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SIDON SETS AND BOHR CLUSTER POINTS

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Abstract. It is shown that a Sidon set cannot have an integer cluster point in the Bohr topology.

1. Introduction and notations. The Bohr compactification of the integers $\mathbb{Z}$, $\beta\mathbb{Z}$, with the Bohr topology is defined as the compact dual group of the discrete circle group $\mathbb{T}$. The continuous complex valued functions on $\beta\mathbb{Z}$ when restricted to $\mathbb{Z}$ itself are the classical almost periodic functions on $\mathbb{Z}$. The Fourier transforms of the discrete measures on the dual circle group $\mathbb{T}$, B_d, is an algebra of continuous functions on $\beta\mathbb{Z}$ which is sup-norm dense in the algebra of all continuous functions on $\beta\mathbb{Z}$.

For E a subset of the integers $\mathbb{Z}$, $B(E)$ will denote the restrictions to E of the Fourier transforms of all regular Borel measures $M(\mathbb{T})$ on $\mathbb{T}$ in the quotient topology. $I(E)$ denotes all measures whose Fourier transforms vanish on E. If μ denotes a measure, then its quotient norm is defined as

$$\|\mu\|_{B(E)} = \inf\{\|\mu + \tau\| : \hat{\tau} = 0 \text{ on } E\},$$

and the infimum is taken over all finite Borel measures τ on T whose Fourier transform vanishes on the set E.

The subspaces of Fourier transforms of discrete measures and continuous measures when restricted to E will be denoted respectively by $B_d(E)$ and $B_c(E)$, each provided with the quotient norm topology. Discrete and continuous finite regular Borel measures on T will be denoted respectively with subscripts τ_d and τ_c.

DEFINITION 1.1. A set E of integers is a *Sidon set* if and only if $B(E) = \ell_\infty(E)$. The latter condition is well known to be equivalent to $A(E) = c_0(E)$, where $A(E)$ is the restriction to E of Fourier transforms of L_1 functions on the compact circle T.

Ramsey [R] has shown that if there is a Sidon set E having an integer cluster point with respect to the Bohr topology, there is another Sidon set (still called E) whose closure in the Bohr topology contains $\mathbb{Z}$ itself.

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2. Bohr cluster points

THEOREM 2.1. *If $E \subset \mathbb{Z}$ is a Sidon set, no integer m may be a cluster point of E in the Bohr topology.*

Proof. Our proof will assume the existence of a Sidon set E whose closure in the Bohr topology contains all of $\mathbb{Z}$. This assumption will lead to a contradiction.

The main result of [W] is the following: If E is a Sidon set, then $B_c(E) = B(E) = \ell_\infty(E)$. A consequence of these equalities is that the following quotient norms on $M(T)/I(E)$ are equivalent:

(1) $\|\hat{\mu}\|_{C(E)} = \sup_{n\in E} |\hat{\mu}(n)|$.
(2) $\|\mu\|_{B(E)} = \inf\{\|\nu\|\}$, where the infimum is taken over all measures ν whose transforms coincide with the transform of μ on E.
(3) $\|\mu\|_{B_c(E)} = \inf\{\|\nu_c\|\}$, where the infimum is taken over all continuous measures ν_c whose transforms coincide with the transform of μ on E.

It follows from the equivalence of the norms defined by (1) and by (3) that there is a positive constant K such that for an arbitrary discrete measure τ_d,

$$\|\nu_c\|_{B_c(E)} \le K \sup_{n\in E} |\hat{\nu}_c(n)| = K \sup_{n\in E} |\hat{\tau}_d(n)| \le K\|\tau_d\| \tag{2.1}$$

for $\hat{\nu}_c|_E = \hat{\tau}_d|_E$.

