorter proof of the Harnack inequality [16] is given. A boundary Harnack principle and a comparison theorem for nonnegative $\mathcal{L}$-solutions which continuously vanish on a part of the lateral boundary are proved.

2.1. *The Harnack inequality.* By means of the Green function estimates and a potential-theoretic argument we present an alternative and shorter proof of the Harnack inequality [16] for nonnegative $\mathcal{L}$-solutions. Our idea is new and can be applied to other similar operators.

THEOREM 2.1. *Let $0 < \alpha_2 < \beta_2 < \alpha_1 < \beta_1 < 1$, $\delta \in]0,1[$ and $R_0 > 0$ be given. Then there exists $k > 0$ such that for all $(x,s) \in \mathbb{R}^n \times \mathbb{R}$, all $R \in]0,R_0]$ and all nonnegative $\mathcal{L}$-solutions u in $B(x,R) \times [s-R^2, s]$, we have*

$$\sup_{Q^-} u \le k \inf_{Q^+} u$$

where $Q^- = B(x,\delta R) \times [s-\beta_1 R^2, s-\alpha_1 R^2]$ *and* $Q^+ = B(x,\delta R) \times [s-\beta_2 R^2, s-\alpha_2 R^2]$. *The constant* k *depends only on* n, μ, *the parameters* α_i, β_i, R_0, *and on* V *in terms of the rate of convergence of* $N_h^\alpha(V)$ *to zero as* $h \to 0$.

Proof. We may assume $(x,s)=(0,0)$. For $R \in]0,R_0]$ put $\Omega = B(0,R)\times]-R^2,0[$ and let G be the $\mathcal{L}$-Green function of Ω. Let Σ be the open set $B(0,\delta' R) \times]-\beta' R^2, -\alpha' R^2[$, where $\alpha' = \alpha_2/2$, $\beta' = (1+\beta_1)/2$ and $\delta' = (1+\delta)/2$. We denote by R_u^Σ the nonnegative $\mathcal{L}$-superparabolic envelope of u with respect to Σ, which is also called the "reduct" of u with respect to Σ, and defined by

$$R_u^\Sigma = \inf\{v : v \text{ a nonnegative } \mathcal{L}\text{-supersolution on } \Omega \text{ with } v \geq u \text{ on } \Sigma\}.$$

Then R_u^Σ is an $\mathcal{L}$-potential on Ω which is harmonic on $\Omega \setminus \partial\Sigma$, so there exists a positive measure supported in $\partial\Sigma$ such that

$$R_u^\Sigma(x,t) = \int_{\partial\Sigma} G(x,t,y,s)\,d\mu(y,s) \quad \text{for all } (x,t)\in\Omega.$$

For $(x,t)\in Q^-$, we have, by Theorem 1.1,

$$\begin{aligned} u(x,t) = R_u^\Sigma(x,t) &= \int_{\partial\Sigma\cap]-R^2,-\alpha_1 R^2[} G(x,t,y,s)\,d\mu(y,s) \\ &\leq k \int_{\partial\Sigma\cap]-R^2,-\alpha_1 R^2[} \frac{\exp\left(-c_1\frac{|x-y|^2}{t-s}\right)}{(t-s)^{n/2}}\,d\mu(y,s). \end{aligned} \tag{19}$$

Note that

$$\begin{aligned} &\partial\Sigma \cap]-R^2,-\alpha_1 R^2[\\ &\qquad = (\partial B(0,\delta' R)\times]-\beta' R^2, -\alpha_1 R^2[) \cup (\overline{B}(0,\delta' R)\times\{-\beta' R^2\}). \end{aligned}$$

If $(y,s)\in \partial B(0,\delta' R)\times]-\beta' R^2,-\alpha_1 R^2[$, then

$$\frac{\exp\left(-c_1\frac{|x-y|^2}{t-s}\right)}{(t-s)^{n/2}} \leq \frac{C}{|x-y|^n} \leq \frac{k'}{R^n}.$$

