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# ALMOST PERFECT DOMAINS

BY

S. BAZZONI and L. SALCE (Padova)

**Abstract.** Commutative rings all of whose quotients over non-zero ideals are perfect rings are called almost perfect. Revisiting a paper by J. R. Smith on local domains with TTN, some basic results on these domains and their modules are obtained. Various examples of local almost perfect domains with different features are exhibited.

**Introduction.** A commutative ring $R$ with 1 is called *almost perfect* if every quotient of $R$ over a non-zero ideal is a perfect ring. In a recent paper [6] we characterized the commutative integral domains $R$ which are almost perfect as those domains such that all $R$-modules have a strongly flat cover, or equivalently, such that all flat $R$-modules are strongly flat. The last property amounts to saying that weakly cotorsion modules (that is, cotorsion in Matlis' sense) are cotorsion (in Enochs' sense).

The goal of this paper is to investigate more deeply almost perfect commutative rings. In the local case, almost perfect domains have already been introduced by J. R. Smith in [17] under the name of domains with TTN. We will reconsider here some of his results. We concentrate on almost perfect domains, because we show in Section 1 that an almost perfect ring that is not a domain is perfect.

J. R. Smith [17] proved that if every torsion module over a domain $R$ is semi-artininan, then $R$ is locally almost perfect. An important property is missing in order to prove the converse, namely, $h$-locality (see [6]). We show that a classical example of almost Dedekind domain constructed by Heinzer–Ohm [12], which fails to be $h$-local, has all its torsion modules semi-artinian. If $R$ is a local almost perfect domain and $Q$ denotes its field of quotients, we prove that the Loewy length of $Q/R$ equals $\omega$ if and only if the maximal ideal of $R$ is almost nilpotent.

In Section 3 we exhibit three different types of construction of almost perfect local domains. The first construction enables us to provide an example of an integrally closed local almost perfect domain which is not a

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valuation domain, different from that obtained by Smith [17]. The second construction, due to Smith [17], deals with simple integral extensions of local almost perfect domains. Our contribution is the converse of the relevant proposition (see Proposition 3.5). The third construction, which considers semigroup rings with coefficients in fields of positive characteristic, leads us to an example of a local almost perfect domain $R$ such that the Loewy length of $Q/R$ is $\omega 2$.

**1. Preliminaries.** H. Bass defined in [3] an arbitrary ring $R$ with 1 to be a *left-perfect* ring if every left $R$-module has a projective cover. Among the many characterizations of left-perfect rings, we recall here the one we use more often when dealing with almost perfect rings: $R$ is left-perfect if and only if $R/J$ is semisimple ($J$ denotes the Jacobson radical of $R$) and $J$ is $T$-nilpotent; the last condition means that, given any sequence $\{x_n\}_{n\in\omega}$ of elements of $J$, a suitable finite product $x_1\cdot\ldots\cdot x_n$ vanishes. We collect the characterizations of commutative perfect rings that we use in the following. Recall that an $R$-module $M$ is called *semi-artinian* if every non-zero quotient of $M$ has a non-zero socle, and a ring $R$ is semi-artinian if it is semi-artinian as an $R$-module.

THEOREM 1.1. *Let $R$ be a commutative ring. The following are equivalent*:

(1) *$R$ is a perfect ring*;

(2) *$R$ satisfies the DCC on principal ideals*;

(3) *$R$ is a finite direct product of local rings with $T$-nilpotent maximal ideals*;

(4) *$R$ is semilocal and every localization of $R$ at a maximal ideal is a perfect ring*;

(5) *$R$ is semilocal and semi-artinian.*

*Furthermore*, *$R$ is a perfect domain if and only if it is a field.*

*Proof.* For the equivalences (1)$\Leftrightarrow$(2)$\Leftrightarrow$(3) see for instance [13, Ch. 8, Theorems 23.20 and 23.24].

The implication (3)$\Rightarrow$(4) is obvious. Concerning the converse implication, let $\{P_1,\ldots,P_n\}$ be the maximal ideals of $R$. Clearly $R/J$ is semisimple. We must prove that $J$ is $T$-nilpotent. Let $\{x_n\}_{n\in\omega}$ be a sequence of elements of $J$. Considering its image in each localization $R_{P_i}$ $(1\le i\le n)$, since every ideal $P_iR_{P_i}$ is $T$-nilpotent, we can find an index $k$ such that the image of the product $x_1\cdot\ldots\cdot x_k$ vanishes in each $R_{P_i}$. This implies that $x_1\cdot\ldots\cdot x_k$ vanishes in $R$.

For the equivalence (1)$\Leftrightarrow$(5) see [18, Proposition 5.1]. The last statement is an immediate consequence of (3). ∎

The following result will be very useful in what follows.

Lemma 1.2. *Let $R$ be a local ring with maximal ideal $P$ such that $R/aR$ is a perfect ring for some $a \in R$. Then $R/a^nR$ is perfect for every $n \in \mathbb{N}$.*

*Proof.* It is clearly enough to assume that $0 \neq a \in P$. Let $\{b_i + a^2R\}_{i\in\mathbb{N}}$ be a sequence of elements in $P/a^2R$ and consider the corresponding sequence $\{b_i + aR\}_{i\in\mathbb{N}}$. By hypothesis there exists an index $m$ such that $b_1 \cdot \ldots \cdot b_m \in aR$. Consider now the sequence $\{b_m + aR, b_{m+1} + aR, \ldots\}$; there exists $k$ such that $b_{m+1} \cdot \ldots \cdot b_{m+k} \in aR$. Thus $b_1 \cdot \ldots \cdot b_m b_{m+1} \cdot \ldots \cdot b_{m+k} \in a^2R$ and so $R/a^2R$ is a perfect ring. An easy induction completes the proof. ∎

The next proposition reduces the study of almost perfect rings to the domain case.

