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THE AUTOMORPHISM GROUP OF THE LEBESGUE MEASURE HAS NO NON-TRIVIAL SUBGROUPS OF INDEX $< 2^\omega$

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Abstract. We show that the automorphism group $\mathrm{Aut}([0,1],\lambda)$ of the Lebesgue measure has no non-trivial subgroups of index $< 2^\omega$.

1. Introduction. In this note, we study the group $\mathrm{Aut}([0,1],\lambda)$ of all measure preserving transformations of the Lebesgue probability measure λ on the unit interval $[0,1]$, modulo null sets (equivalently, $\mathrm{Aut}(X,\mu)$, where X is an uncountable standard Borel space with a non-atomic probability measure μ.) The main theorem of the paper states that there are no nontrivial subgroups of index $< 2^\omega$ in $\mathrm{Aut}([0,1],\lambda)$. The proof borrows some arguments from [2].

We also present a short proof of a result from [1] to the effect that the group $\mathrm{Aut}([0,1],\lambda)$ has the automatic continuity property. In the original proof, a tool called metric ample generics was used. Our argument does not require this ingredient.

2. Results. A *Polish space* is a topological space whose topology is separable and completely metrizable. A *standard Borel space* is a pair $(X,\mathcal{B})$, where X is a set, and $\mathcal{B}$ is a σ-algebra over X such that for some Polish topology on X, the family $\mathcal{B}$ is equal to the σ-algebra of all Borel subsets of X. For example, the usual topology on the unit interval $[0,1]$ gives rise to the standard Borel space of all Borel subsets of $[0,1]$.

For a standard Borel space X, and a probability measure μ on X, the group $\mathrm{Aut}(X,\mu)$ can be endowed with two natural group topologies. Let MALG_μ be the measure algebra of all μ-measurable subsets of X modulo null sets. It is a metric space with metric defined by

$$\rho(A,B) = \mu(A \triangle B).$$

Now, every measure preserving automorphism of X gives rise to an isometry of MALG_μ, so we can consider the pointwise convergence topology (the

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weak topology) and the uniform convergence topology (the uniform topology) on $\mathrm{Aut}(X,\mu)$. The former is Polish non-locally compact, and that is the topology we equip $\mathrm{Aut}(X,\mu)$ with.

It is well known (see [4, Corollary 1.4.3]) that up to Borel isomorphism there exists only one uncountable standard Borel space. Moreover, if μ is a non-atomic probability measure on an uncountable standard Borel space X, then there exists an isomorphism between the measure spaces (X,μ) and $([0,1],\lambda)$, where λ is the Lebesgue measure on $[0,1]$ (see [5, Theorem 17.41]). Therefore, $\mathrm{Aut}(X,\mu)$ is isomorphic to $\mathrm{Aut}([0,1],\lambda)$. In the following, all the results are stated for groups of the form $\mathrm{Aut}(X,\mu)$, where X is an uncountable standard Borel space, and μ is a non-atomic probability measure on X.

Formally, elements of $\mathrm{Aut}(X,\mu)$ are equivalence classes of mappings but most of the time we will work with actual mappings. Thus $g \in \mathrm{Aut}(X,\mu)$ may mean a bijection $g : X \to X$ that preserves μ or an equivalence class of such bijections. This, however, should not cause confusion in the proofs.

For $g \in \mathrm{Aut}(X,\mu)$, the *support* of g is defined as

$$\mathrm{supp}(g) = \{x \in X : g(x) \neq x\}.$$

For a Borel $A \subseteq X$, by $\mathrm{Aut}(A,\mu)$ we mean the group of all $g \in \mathrm{Aut}(X,\mu)$ whose support is contained in A.