If $(y,s)\in \overline{B}(0,\delta' R)\times\{-\beta' R^2\}$, then

$$\frac{\exp\left(-c_1\frac{|x-y|^2}{t-s}\right)}{(t-s)^{n/2}} \leq \frac{1}{(t-s)^{n/2}} \leq \frac{k''}{R^n}.$$

Therefore, from (19) it follows that

$$\sup_{Q^-} u \leq \frac{C}{R^n}\,\mu(\partial\Sigma\cap]-R^2,-\alpha_1 R^2]). \tag{20}$$

For $(x,t) \in Q^+$, we have, by Theorem 1.1,

$$u(x,t) = R_u^{\Sigma}(x,t) \geq \int\limits_{\partial\Sigma\cap\,]-R^2,-\alpha_1 R^2]} G(x,t,y,s)d\mu(y,s) \tag{21}$$
$$\geq \frac{1}{k} \int\limits_{\partial\Sigma\cap\,]-R^2,-\alpha_1 R^2]} \psi(x,y,t-s)\, \frac{\exp\left(-c_2\frac{|x-y|^2}{t-s}\right)}{(t-s)^{n/2}}\, d\mu(y,s).$$

On the other hand, for $(y,s) \in \partial\Sigma\cap\,]-R^2,-\alpha_1 R^2]$ and $(x,t) \in Q^+$, we have

$$(\alpha_1 - \beta_2)R^2 \leq t - s \leq (\beta' - \alpha_2)R^2,$$
$$|x-y|^2 \leq 4R^2, \quad d(y) \geq (1-\delta')R, \quad d(x) \geq (1-\delta)R.$$

Combining the last inequalities with (21), we get

$$\inf_{Q^+} u \geq \frac{1}{CR^n}\, \mu(\partial\Sigma \cap\,]-R^2, -\alpha_1 R^2]). \tag{22}$$

From (20) and (22), we have

$$\sup_{Q^-} u \leq k \inf_{Q^+} u.$$

2.2. *Boundary behavior of nonnegative $\mathcal{L}$-solutions.* In this subsection we prove a boundary Harnack principle and a comparison theorem for nonnegative $\mathcal{L}$-solutions which continuously vanish on a part of the lateral boundary of Ω. The boundary Harnack principle provides a uniform bound for such solutions and the comparison theorem proves that two nonnegative $\mathcal{L}$-solutions continuously vanishing on a portion of the lateral boundary vanish at the same rate on a subportion. These results give precise information on the behavior of these solutions near the lateral boundary. For the unperturbed operator $\mathcal{L}_0$ these results were proved by Fabes, Garofalo and Salsa [5], and found their applications in the proofs of the doubling property of harmonic measure, the existence of kernel functions, etc. In the elliptic setting, these results have been proved in several situations. We refer the reader to [3], and the references given in [5] and [3]. Here we present new methods of proof based on the Green function estimates and potential-theoretic arguments which can be applied to other similar operators.

We first give some notations. For $(Q,s) \in \mathbb{R}^n \times \mathbb{R}$ and $r > 0$, we define the cylinder

$$T_r(Q,s) = \{(x,t) \in \mathbb{R}^n \times \mathbb{R} : |x-Q| < r,\ |t-s| < r^2\}.$$

For $(Q,s) \in \partial_{\mathrm{p}}\Omega$, we put $\Delta_r(Q,s) = \partial_{\mathrm{p}}\Omega \cap \overline{T}_r(Q,s)$. We know that there exists $r_0 > 0$ such that for each $Q \in \partial D$, there is a local coordinate system in which $\partial D \cap B(Q, r_0)$ is the graph of a $C^{1,1}$ function. When $Q \in \partial D$ is represented by $(x_0', \varphi(x_0'))$ in the local coordinate system, we set

$$M_r(Q,s) = (x_0', \varphi(x_0') + r, s + 2r^2),$$
$$M_r^*(Q,s) = (x_0', \varphi(x_0') + r, s - 2r^2).$$

We have the following result.