Proposition 1.3. *Let $R$ be an almost perfect commutative ring. If $R$ is not a domain, then $R$ is a perfect ring.*

*Proof.* First of all we note that $R$ has Krull dimension 0. In fact, a prime ideal $L$ of $R$ is non-zero and it is maximal, since $R/L$ is a domain and a perfect ring, hence a field. We now show that $R$ is semilocal. This is trivial if the Jacobson radical $J$ of $R$ is non-zero, since in this case $R/J$ is a perfect ring by hypothesis, hence semilocal. If $J = 0$, then $R$ is von Neumann regular, since it is zero-dimensional and reduced. Thus, if $0 \neq r$ is a non-invertible element of $R$, then $rR = eR$ for some non-trivial idempotent $e \in R$. So $R = eR \oplus (1-e)R$, where $eR$ and $(1-e)R$ are perfect rings, being non-zero homomorphic images of $R$. We conclude that they are both semilocal and the same holds for $R$. Moreover, it is easily shown that every localization of $R$ at a maximal ideal is almost perfect. Thus, by Theorem 1.1, it is enough to show that a local almost perfect ring $S$ which is not a domain is a perfect ring. Let $P$ be the maximal ideal of $S$. Consider $0 \neq t \in P$; by hypothesis $S/tS$ is a perfect ring and, by the preceding lemma, so is $S/t^2S$. But, since $S$ is not a domain and $P$ is the only prime ideal of $S$, we conclude that $P$ is the nilradical of $S$. So there exists a non-zero $t \in P$ such that $t^2 = 0$. By the above argument we conclude that $S$ is a perfect ring. ∎

The last result is reminiscent of similar situations valid for valuation and local rings: as proved by Gill [10] (see also [9, II 6.4]), an almost maximal valuation ring which is not a domain is indeed a maximal valuation ring; and a local almost henselian ring that is not a domain is henselian (see [19] or [9, II 7, p. 85]).

From now on, we will consider almost perfect domains only. We collect the basic facts on almost perfect domains already proved in [6] in the next theorem.

THEOREM 1.4. (1) *If $R$ is an almost perfect domain, then $R$ is a Matlis domain (that is, the projective dimension of $Q$ equals* 1) *and has Krull dimension* 1 [6, Proposition 4.5].

(2) *The domain $R$ is almost perfect if and only if it is $h$-local and every localization of $R$ at a maximal ideal is almost perfect* [6, Theorem 4.4].

(3) *$R$ is a coherent and almost perfect domain if and only if it is* 1*-dimensional and Noetherian* [6, Propositions 4.3 and 4.5].

Point (2) in the preceding theorem reduces the investigation of almost perfect domains and their torsion modules to the local case. Point (3) shows that the only Prüfer (resp., valuation) domains which are almost perfect are the Dedekind (resp., DVR) domains.

**2. Almost perfect domains and semi-artinian modules.** In [17] J. R. Smith proved that a local domain $R$ is almost perfect if and only if every $R$-module which is not torsion-free contains a simple submodule; this condition is clearly equivalent to the fact that all (cyclic) torsion $R$-modules are semi-artinian. We now give an example that shows that this is no longer the case for non-local almost perfect domains.

EXAMPLE 2.1. Let $R$ be the almost Dedekind domain constructed in Example 2.2 of [12] (see also [9, III 5.5]). $R$ is not $h$-local, hence it is not almost perfect, and has countably many maximal ideals $P_n$ which are finitely generated, and exactly one maximal ideal $P_\infty$ which is not finitely generated. We claim that for every non-zero ideal $I$ of $R$, $R/I$ has non-zero socle, hence $Q/R$ is semi-artinian (see next Theorem 2.2). In fact, let $P$ be a maximal ideal of $R$ containing $I$. Since $R$ is locally a DVR, it is locally almost perfect, so $R_P/I_P$ contains a simple $R_P$-module; thus there exists $r \in R$ such that $0 \neq r + I_P$ and $rP \subseteq I_P$. Assume first that $I$ is contained in at least one finitely generated maximal ideal $P_n$; setting $P = P_n$ in the previous argument, we can find $s \notin P_n$ such that $rsP_n \subseteq I$ and thus $rs + I$ generates a non-zero simple submodule of $R/I$. If $I$ is contained only in the maximal ideal $P_\infty$, then the equality $I_{P_\infty} \cap R = I$ holds locally, since $R$ has Krull dimension one; hence, the equality holds in $R$. Thus, setting $P = P_\infty$ in the previous argument, we obtain $rP_\infty \subseteq I_{P_\infty} \cap R = I$, hence $R/I$ contains a simple $R$-module.

We now want to establish a connection between almost perfect domains and semi-artinian modules. In [17] Smith claimed that every torsion module over a domain $R$ is semi-artinian if and only if $R$ is locally almost perfect. The necessity of this condition is obviously true, while the proof of the converse is not correct. We do not know if the converse holds in general. In connection with this result, Enochs showed in [7] that the domains $R$ such that direct products of torsion-free covers are still torsion-free covers are exactly the domains $R$ such that every torsion $R$-module is semi-artininan.

We need the following result.

THEOREM 2.2 ([8, Theorem 4.4.1]). *Let $R$ be a commutative domain. The following are equivalent*:

(1) *every non-zero torsion module contains a simple module*;

(2) *every torsion $R$-module is semi-artinian*;

(3) *for every non-zero ideal $I$ of $R$, $R/I$ contains a simple module*;

(4) *for every non-zero $R$-submodule $A$ of $Q$, $Q/A$ contains a simple module*;

(5) *$Q/R$ is semi-artinian.*

We can now easily prove the next result.

THEOREM 2.3. *Let $R$ be a commutative domain. The following are equivalent*:

(1) *$R$ is almost perfect*;

(2) *$R$ is $h$-local and $R$ satisfies one of the equivalent conditions of Theorem* 2.2.

*Proof.* (1)$\Rightarrow$(2) follows by Theorem 1.4(2) and, as noted above, by the equivalence (1)$\Leftrightarrow$(5) in Theorem 1.1.

(2)$\Rightarrow$(1). Assume $R$ is $h$-local and satisfies condition (3) of Theorem 2.2. Let $0 \neq I \leq R$. Since $R$ is $h$-local, $R/I$ is a semilocal ring and every quotient of $R/I$ has non-zero socle, by hypothesis. Hence, $R/I$ is semi-artinian; thus, by Theorem 1.1, $R/I$ is a perfect ring, i.e., $R$ is almost perfect. ∎

In particular, since an almost perfect domain $R$ is $h$-local, $Q/R$ is the direct sum of the submodules $Q/R_P$ with $P$ ranging over the maximal ideals of $R$ (see [14]).

The following consequence, partly due to J. R. Smith, is immediate.