LEMMA 2.1. *Let $A, B \subseteq X$ be Borel sets with positive measure, and such that*

$$2\mu(A \setminus B), 2\mu(B \setminus A) \leq \mu(A \cap B).$$

Then

$$\mathrm{Aut}(A \cup B,\mu) = (\mathrm{Aut}(A,\mu)\mathrm{Aut}(B,\mu))^2.$$

Proof. Put $C = A \cap B$, and fix $g \in \mathrm{Aut}(A \cup B,\mu)$. After possibly interchanging A and B, we can assume that there exist $D \subseteq C$ such that

$$g[D] \subseteq A, \quad \mu(D) = \mu(A \setminus B), \quad \mu(D) \leq \mu(A \setminus D) = \mu(A \setminus g[D]).$$

Observe that there exists $h \in \mathrm{Aut}(A,\mu)$ such that $h(g(d)) = d$ for $d \in D$, and $x \in \mathrm{Aut}(A,\mu)$ such that $x^{-1}[A \setminus B] = D$. Then

$$xhgx^{-1}(a) = a$$

for every $a \in A \setminus B$, so

$$xhgx^{-1} \in \mathrm{Aut}(B,\mu),$$

and

$$g \in h^{-1}x^{-1}\mathrm{Aut}(B,\mu)x \leq \mathrm{Aut}(A,\mu)\mathrm{Aut}(B,\mu)\mathrm{Aut}(A,\mu).$$

On the other hand, if we have interchanged A and B, in the same manner we get

$$g \in \mathrm{Aut}(B,\mu)\mathrm{Aut}(A,\mu)\mathrm{Aut}(B,\mu). \blacksquare$$

Lemma 2.2. *Let $A, B \subseteq X$ be Borel sets such that $A \cap B$ has positive measure. Then there exists $n \in \mathbb{N}$ such that*

$$\mathrm{Aut}(A \cup B, \mu) = (\mathrm{Aut}(A, \mu)\mathrm{Aut}(B, \mu))^n.$$

Proof. If $\mu(B \setminus A) = 0$, then there is nothing to prove. Otherwise, fix $B_0 \subseteq B$ with $A \cap B \subseteq B_0$, and

$$2\mu(B_0 \setminus A) = \mu(A \cap B)$$

or, if there is no such B_0, take $B_0 = B$. Similarly, let $\bar{A}_0 \subseteq A$ be such that $A \cap B \subseteq \bar{A}_0$ and

$$2\mu(\bar{A}_0 \setminus B) = \mu(A \cap B)$$

or $\bar{A}_0 = A$. By Lemma 2.1,

$$\mathrm{Aut}(\bar{A}_0 \cup B_0, \mu) = (\mathrm{Aut}(\bar{A}_0, \mu)\mathrm{Aut}(B_0, \mu))^2 \le (\mathrm{Aut}(A, \mu)\mathrm{Aut}(B, \mu))^2,$$

so we can put $B_1 = \bar{A}_0 \cup B_0$, and find some $\bar{A}_1 \subseteq A$ such that

$$2\mu(\bar{A}_1 \setminus B_1) = \mu(\bar{A}_1 \cap B_1)$$

or $\bar{A}_1 = A$. After some finite number k of iterations of the above step applied to $\bar{A}_i$ and B_i, we will obtain $\bar{A}_k = A$. Then

$$\bar{A}_k \cup B_k = A \cup B_0,$$

and

$$\mathrm{Aut}(A \cup B_0, \mu) \le (\mathrm{Aut}(A, \mu)\mathrm{Aut}(B, \mu))^{2k}.$$

Now apply the same procedure to $A \cup B_0$ and B. Clearly, after a finite number of steps, we will obtain n such that

$$\mathrm{Aut}(A \cup B, \mu) \le (\mathrm{Aut}(A, \mu)\mathrm{Aut}(B, \mu))^n. \ ■$$

Theorem 2.3. *The group* $\mathrm{Aut}(X, \mu)$ *does not contain any non-trivial subgroups of index* $< 2^\omega$.

Proof. Put $X = S^1$, let μ be the Haar measure on S^1, and let $G = \mathrm{Aut}(X, \mu)$. Let $H \le G$ be a subgroup with $[G : H] < 2^\omega$, and let $\{B_n\}_{n\in\mathbb{N}}$ be a partition of S^1 into infinitely many half-open intervals. Put $G_n = \mathrm{Aut}(B_n, \mu)$, $P = \prod_n G_n$, that is, P is the direct product of the groups G_n, $n \in \mathbb{N}$. We can regard P as a subgroup of G, consisting of all the elements $g \in G$ that setwise fix every B_n, $n \in \mathbb{N}$.