THEOREM 2.2.1 (Boundary Harnack principle). *Let* $(Q,s) \in \partial D \times]0,T[$ *and* $r \in]0, r_0 \wedge \sqrt{T-s}[$. *Then there exists a constant* $k > 0$, *depending only on* n, μ, T, D, *and on* V *in terms of the rate of convergence of* $N_h^\alpha(V)$ *to zero as* $h \to 0$, *such that for all nonnegative* $\mathcal{L}$*-solutions* u *on* $\Omega \setminus T_{r/2}(Q,s)$ *continuously vanishing on* $\partial_{\mathrm{p}}\Omega \setminus T_{r/2}(Q,s)$, *we have*

$$u(M) \le k u(M_r(Q,s)) \quad \text{for all } M \in \Omega \setminus T_r(Q,s).$$

Proof. Without loss of generality we assume $Q = 0$. We first prove the result for $u = G_A \equiv G(\cdot, A)$ with $A \in \Omega \cap T_{r/2}(0,s)$. We write

$$A = (0,s) + (y,\tau) \quad \text{with } |y| < r/2 \text{ and } |\tau| < r^2/4,$$
$$M = (0,s) + (x,t) \quad \text{with } |x| \ge r \text{ or } |t| \ge r^2,$$

and we put $M_r = M_r(0,s)$. When $t \le \tau$, we know $G_A(M) = 0$. In what follows we assume $t > \tau$.

By the Green function estimates (Theorem 1.1), we have

$$G_A(M) \le k \frac{d(y)}{(t-\tau)^{(n+1)/2}} \exp\left(-c_1 \frac{|x-y|^2}{t-\tau}\right).$$

If $|t| \ge r^2$, then

$$G_A(M) \le k \frac{d(y)}{\left(\frac{3}{4}r^2\right)^{(n+1)/2}} = k' \frac{d(y)}{r^{n+1}}. \tag{23}$$

If $|x| \ge r$, then

$$G_A(M) \le C \frac{d(y)}{|x-y|^{n+1}} \le C' \frac{d(y)}{r^{n+1}}. \tag{24}$$

On the other hand, by Theorem 1.1, we also have

$$G_A(M_r) \ge \frac{1}{k} \frac{d(y) r}{(2r^2 - \tau)^{n/2+1}} \exp\left(-c_2 \frac{|y'|^2 + |r - y_n|^2}{2r^2 - \tau}\right).$$

Using the fact that $|y'|^2 + |r - y_n|^2 \le \frac{5}{2}r^2$ and $\frac{7}{4}r^2 \le 2r^2 - \tau \le \frac{9}{4}r^2$, we obtain

$$G_A(M_r) \ge k'' \frac{d(y)}{r^{n+1}}. \tag{25}$$

Combining (23)–(25) gives

$$G_A(M) \le k G_A(M_r).$$

Note that the same estimate holds when the pole A lies in $\Omega \cap T_{\varepsilon r}(Q,s)$ with $0<\varepsilon<1$. The constant k then depends also on ε.

For the general case, let $\Sigma = \Omega \setminus T_{2r/3}(Q,s)$. The function R_u^{Σ} is an $\mathcal{L}$-potential on Ω with harmonic support in $\Omega \cap \partial\Sigma$, and so there exists a positive measure μ supported in $\Omega \cap \partial\Sigma$ such that

$$R_u^{\Sigma} = \int_{\Omega\cap\partial\Sigma} G_A \, d\mu(A).$$

For $M \in \Omega \setminus T_r(Q,s)$, we have

$$\begin{aligned} u(M) = R_u^{\Sigma}(M) &= \int_{\Omega\cap\partial\Sigma} G_A(M)\, d\mu(A) \\ &\le k \int_{\Omega\cap\partial\Sigma} G_A(M_r)\, d\mu(A) = kR_u^{\Sigma}(M_r) = ku(M_r). \end{aligned}$$