COROLLARY 2.4. *Let $R$ be a commutative local domain. The following are equivalent*:

(1) *$R$ is almost perfect*;

(2) *$Q/R$ is semi-artinian*;

(3) *every non-zero torsion module is semi-artinian.*

Summarizing the situation illustrated above, we have the following implications for a domain $R$:

$$R \text{ almost perfect} \;\Rightarrow\; Q/R \text{ semi-artinian} \;\Rightarrow\; R \text{ locally almost perfect.}$$

The first implication cannot be reversed, as shown by Example 2.1, while we do not know whether the second implication can be reversed. The two implications are trivially equivalences in the local case.

We now give some conditions on a local 1-dimensional domain $R$ in order that $R$ is almost perfect.

PROPOSITION 2.5. *Let $R$ be a local* 1*-dimensional domain with maximal ideal $P$. Then*

(1) $Q/R = \bigcup_n a^{-n}R/R$ *for every* $0 \neq a \in P$, *so*, *in particular*, $Q/R$ *is countably generated*;

(2) *$R$ is almost perfect if and only if there exists* $0 \neq a \in P$ *such that* $R/aR$ *is perfect*;

(3) *if $P$ is a principal ideal of its endomorphism ring*, *then $R$ is almost perfect.*

*Proof.* (1) Obvious.

(2) The necessity of the condition is trivial. So assume $R/aR$ is perfect for a non-zero element $a \in P$. By Lemma 1.2, $R/a^nR$ is perfect for every $n \in \mathbb{N}$, hence $a^{-n}R/R$ is semi-artinian. As noted above, $Q/R = \bigcup_n a^{-n}R/R$, since $R$ is 1-dimensional. Thus $Q/R$ is also semi-artinian, and we conclude by Corollary 2.4.

(3) Let $P = aE$, where $E$ is the endomorphism ring of $P$. Then $P^2 = aP \subseteq aR$; hence the maximal ideal of $R/aR$ is nilpotent. Then (2) yields the conclusion. ∎

We now give an application of the preceding proposition. First recall that an ideal of a domain is *stable* if it is invertible over its endomorphism ring, and a domain is *finitely stable* if every finitely generated ideal of $R$ is stable. In [16] it is proved that every overring of a finitely stable local domain is semilocal. Thus, if $R$ is a finitely stable local domain of Krull dimension one with stable maximal ideal $P$, then $E = \mathrm{End}(P)$ is semilocal, so $P$ is principal over $E$; whence, by the above proposition, $R$ is almost perfect.

A condition which implies that an almost perfect domain is Noetherian is illustrated in the next result. Recall that a domain is called *divisorial* if for every fractional non-zero ideal $I$, $R : (R : I) = I$.

PROPOSITION 2.6. (1) *If $R$ is almost perfect and divisorial*, *then $R$ is Noetherian.*

(2) *If $R$ is almost perfect and $Q/R$ is an injective module*, *then $R$ is Noetherian.*

*Proof.* (1) It is well known (see [15] or [11]) that a divisorial domain is $h$-local and locally divisorial, thus by Theorem 1.4(2) we may assume that $R$ is local. Since $R$ is divisorial, $R/I$ is a module satisfying the $AB$-$5^*$ condition for every non-zero ideal $I$ of $R$ (see [5, Sec. 2]). Since $R/I$ is semi-artinian it has non-zero essential socle; moreover its socle is finitely generated, since $R/I$ is a module satisfying the $AB$-$5^*$ condition (see [1, Theorem 1.2]). This implies that, for every $0 \neq r \in R$, every epimorphic image of $R/rR$ is finitely cogenerated, hence $R/rR$ is an artinian ring. We conclude that the maximal ideal of $R$ is finitely generated, thus $R$ is Noetherian. Recall that, by Theorem 1.4(1), every non-zero prime ideal is maximal.

(2) Since $R$ is $h$-local, $R$ is Noetherian if and only if it is locally Noetherian, so we may assume that $R$ is local. Since $Q/R$ is semi-artinian and injective, $Q/R$ is an injective cogenerator of $\operatorname{Mod} R$. By [15, Theorem 2.1], $R$ is a reflexive domain, hence in particular a divisorial domain. Thus the preceding proposition applies. ■

We now recall the notions of Loewy series and Loewy length of a module. For every module $M$ let $s(M)$ denote the socle of $M$, that is, the sum of the simple submodules of $M$; then we can define a continuous ascending chain of submodules of $M$ indexed by the ordinal numbers, in the following way:

$$\begin{aligned} s_0(M) &= s(M), \\ s_{\alpha+1}(M)/s_\alpha(M) &= s(M/s_\alpha(M)), \\ s_\alpha(M) &= \sum_{\beta<\alpha} s_\beta(M) \quad \text{for a limit ordinal } \alpha. \end{aligned}$$

It is well known that $M$ is semi-artinian if and only if $M = s_\lambda(M)$ for some ordinal $\lambda$ (see, for instance, [18]). The minimum ordinal $\lambda$ such that $M = s_\lambda(M)$ is called the *Loewy length* of $M$ and is denoted by $l(M)$. If $R$ is a one-dimensional domain, the length of the Loewy series of $Q/R$ is an ordinal $\lambda$ of countable cofinality, by Proposition 2.5. Obviously, $\lambda$ is a limit ordinal. In fact, assume by way of contradiction that $\lambda = \alpha+1$. Let $x+R \in s_{\alpha+1}(Q/R)\setminus s_\alpha(Q/R)$, $p \in P$ and $y = p^{-1}x$. Then $y+R \notin s_{\alpha+1}(Q/R)$, since otherwise $x+R = p(y+R) \in s_\alpha(Q/R)$, so we get the desired contradiction.

We conclude this section by showing that, for a local almost perfect domain $R$, the Loewy length of $Q/R$ is related to the property of the maximal ideal $P$ of being almost nilpotent, i.e., for every non-zero ideal $I$, $P^n \subseteq I$ for some $n \in \mathbb{N}$.

PROPOSITION 2.7. *Let $R$ be a local almost perfect domain with maximal ideal $P$. The following are equivalent.*

(1) *The Loewy length $l(Q/R)$ of $Q/R$ is $\omega$*;

(2) *$P$ is almost nilpotent*;

(3) *there exists an element $0 \neq a \in P$ such that $P^n \subseteq aR$ for some natural number $n$.*

*Proof.* (1)⇒(2). By Corollary 2.4, $Q/R$ is semi-artinian. Clearly $s_n(Q/R) = (R : P^{n+1})/R$, for every $n \geq 1$. If $l(Q/R) = \omega$, then for every $0 \neq a \in P$ there exists a natural number $n$ such that $a^{-1} \in R : P^n$; consequently, $P^n \subseteq aR$, i.e., $P$ is almost nilpotent.