For any discrete measure τ_d and a measure ν such that $\hat{\nu}|_E = 0$ we may write $\|\tau_d + \nu\| = \|\tau_d + \nu_d\| + \|\nu_c\|$, where $\nu = \nu_d + \nu_c$. We now make a particular choice by setting $\nu_d = -\tau_d$ and choose ν_c to satisfy $\hat{\nu}_c|_E = \hat{\tau}_d|_E$. It follows that

$$\|\tau_d\|_{B(E)} \le \|\nu_c\|. \tag{2.2}$$

Since this inequality holds for any ν_c such that $\hat{\nu}_c|_E = \hat{\tau}_d|_E$ we obtain

$$\|\tau_d\|_{B(E)} \le \|\nu_c\|_{B_c(E)}. \tag{2.3}$$

The usual quotient norm on discrete measures defined by $\|\tau_d\|_{B_d(E)} = \inf \|\tau_d + \nu_d\|$ is equal to the measure norm $\|\tau_d\|$, since by assumptions there are no non-zero measures ν_d whose transforms vanish on E, i.e. $B_d(E) = B_d(\mathbb{Z})$.

The inclusion mapping $B_d(\mathbb{Z}) \hookrightarrow B_c(E)$ is one-to-one, and in view of (2.1) it is continuous. However, the image of $B_d(\mathbb{Z})$ in $B_c(E)$ may not be closed. In fact, since all Fourier transforms of discrete measures are sup-norm dense in AP, the sup-norm closure of the image of $B_d(\mathbb{Z})$ in $B_c(E)$ contains the restriction to E of all AP (almost periodic) functions. If f is an AP function whose restriction to E is not in the image of $B_d(E)$, there is a sequence $\{\nu_c^n\}$ of continuous measures such that for each n there exists a discrete measure τ_d^n such that $\hat{\tau}_d^n|_E = \hat{\nu}_c^n|_E$ and a continuous measure ν_c^0 such that $\hat{\nu}_0|_E = f|_E$ and $\|\nu_c^n - \nu_c^0\|_{B_c(E)} \to 0$. The sequence $\{\tau_d^n\}$ of discrete measures may not converge in measure, but by inequality (2.3) it does converge to some measure μ in the $\| \ \|_{B(E)}$-norm and satisfies $\hat{\mu}|_E = \hat{\nu}_c^0|_E = f|_E$.

A result of Eberlein [E] is that any weakly almost periodic (WAP) function F has a unique representation as the sum of an almost periodic (AP) function plus a WAP function whose mean square is equal to 0. The Bohr compactification of the integers, $\beta\mathbb{Z}$, may be regarded as a subset of the WAP compactification of the integers, $w\mathbb{Z}$. This follows from the following observation. The ideal of all continuous functions on $\beta\mathbb{Z}$ vanishing at the point $x_0 \in \beta\mathbb{Z}$, I_{x_0}, is a maximal ideal in $C(\beta\mathbb{Z})$. By Eberlein's theorem [E], $I_{x_0} \oplus \mathrm{WAP}_0$ is a maximal ideal in $C(w\mathbb{Z})$ where WAP_0 denotes the ideal of WAP functions whose mean square is equal to zero. The compact abelian group $\beta\mathbb{Z}$ enjoys a Haar measure m supported on $\beta\mathbb{Z} \setminus \mathbb{Z}$. The mean square value of a function $f \in \mathrm{AP}$ may be expressed as the integral of $|f|^2$ with respect to m. A function h in $C(w\mathbb{Z})$ whose mean square is zero must vanish on $\beta\mathbb{Z} \setminus \mathbb{Z}$.

For any given AP function f on $\mathbb{Z}$ we have shown that there exists a continuous measure ν_c whose Fourier transform satisfies $\hat{\nu}_c|_E = f|_E$. For simplicity take the AP function f to be identically 1 on $\mathbb{Z}$. A consequence of the above paragraph and the fact that E is dense in $\beta\mathbb{Z}$ is that the only almost periodic extension to $\mathbb{Z}$ of f restricted to E is f itself, and therefore any WAP extension of f from E must have the form $f+h$ where h has mean square zero. It follows that $\hat{\nu}_c$ and $f+h$ must have identical mean square values. The mean square value of $\hat{\nu}_c$ is equal to 0; that of $f+h$ is equal to 1. This contradiction completes the proof. ∎

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