THEOREM 2.2.2 (Comparison theorem). *Let* $(Q,s) \in \partial D \times]0,T[$ *and* $r \in]0, r_0/2 \wedge \sqrt{T-s}[$. *Then there exists a constant* $k>0$, *depending only on* n, μ, T, D, *and on* V *in terms of the rate of convergence of* $N_h^{\alpha}(V)$ *to zero as* $h \to 0$, *such that for all nonnegative* $\mathcal{L}$*-solutions* u *and* v *on* $\Omega \cap T_{2r}(Q,s)$ *continuously vanishing on* $\Delta_{2r}(Q,s)$, *we have*

$$\frac{u(M)}{v(M)} \le k\, \frac{u(M_r(Q,s))}{v(M_r^*(Q,s))} \qquad \textit{for all } M \in \Omega \cap T_r(Q,s).$$

Proof. Without loss of generality we assume $Q=0$. We first prove the estimate for $u=G_A$ and $v=G_B$ with $A,B \in \Omega \cap \partial T_{3r/2}(0,s)$. We write

$$\begin{aligned} M &= (0,s)+(x,t) \quad \text{with } |x|<r \text{ and } |t|<r^2, \\ A &= (0,s)+(y,\tau) \quad \text{with } \begin{cases} |y| = \frac{3}{2}r \text{ and } |\tau| \le \frac{9}{4}r^2, \text{ or} \\ |y| \le \frac{3}{2}r \text{ and } |\tau| = \frac{9}{4}r^2, \end{cases} \\ B &= (0,s)+(z,\varrho) \quad \text{with } \begin{cases} |z| = \frac{3}{2}r \text{ and } |\varrho| \le \frac{9}{4}r^2, \text{ or} \\ |z| \le \frac{3}{2}r \text{ and } |\varrho| = \frac{9}{4}r^2. \end{cases} \end{aligned}$$

We will estimate $\frac{G_A(M)G_B(M_r^*)}{G_A(M_r)G_B(M)}$. By Theorem 1.1, we have

$$\begin{aligned} \frac{G_A(M)}{G_A(M_r)} &\le k\left(\frac{2r^2-\tau}{t-\tau}\right)^{n/2+1} \exp\left(-c_1\frac{|x-y|^2}{t-\tau}\right) \exp\left(c_2\frac{|(0,r)-y|^2}{2r^2-\tau}\right) \\ &\quad \times \frac{\min(\sqrt{t-\tau},|x|)\min(\sqrt{t-\tau},|y|)}{\min(\sqrt{2r^2-\tau},r)\min(\sqrt{2r^2-\tau},|y|)}. \end{aligned}$$

CASE 1: $|y| \le \frac{3}{2}r$ and $\tau = -\frac{9}{4}r^2$. We have

$$\frac{5}{4}r^2 \le t-\tau \le \frac{13}{4}r^2, \qquad 2r^2-\tau = \frac{17}{4}r^2.$$

Hence

$$\frac{G_A(M)}{G_A(M_r)} \le k' \min\left(1, \frac{|x|}{r}\right). \tag{26}$$

CASE 2: $|y| = \frac{3}{2}r$ and $|\tau| \le \frac{9}{4}r^2$. We have

$$-\frac{9}{4}r^2 \le \tau < t < r^2, \qquad r^2 \le 2r^2 - \tau \le \frac{17}{4}r^2.$$

Then

$$\begin{aligned}\frac{G_A(M)}{G_A(M_r)} &\le k'\left(\frac{2r^2-\tau}{|x-y|^2}\right)^{n/2+1} \exp\left(c_2\,\frac{|(0,r)-y|^2}{2r^2-\tau}\right) \\ &\quad \times \frac{\min(\sqrt{t-\tau},|x|)\min(\sqrt{t-\tau},|y|)}{\min(\sqrt{2r^2-\tau},r)\min(\sqrt{2r^2-\tau},|y|)} \\ &\le k''\min\left(1,\frac{|x|}{r}\right).\end{aligned} \tag{27}$$

In a similar way we prove that

$$\frac{G_B(M_r^*)}{G_B(M)} \le \frac{k'}{\min(1,|x|/r)}. \tag{28}$$

Combining (26)–(28) gives

$$\frac{G_A(M)G_B(M_r^*)}{G_A(M_r)G_B(M)} \le k$$

for all $M \in \Omega \cap T_r(Q,s)$ and $A, B \in \Omega \cap T_{3r/2}(Q,s)$.