(2)⇒(3). Obvious.

(3)$\Rightarrow$(1). By hypothesis, $a^{-1} \in R : P^n$, so $a^{-m} \in R : P^{nm}$ for all $m \in \mathbb{N}$. Since the elements $a^{-m} + R$ generate $Q/R$, by Proposition 2.5, we conclude that $l(Q/R) = \omega$. ∎

Obviously, the equivalent conditions in Proposition 2.7 are satisfied if $R$ is a local 1-dimensional Noetherian domain, since, for $0 \neq a \in R$, the factor ring $R/aR$ is artinian. We can add the following information on the Loewy length of the module $Q/R$.

PROPOSITION 2.8. *Let $R$ be a local almost perfect domain with maximal ideal $P$ and let $0 \neq a \in P$. Then*

$$l(Q/R) = \sup_{n \in \mathbb{N}} \{ l(a^{-n}R/R) \}.$$

*Proof.* Clearly, if $N$ is a submodule of a semi-artinian module $M$, then $s_\alpha(N) = s_\alpha(M) \cap N$ for every ordinal $\alpha$. Using the fact that $Q/R = \bigcup_n a^{-n}R/R$ it is easy to conclude the proof by induction. ∎

We remark that since the Loewy length of $a^{-n}R/R$ is a non-limit ordinal (say $\alpha_n + 1$) where $\alpha_n$ is the Loewy length of $a^{-n}P/R$, $\{\alpha_n\}_n$ is an increasing sequence of ordinals and it is not stationary, since $l(Q/R)$ is a limit ordinal.

**3. Constructions of almost perfect domains.** In this section we exhibit three different ways of obtaining (non-Noetherian) almost perfect local domains: using the D+M construction; considering suitable simple integral extensions of almost perfect domains; using semigroup rings over submonoids of the non-negative real numbers.

*$D + M$ construction.* This construction allows us to answer to a question raised in [17], namely, whether an integrally closed almost perfect local domain is necessarily a valuation domain. This question is natural, since the integral closure of a local domain is the intersection of all the valuation domains dominating it and contained in its field of quotients. In [17, p. 243] a counterexample is given, but, as noted by Griffin in the review (see [MR 40♯ 130]), its proof is inadequate. As a consequence of the next Lemma 3.1, we will give an easy example of an integrally closed local almost perfect domain which is not a valuation domain.

The construction is as follows.

Let $V = K[[Y]]$ be the power series ring in the indeterminate $Y$ with coefficients in the field $K$. For every local subring $D$ of $K$ with maximal ideal $P$ consider the domain $R = D + M$, where $M = YK[[Y]]$. Clearly, $R$ is a local domain with maximal ideal $P + M$.

LEMMA 3.1. *Let $R$ be as defined above. Then $R$ is almost perfect if and only if $D$ is a field.*

*Proof.* Assume $D$ is a field. Then $R$ is 1-dimensional and its maximal ideal is $M$. By Proposition 2.5, it is enough to show that, given $0 \neq a \in M$, a suitable power of $M$ is contained in $aR$. We have $a = Y^n v$ for some unit $v \in V$ and some integer $n \geq 1$, and it is immediate to see that $M^{n+1} \subseteq aM \subseteq aR$. Conversely, if $D$ is not a field, then $R$ has Krull dimension $\geq 2$, so it is not almost perfect, by Theorem 1.4(1). ∎

EXAMPLE 3.2. Let $F$ be a field, $X$ an indeterminate over $F$ and let $K = F(X)$. By Lemma 3.1, the domain $R = F + M$ (where $M = YK[[Y]]$) is an almost perfect local domain. $R$ is integrally closed, since $F$ is algebraically closed in $K$; moreover $R$ is not a valuation domain, since $F \subsetneq K$ (see [4, Theorem 2.1]).

EXAMPLE 3.3. In the preceding notation, the domain $R = D + M$ is Noetherian if and only if the degree $[K : D]$ is finite (see, for instance [4, Theorem 2.1]). Thus, if $[K : D]$ is infinite, $R$ is an example of a non-Noetherian almost perfect domain. Note that in this case the residue field $D$ of $R$ is properly contained in the residue field $K$ of $V = K[[Y]]$, which is a valuation domain dominating $R$.

*Integral extensions.* This second type of construction is due to J. R. Smith [17, Sec. 5], who proved the sufficiency in the next Proposition 3.5. We first consider a general situation on local domains.

Let $R$ be a local domain. Let $x$ be an element of a field extension $F$ of the quotient field $Q$ of $R$. Assume $x$ is integral over $R$ and let

$$f(X) = X^{n+1} + r_n X^n + \ldots + r_1 X + r_0 \qquad (r_i \in R)$$

be a monic polynomial of $R[X]$ of minimum degree such that $f(x) = 0$. Denote by $P$ the maximal ideal of $R$ and by $K$ the residue field $R/P$. If $- : R \to K$ is the canonical homomorphism and $f \in R[X]$, $\bar{f}$ will denote the polynomial in $K[X]$ whose coefficients are the images under $-$ of the coefficients of $f$. The following lemma likely exists in the literature, but for the sake of completeness we give its proof; it describes the maximal ideals of the extension ring $R[x]$.

LEMMA 3.4. *Let the notations be as above. The ring $R[x]$ is semilocal and its maximal ideals are the ideals $P_i = f_i(x)R[x] + P[x]$, where $\bar{f}_i$ are the irreducible factors of $\bar{f}$ in $K[X]$.*