We will show that $G_n \le H$ for some n.

If H_n is the projection of $H \cap P$ onto B_n, that is,

$$H_n = \{h \in H \cap P : \mathrm{supp}(h) \subseteq B_n\},$$

then $H \cap P \le \prod_n H_n$. However,

$$\prod [G_n : H_n] = \Big[P : \prod_n H_n\Big] \le [P : H \cap P] \le [G : H] < 2^\omega,$$

so $H_n = G_n$ for almost all n. In other words, the setwise stabilizer $H_{\{B_n\}}$ of B_n acts on B_n as the full automorphism group of (B_n, μ). Observe that then $H \cap G_n$ is normal in G_n: for every $g \in G_n$, there exists $g_0 \in H$ such that $g_0 \restriction B_n = g$. Therefore, $g_0 h g_0^{-1} \in H$ for $h \in H \cap G$, and

$$g h g^{-1} \restriction B_n = g_0 h g_0^{-1} \restriction B_n.$$

But $\mathrm{supp}(h) \subseteq B_n$, so actually $g_0 h g_0^{-1} \in H \cap G_n$. Moreover, since

$$[G_n : H \cap G_n] \leq [G : H] < 2^\omega,$$

$H \cap G_n$ is non-trivial. However, G_n is simple (see [3]), so $H \cap G_n = G_n$ for almost all n. Fix such an $n = n_0$, and put $B = B_{n_0}$.

Now, $[\mathrm{Rot}(S^1) : H \cap \mathrm{Rot}(S^1)] < 2^\omega$ in $\mathrm{Rot}(S^1)$, where $\mathrm{Rot}(S^1) \leq G$ stands for the group of all rotations of S^1, and $\mathrm{Rot}(S^1)$ has size 2^ω. Since the rational rotations form a countable subgroup of $\mathrm{Rot}(S^1)$, there exists an irrational rotation ϕ in H. It is well known that irrational rotations are topologically transitive, that is, for every irrational rotation $\phi \in \mathrm{Rot}(S^1)$ and $x \in S^1$, the set $\{\phi^n(x)\}_{n \in \mathbb{N}}$ is dense in S^1. Therefore, there exist natural numbers k, n such that

$$\phi^{ki}[B] \cap \phi^{k(i+1)}[B] \neq \emptyset \quad \text{ for } i < n,$$

and

$$\bigcup_{i \leq n} \phi^{ki}[B] = S^1.$$

By Lemma 2.2,

$$\mathrm{Aut}(\phi^{ki}[B] \cup \phi^{k(i+1)}[B], \mu) \leq H$$

for every $i < n$, and so

$$H \leq G \leq H. \ \blacksquare$$

2.1. Automatic continuity of $\mathrm{Aut}(X, \mu)$**.** In [1], the authors prove that $\mathrm{Aut}(X, \mu)$ has the automatic continuity property, that is, every homomorphism $\phi : \mathrm{Aut}([0, 1], \lambda) \to H$ into a separable group H is continuous (Theorem 6.2 in [1]). Their argument makes use of metric ample generics, a notion introduced in that paper. Below we sketch a proof which does not require this ingredient.

THEOREM 2.4 (Theorem 6.2 in [1]). *The group* $\mathrm{Aut}(X, \mu)$ *has the automatic continuity property.*

Proof. Let us assume that X is the Cantor set $2^{\mathbb{N}}$, and μ is the Lebesgue measure on $2^{\mathbb{N}}$.

First of all, it is known that in order to prove that a separable group G has the automatic continuity property, it suffices to prove that it has the Steinhaus property (see [8]). A group G has the *Steinhaus property* if there exists $n \in \mathbb{N}$ such that for every symmetric set $W \subseteq G$ whose countably many translates cover G, the set W^n contains a neighborhood of the identity.