For the general case we consider the set $\Sigma = \Omega \cap T_{3r/2}(Q,s)$. The functions R_u^Σ and R_v^Σ are two $\mathcal{L}$-potentials on Ω with harmonic support in $\Omega \cap \partial T_{3r/2}(Q,s)$, and so there exist two positive measures σ and ν supported in $\Omega \cap \partial T_{3r/2}(Q,s)$ such that

$$R_u^\Sigma = \int\limits_{\Omega\cap\partial T_{3r/2}(Q,s)} G_A\,d\sigma(A), \qquad R_v^\Sigma = \int\limits_{\Omega\cap\partial T_{3r/2}(Q,s)} G_B\,d\nu(B).$$

From the previous inequality, we deduce

$$\iint G_A(M)G_B(M_r^*)\,d\sigma(A)\,d\nu(B) \le k\iint G_B(M)G_A(M_r)\,d\sigma(A)\,d\nu(B),$$

which means $R_u^\Sigma(M)R_v^\Sigma(M_r^*) \le kR_v^\Sigma(M)R_u^\Sigma(M_r)$, and the equalities $R_u^\Sigma = u$ on Σ, $R_v^\Sigma = v$ on Σ give the required estimate.

2.3. *Comparison of parabolic measures.* The comparison of harmonic measures has been studied by several authors in the elliptic and parabolic settings. We refer the reader to [12] and the references given there. In particular, for the general Schrödinger operator $L = -\operatorname{div}(A(x)\nabla_x) + q(x)$, in a bounded Lipschitz domain in $\mathbb{R}^n$, $n \ge 3$, with potential in the elliptic Kato class satisfying the gauge condition this problem was studied by Cranston, Fabes and Zhao [3].

Basing on the Green function estimates (Theorem 1.1), Corollary 1.2 and a potential-theoretic approximation argument, we are able to show, as in [12], the comparability of the $\mathcal{L}$-parabolic measure, the adjoint $\mathcal{L}$-parabolic measure and the surface measure on the lateral boundary of Ω. Since the argument is standard, we do not give the details of the proof but we only state the result. We first introduce the definition of the parabolic measure.

For any $\varphi \in C(\partial_{\mathrm{p}}\Omega)$, there exists a unique solution $u = H_\varphi^\Omega$ of the Dirichlet problem $\mathcal{L}u = 0$ on Ω and $u|_{\partial_{\mathrm{p}}\Omega} = \varphi$. For all $M \in \Omega$, the map $M \mapsto H_\varphi^\Omega(M)$ is a linear positive functional on $C(\partial_{\mathrm{p}}\Omega)$, and so there exists a unique Borel measure μ_M on $\partial_{\mathrm{p}}\Omega$ such that

$$H_\varphi^\Omega(M) = \int_{\partial_{\mathrm{p}}\Omega} \varphi(\xi)\, d\mu_M(\xi).$$

μ_M will be called the *$\mathcal{L}$-parabolic measure* at M. The $\mathcal{L}^*$-parabolic measure μ_M^* at M, where $\mathcal{L}^*$ is the adjoint of $\mathcal{L}$, is defined in a similar way.

Let σ be the surface measure on $\partial\Omega$. We have the following.

THEOREM 2.3. *Let V be in the parabolic Kato class. Let $(Q,s) \in \partial D \times]0,T[$, let $r \in]0,r_0]$ be such that $M = (Q,s) + ((0,r),2r^2) \in \Omega$, $M^* = (Q,s) + ((0,r),-2r^2) \in \Omega$, and set $F = (\partial D \times]0,T[) \cap T_r(Q,s)$. There exists a positive constant k, which depends only on n, μ, D, T, and on V in terms of the rate of convergence of $N_h^\alpha(V)$ to zero as $h \to 0$, such that*

$$k^{-1}\sigma|_F \le \mu_M|_F \le k\sigma|_F, \qquad k^{-1}\sigma|_F \le \mu_{M^*}^*|_F \le k\sigma|_F.$$

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