*Proof.* It is immediate to check that $-$ induces an isomorphism

$$\phi : \frac{R[X]}{P[X] + (f)} \to \frac{K[X]}{(\bar{f})}.$$

Let now $I$ be the ideal of $R[X]$ consisting of the polynomials which have $x$ as a root. If $S = R[x]$, then $S \cong R[X]/I$ and $S/PS \cong R[X]/(P[X] + I)$. We claim that $P[X] + (f) = P[X] + I$. Let $g(X) \in I$; by the division algorithm we can write $g(X) = f(X)h(X) + l(X)$ for some polynomials $h, l \in R[X]$ such that the degree of $l$ is less than the degree of $f$. Thus $l(x) = 0$ and we show that $l(X) \in P[X]$. Write $l(X) = \sum_i a_i X^i + \sum_j b_j X^j$ with $a_i$ units of $R$ and $b_j \in P$; then $\sum_i a_i x^i = -(\sum_j b_j x^j) \in P[x]$. Let $i_0$ be the maximum integer such that $a_{i_0}$ is a unit; then $i_0 < n$, otherwise $a_{i_0}^{-1} l(X)$ would be a monic polynomial of degree $\le n$ with $x$ as a root. Consider the element $x^{n-i_0}(\sum_i a_i x^i)$; it is a linear combination with unit coefficients of powers of $x$ up to the $n$th power, and since it coincides with $-x^{n-i_0}(\sum_j b_j x^j)$, it belongs to $P[x]$. Using the relation $x^{n+1} = -(r_n x^n + \ldots + r_1 x + r_0)$, we can write $x^{n-i_0}(\sum_i a_i x^i) = \sum_{0 \le m \le n} c_m x^m$ with the coefficients $c_m \in P$. Thus we obtain a linear combination of powers $x^m$, $m \le n$, whose leading coefficient is $a_{i_0} - c_n$. Since $R$ is local and $c_n \in P$, $a_{i_0} - c_n$ is a unit, thus $x$ is a root of a monic polynomial of degree $n$, a contradiction. We conclude that $S/PS$ is isomorphic to $K[X]/(\bar{f})$. So the maximal ideals of $S/PS$ correspond to the irreducible factors $\bar{f}_i$ of $\bar{f}$ in $K[X]$. Moreover, since $S$ is integral over $R$, every maximal ideal of $S$ contains $PS$, thus the conclusion follows. ∎

We can now state Smith's result and prove its converse.

PROPOSITION 3.5. *Let $R$ be an almost perfect local domain. Then, in the above notation, $R[x]$ is an almost perfect domain if and only if all the coefficients of $f$ (except the leading one) are in the maximal ideal of $R$, in which case $R[x]$ is local.*

*Proof.* The sufficiency has been proved in [17, Sec. 5]. If $S = R[x]$ is almost perfect, then a suitable power of $x$ is contained in $PS$ and, in view of the isomorphism $S/PS \cong K[X]/(\bar{f})$ established in Lemma 3.4, some power of the element $X \in K[X]$ is contained in the ideal $\bar{f}K[X]$. Clearly this is possible only if $\bar{f} = X^m$ for some $m \ge 1$, i.e., all the coefficients of $f \in R[X]$ except the leading one are in $P$. In this case the maximal ideal of $S/PS$ corresponds to the only irreducible factor $X$ of $\bar{f}$ in $K[X]$; thus $S$ is local with maximal ideal $xS + PS = P + Rx + \ldots + Rx^n$. ∎

It is an easy exercise to show that if the sufficient condition of the preceding proposition is satisfied and the almost perfect local domain $R$ has almost nilpotent maximal ideal, then also $R[x]$ has almost nilpotent maximal ideal.

*Semigroup rings.* This type of construction is similar to that used in [17]; it differs by the fact that we use semigroup rings instead of formal power series rings with exponent in a semigroup.

For every submonoid $\Sigma$ of the non-negative real numbers, let $\Sigma_+$ denote the set of the strictly positive elements of $\Sigma$. If $K$ is a field, consider the semigroup ring $K[\Sigma]$. It is well known that since $\Sigma$ is a cancellative torsion-free semigroup, $K[\Sigma]$ is a domain. Let $R$ be the localization of $K[\Sigma]$ at the maximal ideal $M$ generated by the set $\{X^\sigma \mid \sigma \in \Sigma_+\}$, and let $P$ be its maximal ideal.

Let $\Gamma$ be the subgroup of the additive group of real numbers generated by $\Sigma$. Then the same construction as above, replacing $\Sigma$ by $\Gamma$, produces a valuation domain $V$ contained in the field of quotients $Q$ of $R$, which dominates $R$, in the sense that the maximal ideal of $V$ intersects $R$ in $P$. The value of a polynomial with exponents in $\Gamma$ is, as usual, the minimum of the support.

In the above notation we have the following

LEMMA 3.6. *Assume the field $K$ has characteristic $p \neq 0$.*

(1) *$R$ is 1-dimensional, provided that $\Sigma$ satisfies the condition*:

(I) *if $\sigma, \tau \in \Sigma$ are such that $\sigma - \tau > 0$, then $m(\sigma - \tau) \in \Sigma$ for all $m \in \mathbb{N}$ large enough.*

(2) *If $\Sigma$ satisfies* (I) *and the condition*:

(II) *there exist an element $\sigma_0 \in \Sigma_+$ and an integer $n \in \mathbb{N}$ such that $\sigma_1 + \ldots + \sigma_n - \sigma_0 \in \Sigma$, for every choice of $n$ elements $\sigma_i \in \Sigma_+$,*

*then $P$ is almost nilpotent*; *in particular*, *$R$ is almost perfect.*

*Proof.* 1. Let $0 \neq f \in P$; we show that a suitable power of $f$ is associated with an element of the form $X^\sigma$ for some $\sigma \in \Sigma_+$. Without loss of generality we may assume that $f \in K[\Sigma]$, so that we can write $f = a_1 X^{\sigma_1} + \ldots + a_n X^{\sigma_n}$ with $\sigma_1 < \ldots < \sigma_n \in \Sigma$ and $a_i \in K$, $a_1 = 1$. Since $K$ has characteristic $p$, we have

$$f^{p^m} = X^{p^m \sigma_1}(1 + a_2^{p^m} X^{p^m(\sigma_2 - \sigma_1)} + \ldots + a_n^{p^m} X^{p^m(\sigma_n - \sigma_1)})$$

and if $m$ is large enough, condition (I) guarantees that $p^m(\sigma_i - \sigma_1) \in \Sigma$ for every $i = 2, \ldots, n$. Thus, for $m$ large enough, $f^{p^m} = X^{p^m \sigma_1} u$, where $u$ is a unit of $R$, hence $f^{p^m}$ is associated with $X^\sigma, \sigma \in \Sigma_+$. Let now $f, g \in P$. Then suitable powers of $f$ and $g$ are associated with the same $X^\tau, \tau \in \Sigma_+$. We conclude that $fR$ and $gR$ have the same radical; hence $R$ has Krull dimension one.