It turns out that to show that $\mathrm{Aut}(2^{\mathbb{N}}, \mu)$ has the Steinhaus property, it suffices to show that $\mathrm{Aut}(2^{\mathbb{N}}, \mu)$, equipped with the uniform topology, has the Steinhaus property, and so does the group $\mathrm{Hom}(2^{\mathbb{N}}, \mu)$ of measure preserving homeomorphisms of $2^{\mathbb{N}}$, equipped with the uniform convergence topology.

To see this, write $G_0 = \mathrm{Aut}(2^{\mathbb{N}}, \mu)$ with the uniform topology, $H = \mathrm{Hom}(2^{\mathbb{N}}, \mu)$, and suppose that both G_0 and H have the Steinhaus property. Take a symmetric set $W \subseteq G_0$ such that $G_0 = \bigcup_n g_n W$ for some $\{g_n\} \subseteq G_0$. Then it is easy to see that there exists a countable sequence $\{h_n\} \subseteq H$ such that $H = \bigcup_n h_n (W')^2$, where $W' = W \cap H$. If H has the Steinhaus property, then there exists $k \in \mathbb{N}$ such that $U \subseteq (W')^k$, where U is some neighborhood of the identity in H. If G_0 has the Steinhaus property, then there exists $m \in \mathbb{N}$ such that $V \subseteq W^m$, where V is some neighborhood of the identity in the uniform topology. Therefore, there exists $n \in \mathbb{N}$ such that $UV \subseteq W^n$, and this n does not depend on the choice of W.

Now, it is well known that measure preserving transformations of $2^{\mathbb{N}}$ can be uniformly approximated by measure preserving homeomorphisms of $2^{\mathbb{N}}$. In other words, UV is open in the weak topology if U is a neighborhood of the identity in $\mathrm{Hom}(2^{\mathbb{N}}, \mu)$, and V is a neighborhood of the identity in $\mathrm{Aut}(2^{\mathbb{N}}, \mu)$ in the uniform topology. Thus W^n contains a neighborhood of the identity in the weak topology on $\mathrm{Aut}(2^{\mathbb{N}}, \mu)$.

The Steinhaus property for $\mathrm{Aut}(2^{\mathbb{N}}, \mu)$ with the uniform topology was proved in [7] (even though [7, Theorem 3.1] concerns certain subgroups of $\mathrm{Aut}(2^{\mathbb{N}}, \mu)$, its proof works verbatim for $\mathrm{Aut}(2^{\mathbb{N}}, \mu)$). Regarding $\mathrm{Hom}(2^{\mathbb{N}}, \mu)$, it follows from the fact that this group has ample generics, which was shown in [6]. ∎

REFERENCES

[1] I. Ben Yaacov, A. Berenstein and J. Melleray, *Polish topometric groups*, Trans. Amer. Math. Soc. 365 (2013), 3877–3897.

[2] J. D. Dixon, P. M. Neuman and S. Thomas, *Subgroups of small index in infinite symmetric groups*, Bull. London Math. Soc. 18 (1986), 580–586.

[3] A. Fathi, *Le groupe des transformations de* $[0, 1]$ *qui preservent la mesure de Lebesgue est un groupe simple*, Israel J. Math. 29 (1978), 302–308.

[4] S. Gao, *Invariant Descriptive Set Theory*, Chapman and Hall, 2008.

[5] A. Kechris, *Classical Descriptive Set Theory*, Springer, 1995.

[6] A. Kechris and C. Rosendal, *Turbulence, amalgamation, and generic automorphisms of homogeneous structures*, Proc. London Math. Soc. 94 (2007), 302–350.

[7] J. Kittrell and T. Tsankov, *Topological properties of full groups*, Ergodic Theory Dynam. Systems 30 (2010), 525–545.

[8] C. Rosendal and S. Solecki, *Automatic continuity of homomorphisms and fixed points on metric compacta*, Israel J. Math. 162 (2007), 349–371.

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