2. By point 1, we know that $R$ is one-dimensional. By Proposition 2.7, it is enough to show that $P/(X^{\sigma_0} R)$ is nilpotent. We show that if $n$ is a natural number satisfying condition (II), then $P^n \subseteq X^{\sigma_0} R$. In fact, consider $n$ elements $f_1, \ldots, f_n$ in $P$; we may assume that the elements $f_i$ are polynomials with coefficients in $K$ and exponents in $\Sigma_+$. The product $f_1 \ldots f_n$ is a sum of monomials of the form $aX^{\sigma_{i_1} + \ldots + \sigma_{i_n}}$, $a \in K$; so, by condition (II), $f_1 \ldots f_n X^{-\sigma_0}$ is an element of $K[\Sigma]$. Hence $P^n \subseteq X^{\sigma_0} R$. ∎

The argument making use of the characteristic $p$ in the preceding lemma is borrowed from [2]. The situation described above for an almost perfect local domain and the valuation domain dominating it, starting from a semigroup ring over a submonoid $\Sigma$ of the non-negative real numbers and the subgroup $\Gamma$ generated by it, is quite general. In fact, it is well known that every local domain $R$ of Krull dimension one is dominated by an archimedean valuation domain $V$ contained in its field of quotients. Furthermore, J. R. Smith proved in [17, Sec. 2] the following two facts, which we collect in a single proposition.

PROPOSITION 3.7 ([17, pp. 235–236]). *Let $V$ be a rank one valuation domain with valuation $v$ onto the subgroup $\Gamma$ of the reals, dominating the local domain $R$ with maximal ideal $P$.*

(1) *If $R$ is almost perfect, then there exists a strictly positive element $\gamma_0 \in \Gamma$ such that $v(a) \geq \gamma_0$ for every $a \in P$.*

(2) *If there exists $0 < \gamma_1 \in \Gamma$ such that $\{x \in V \mid v(x) \geq \gamma_1\} \subseteq P$, then $R$ is almost perfect.*

Smith gave an example showing that condition (2) in Proposition 3.7 is not necessary. We now give a different example.

EXAMPLE 3.8. Let $\{M_i \mid i \in \mathbb{N}\}$ be a strictly increasing sequence of positive integers with $M_1 \geq 1$. Define a sequence of semigroups as follows: $\Sigma_0 = \{0\}$, $\Sigma_n = \{m/2^n \mid m \in \mathbb{N}, m/2^n \geq M_n\}$. Each $\Sigma_n$ is a monoid contained in $\Sigma_{n+1}$, and $\Sigma = \bigcup_n \Sigma_n$ is a semigroup generating the group $\mathbb{Z}[1/2] = \{m/2^n \mid m \in \mathbb{Z}, n \in \mathbb{N}\}$. $\Sigma$ has the following properties.

(a) For every $0 < \gamma \in \mathbb{Z}[1/2]$, $m\gamma \in \Sigma$ for all $m$ large enough.

In fact, let $\gamma = a/2^n$ where $a$ is an odd natural number. Take $m_0 \in \mathbb{N}$ such that $m_0 a \geq 2^n M_n$. Then for every $m \geq m_0$, $ma \geq 2^n M_n$, thus $m\gamma \in \Sigma_n$.

(b) Let $\sigma_0 = M_1$; then $\sigma + \tau - \sigma_0 \in \Sigma$, for all $\sigma, \tau$ in $\Sigma_+$.

In fact, by construction $\tau \geq M_1$ for every $\tau \in \Sigma_+$. Assume $\sigma \in \Sigma_n$, $\tau \in \Sigma_m$ with $m \leq n$; then $\sigma + \tau - M_1 \geq \sigma$ and clearly $\sigma + \tau - \sigma_0$ can be written in the form $m/2^n$; hence $\sigma + \tau - M_1 \in \Sigma_n$.

(c) There are arbitrarily large positive elements of $\mathbb{Z}[1/2]$ not in $\Sigma$.

In fact, for every $0 < h \in \mathbb{N}$ choose $n \geq 2$ such that $M_n > h$. Then there is an odd natural number $a$ satisfying $2^n h < a < 2^n M_n$; it is immediate to check that $a/2^n \notin \Sigma$.

Let now $R$ be the localization of $K[\Sigma]$ at the maximal ideal generated by $\{X^\sigma \mid \sigma \in \Sigma_+\}$, where $K$ is a field of characteristic $p$ and $\Sigma$ is the semigroup defined above. Then $R$ is an almost perfect local domain, since the conditions in Lemma 3.6 are obviously satisfied; in view of property (c), $R$ does not satisfy condition (2) of Proposition 3.7. Note that the maximal ideal of $R$ is almost nilpotent, by property (b) and Lemma 3.6.

We now exhibit an example of a semigroup $\Sigma$ giving rise to an almost perfect domain whose maximal ideal is not almost nilpotent. Let $\mathcal{P}$ be the set of prime numbers. For every finite subset $F$ of $\mathcal{P}$ set $\varrho_F = 1+\sum_{p\in F} 1/p$. Let $\Sigma$ be the submonoid of the additive group of reals generated by $\{0, 1, \varrho_F \mid F \subseteq \mathcal{P}\}$, and $\Sigma_1$ the submonoid of $\Sigma$ generated by $\{0, 1, 1 + 1/p \mid p \in \mathcal{P}\}$.

Using this notation, we give some lemmas concerning properties satisfied by the monoid $\Sigma$. First of all we recall a well known result on integral numerical semigroups, which is folklore.

LEMMA 3.9. *Let $a_1, \dots, a_n$ be coprime positive integers ($n \geq 2$). Then there exists $N \in \mathbb{N}$ such that the semigroup generated by $a_1, \dots, a_n$ contains all the integers $m \geq N$.*

LEMMA 3.10. *If $\sigma, \tau \in \Sigma$ are such that $\sigma - \tau > 0$, then $m(\sigma - \tau) \in \Sigma_1$ for all $m$ large enough.*

*Proof.* The group $\Gamma$ generated by $\Sigma$ consists of the rational numbers with square-free denominator. Thus it is enough to prove that, given finitely many prime numbers $p_1, \dots, p_n$, there exists $N_0 \in \mathbb{N}$ such that $m/(p_1 \dots p_n) \in \Sigma_1$ for all $m \geq N_0$. Consider the subsemigroup of $\Sigma_1$ generated by $\{1, 1 + 1/p_1, \dots, 1 + 1/p_n\}$ and write all these elements in the form $a_i/(p_1 \dots p_n)$ ($i = 0, 1, \dots, n$). It is immediate to check that the elements $a_i$ ($i = 0, 1, \dots, n$) are coprime, hence we conclude by Lemma 3.9. ∎

LEMMA 3.11. *If $m(1 + 1/p) - 1 \in \Sigma$ for some $m \in \mathbb{N}$, then $m \geq p$.*

*Proof.* Let $\sigma = m(1+1/p)-1 = \alpha_0 + \sum_{1\leq i\leq k} \alpha_i \varrho_{F_i}$, where the $F_i$ are finite subsets of $\mathcal{P}$ and $\alpha_i \in \mathbb{N}$. Rewrite $\sigma$ in the form $\sigma = \beta_0+\sum_{1\leq j\leq h} \beta_j/(pj)$, where $0 \leq \beta_0 \in \mathbb{N}$ and $0 \leq \beta_j < p_j$ for all $j$. Multiplying both sides by $p_1 \dots p_h$, we see that $p_j \mid \beta_j$ for all $j \leq h$ such that $p_j \neq p$, hence $\beta_j = 0$. If $p_j \neq p$ for every $j$, then $\sigma = \beta_0 \in \mathbb{N}$, which easily implies that $m \geq p$. Otherwise $\sigma = \alpha_0 + \alpha_1(1 + 1/p)$, which implies $(m - \alpha_1)(1 + 1/p) = \alpha_0 + 1$; this shows that $p$ divides $m - \alpha_1 > 0$; hence again $m \geq p$. ∎

LEMMA 3.12. *Let $\sigma = \sum_{1\leq i\leq k} m_i\varrho_{F_i} \in \Sigma_+$ with $1 \leq m_i \in \mathbb{N}$, $k \geq 2$ and $F_i$ a finite subset of $\mathcal{P}$ for each $i$. If $\bigcap F_i = \emptyset$, then $\sigma - 1 \in \Sigma_+$.*

*Proof.* We may assume that $m_k \geq m_i$ for all $i \leq k$. For every $p \in F_k$ there exists $i < k$ such that $p \notin F_i$. In the expression of $\sigma$ substitute $m_i\varrho_{F_i}$ by $m_i\varrho_{E_i}$, where $E_i = F_i \cup \{p\}$, and $m_k\varrho_{F_k}$ by $m_k\varrho_{E_k} + (m_k - m_i)(1/p)$, where $E_k = F_k \setminus \{p\}$. Repeat this process for all $p \in F_k$, thus obtaining for new finite subsets $F'_i$ of $\mathcal{P}$:

$$\sigma = \sum_{1\leq i\leq k-1} m_i\varrho_{F'_i} + m_k + \sum_{p\in F_k} (m_k - m_i)\,\frac{1}{p}.$$

Since $m_k - m_i \leq m_k - 1$ for all $i$, we have

$$m_k - 1 + \sum_{p\in F_k} (m_k - m_i)\,\frac{1}{p} \in \Sigma;$$

in fact, if $1 \le r_1 \le \ldots \le r_t \le r$, then

$$r + \frac{r_1}{p_1} + \ldots + \frac{r_t}{p_t}$$
$$= r_1\left(1+\sum_{1\le i\le t}\frac{1}{p_i}\right)+(r_2-r_1)\left(1+\sum_{2\le i\le t}\frac{1}{p_i}\right)+\ldots+(r_t-r_{t-1})\left(1+\frac{1}{p_t}\right)+r-r_t.$$

Thus we have proved that $\sigma - 1 \in \Sigma_+$. ∎

Let now $\Sigma$ be the monoid defined above, $K[\Sigma]$ the semigroup ring over $\Sigma$ with coefficients in the field $K$ of positive characteristic, and $R$ the localization of $K[\Sigma]$ at the maximal ideal $M$ generated by the set $\{X^\sigma \mid \sigma \in \Sigma_+\}$; let $P$ denote the maximal ideal of $R$, and $Q$ its field of quotients. By Lemmas 3.6 and 3.10, $R$ is 1-dimensional. In this notation we shall prove the following

PROPOSITION 3.13. *The ring $R$ constructed above is an almost perfect local domain whose maximal ideal $P$ is not almost nilpotent.*

*Proof.* If $P$ is almost nilpotent, then $P^n \subseteq XR$ for some $n \in \mathbb{N}$. This implies that $X^{n(1+1/p)} \in XR$ for every $p \in \mathcal{P}$, therefore $n(1+1/p) - 1 \in \Sigma$ for all $p \in \mathcal{P}$. But this is impossible, by Lemma 3.11, hence $P$ is not almost nilpotent. In order to conclude, in view of Proposition 2.5 it is enough to prove that $R/XR$ is a perfect ring. Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of elements in the maximal ideal of $R$; we must prove that a suitable product $f_1 \cdot \ldots \cdot f_m$ belongs to $XR$. We can assume that $f_n \in K[\Sigma_+]$ for all $n$. For every $m \in \mathbb{N}$, the product $f_1 \cdot \ldots \cdot f_m$ is a finite sum of monomials of the form

$$aX^{\sigma_1+\ldots+\sigma_m} \qquad (a \in K,\ \sigma_i \in \Sigma_+),$$

where $X^{\sigma_i}$ is a monomial of $f_i$ (apart from a factor in $K$). If we prove the next claim, we are done.

CLAIM. *There exists an integer $n_0 \in \mathbb{N}$ depending on the sequence $\{f_n\}_{n\in\mathbb{N}}$ such that for all $m \ge n_0$ and for all monomials $aX^{\sigma_1+\ldots+\sigma_m}$ appearing in the product $f_1 \cdot \ldots \cdot f_m$ (where $X^{\sigma_i}$ is a monomial of $f_i$, apart from a factor in $K$), $\sigma_1 + \ldots + \sigma_m - 1 \in \Sigma$.*

In fact, the claim implies that $f_1 \cdot \ldots \cdot f_m \in XR$ for $m \ge n_0$, since all the monomials $X^{\sigma_1+\ldots+\sigma_m}$ belong to $XR$. We will show that $n_0$ depends only on the monomials of $f_1$; since $f_1$ is a finite sum of monomials, it is enough to find an integer $n_0$ for a fixed exponent $\sigma_1$ of a monomial $X^{\sigma_1}$ of $f_1$ (taking the maximum of the integers corresponding to all the monomials of $f_1$ we get the claim). Recall that every $\sigma_i$ is an integral linear combination of the generators 1 and $\varrho_F$. If 1 appears among the generators of $\sigma_1$, then $\sigma_1 - 1 \in \Sigma$ and we are done. Assume this is not the case. Let $\sigma_1 = a_1\varrho_{F_1} + \ldots + a_k\varrho_{F_k}$. List the primes involved in the generators $\varrho_{F_i}$ $(i \le k)$: $\{p_1, \ldots, p_s\}$. By the proof of Lemma 3.10, there exists an integer $N_0 \in \mathbb{N}$ such that, for all

$m \geq N_0$, $m/p \in \Sigma_1$ for all $p \in \{p_1, \ldots, p_s\}$. Let $n_0 = N_0 + p_1 + \ldots + p_s$; then $m/p - 1 \in \Sigma$ for all $m \geq n_0$ and for all $p \in \{p_1, \ldots, p_s\}$. Now $n_0$ is the integer we are looking for. In fact, let $m \geq n_0$ and let $\mathcal{F}$ be the family of all the finite subsets $F$ of $\mathcal{P}$ indexing the generators $\varrho_F$ of the elements $\sigma_1, \ldots, \sigma_m$. If $\bigcap_{F \in \mathcal{F}} F = \emptyset$, then the conclusion follows from Lemma 3.12. If $p \in \bigcap_{F \in \mathcal{F}} F$, then $1/p$ appears in all the generators $\varrho_F$ $(F \in \mathcal{F})$. Let $\gamma$ be the sum of all the $1/p$'s, each multiplied by its own positive integral coefficient; then $\gamma = k/p$ for a $k \geq n_0$, hence $\gamma - 1 \in \Sigma$. If we eliminate $1/p$ in all the generators $\varrho_F$, we get new generators $\varrho_{F'}$ and a new family $\mathcal{F}'$ of finite subsets of $\mathcal{P}$ giving rise to an element $\tau \in \Sigma$. Thus, $\sigma_1 + \ldots + \sigma_m - 1 = \gamma - 1 + \tau \in \Sigma$, as claimed. ∎

The information on the domain $R$ constructed above is improved by the next proposition. First a technical lemma is required.

LEMMA 3.14. *Given primes $p_1, \ldots, p_k$ and positive integers $m_1, \ldots, m_k$, there exists a positive integer $n$ such that $m_1/p_1 + \ldots + m_k/p_k + \sigma_1 + \ldots + \sigma_n \in \Sigma$ for any $\sigma_i \in \Sigma_+$.*

*Proof.* It is clearly enough to prove the claim for $k = 1$ and $m_1 = 1$. By Lemma 3.10, there exists an $n \in \mathbb{N}$ such that $m/p_1 \in \Sigma$ for all $m \geq n$. Consider $n$ elements $\sigma_i \in \Sigma_+$; they are linear combinations of the generators $\varrho_F$; if $p_1 \notin F$ for some $F$, then $1/p_1$ can be added to $\varrho_F$, thus yielding an element of $\Sigma$. Otherwise, if $p_1 \in \bigcap F$ then $1/p_1$ is repeated at least $n$ times so $1/p_1 + \sigma_1 + \ldots + \sigma_n = m/p_1 + \sigma'_1 + \ldots + \sigma'_n$, where $m \geq n$ and the $\sigma'_i$'s are obtained by subtracting $1/p_1$ in the generators of the $\sigma_i$'s, thus $m_1/p_1 + \sigma_1 + \ldots + \sigma_n \in \Sigma$. ∎

PROPOSITION 3.15. *Let $R$ be the almost perfect local domain constructed above. Then the Loewy length of $Q/R$ is $\omega 2$.*

*Proof.* By the preceding proposition, $P$ is not almost nilpotent, so, by Proposition 2.7, $l(Q/R) > \omega$. By Proposition 2.8, $l(X^{-r}R/R) \geq \omega + 1$ for all $r \in \mathbb{N}$, and to conclude it is enough to show that $l(X^{-r}R/R) = \omega + r$ for all positive integers $r$. Actually, we will show, by induction on $r$, that

$$X^{-r} + R \in s_{\omega+r}(Q/R) \setminus s_{\omega+r-1}(Q/R). \tag{1}$$

Clearly $X^{-1} + R \notin s_\omega(Q/R)$, otherwise $l(X^{-1}R/R) \leq \omega$. Furthermore, $X^{-1} + R \in s_{\omega+1}(Q/R)$, since $pX^{-1} + R \in s_\omega(Q/R)$ for all $p \in P$; in order to see this, it is enough to show that $X^{\varrho_F - 1} + R \in s_\omega(Q/R)$ for all finite subsets $F$ of $\mathbb{N}$, i.e., for every finite set of primes $\{p_1, \ldots, p_k\}$, there exists $n \in \mathbb{N}$ such that

$$X^{1/p_1 + \ldots + 1/p_k} P^n \subseteq R.$$

This amounts to showing that $1/p_1 + \ldots + 1/p_k + \sigma_1 + \ldots + \sigma_n \in \Sigma$ for any $\sigma_i \in \Sigma_+$, and it is clearly enough to prove this fact for $k = 1$. The preceding lemma provides the proof.

Assume now that $r > 1$ and (1) proved for $r - 1$. Obviously $X^{-r} + R \notin s_{\omega+r-1}(Q/R)$, otherwise $X(X^{-r} + R) = X^{-r+1} + R \in s_{\omega+r-2}(Q/R)$, absurd. To prove that $X^{-r} + R \in s_{\omega+r}(Q/R)$ it is enough to see that $X^{-r}P^r + R \in s_\omega(Q/R)$, i.e., for every choice of $r$ generators $\varrho_{F_i}$ of $P$, there exists $n \in \mathbb{N}$ such that

$$X^{-r} X^{\varrho_{F_1} + \ldots + \varrho_{F_r}} P^n \subseteq R.$$

This amounts to showing that, for each choice of a finite number of primes $p_i$'s, there is $n \in \mathbb{N}$ such that

$$1/p_1 + \ldots + 1/p_k + \sigma_1 + \ldots + \sigma_n \in \Sigma$$

for any $\sigma_i \in \Sigma$. This fact follows from the preceding lemma. ∎

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Dipartimento di Matematica Pura e Applicata
Università di Padova
Via Belzoni 7
35131 Padova, Italy
E-mail: bazzoni@math.unipd.it
salce@math.unipd.it

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