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ON COUNTABLE FAMILIES OF SETS WITHOUT THE BAIRE PROPERTY

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Abstract. We suggest a method of constructing decompositions of a topological space X having an open subset homeomorphic to the space $(\mathbb{R}^n, \tau)$, where n is an integer ≥ 1 and τ is any admissible extension of the Euclidean topology of $\mathbb{R}^n$ (in particular, X can be a finite-dimensional separable metrizable manifold), into a countable family $\mathcal{F}$ of sets (dense in X and zero-dimensional in the case of manifolds) such that the union of each non-empty proper subfamily of $\mathcal{F}$ does not have the Baire property in X.

1. Introduction. Recall that a set A of a topological space X is said to have the *Baire property in* X if $A = (O \setminus M) \cup N$, where O is an open set of X and M, N are meager sets of X. Let 2^X be the family of all subsets of X, and $\mathcal{B}_p(X)$ the subfamily of 2^X consisting of sets with the Baire property. It is well known that the family $\mathcal{B}_p(X)$ is a σ-algebra of sets. However, in the case when $\mathcal{B}_p^C(X) = 2^X \setminus \mathcal{B}_p(X) \neq \emptyset$, the union of two sets from $\mathcal{B}_p^C(X)$ does not need to belong to $\mathcal{B}_p^C(X)$.

In [Ch] (see also [ChN] for generalizations) it was shown that the union of finitely many Vitali sets of the real line $\mathbb{R}$ (see [V]) contains no set of type $O \setminus M$, where O is a non-empty open set and M a meager one. Since each Vitali set is not meager, this easily implies that such unions do not have the Baire property. Let us note that these facts cannot be extended to all countable unions of Vitali sets. It is easy to see that a set A in $\mathbb{R}$ is the union of a (countable) family of Vitali sets iff $|A \cap (x + \mathbb{Q})| \neq \emptyset$ for each $x \in \mathbb{R}$, where $\mathbb{Q}$ is the set of rational numbers, i.e. A contains a Vitali set. Moreover, such a family can be chosen infinite and disjoint iff $|A \cap (x + \mathbb{Q})| = \aleph_0$ for each $x \in \mathbb{R}$. This implies that every element of $\mathcal{B}_p(\mathbb{R})$ with non-empty interior (in particular, any non-empty open set) is the union of an infinite countable disjoint family of Vitali sets.

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In [Ch] the following result was proved:

If S is a Vitali set of $\mathbb{R}$ and $\mathcal{A}$ is any non-empty proper subset of $\mathbb{Q}$ then

$$U(S,\mathcal{A}) = \bigcup\{r+S : r\in\mathcal{A}\} \in \mathcal{B}_p^C(\mathbb{R}).$$

However, the set $U(S,\mathcal{A})$ can contain the difference $O\setminus M$, where O is a non-empty open subset of $\mathbb{R}$ and M is meager. This happens iff

$$\mathrm{Cl}_{\mathbb{R}}\bigcup\{(r+S)'' : r\in\mathbb{Q}\setminus\mathcal{A}\} \neq \mathbb{R}$$

(we recall the operation $(\cdot)''$ in the next section).

Note that for a Vitali set S such that S is dense in $\mathbb{R}$ and $S=S''$ (see examples in [Ch]) we have $\mathrm{Cl}_{\mathbb{R}}\bigcup\{(r+S)'' : r\in\mathbb{Q}\setminus\mathcal{A}\} = \mathbb{R}$ for each non-empty proper subset $\mathcal{A}$ of $\mathbb{Q}$. On the other hand for any Vitali set $S\subset(-1,1)$ and $\mathcal{A}=\mathbb{Q}\cap(-2,2)$ we have $U(S,\mathcal{A})\supset(-1,1)$.

In this paper we suggest a method of constructing decompositions of a topological space X having an open subset homeomorphic to the space $(\mathbb{R}^n,\tau)$, where n is an integer ≥ 1 and τ is any admissible extension (see Section 3 for the definition) of the Euclidean topology of $\mathbb{R}^n$ (in particular, X can be a finite-dimensional separable metrizable manifold) into a countable family $\mathcal{F}$ of sets (dense in X and zero-dimensional in the case of manifolds) such that the union of each non-empty proper subfamily of $\mathcal{F}$ does not have the Baire property in X.

For the notions we refer to [E1] and [Ku].

2. Auxiliary results. We will use some notations from [Ch].

For each non-meager set R of a topological space X, $O_R=\mathrm{Int}_X(\mathrm{Cl}_X R)$, $R'=\{x\in R\cap O_R :$ there is an open neighborhood V of x such that $V\cap R$ is meager$\}$ and $R''=(R\cap O_R)\setminus R'$.

Let us observe that by ([Ku, Theorem 1, p. 87]) the set R' is meager in X. This implies, in particular, that the set R'' is non-meager in X.

Remark 2.1. Recall [Ch, Theorem 2.1]:

Let X be a hereditarily Lindelöf topological space, $\mathcal{A}$ be a non-empty set with $|\mathcal{A}|\leq\aleph_0$ and $R(\alpha)$ a non-meager subset of X for each $\alpha\in\mathcal{A}$. Then $U=\bigcup\{R(\alpha):\alpha\in\mathcal{A}\}\in\mathcal{B}_p(X)$ iff $O_{R''(\alpha)}\setminus U$ is meager in X for each $\alpha\in\mathcal{A}$.

Now we note that by the two sentences before this remark the equivalence above holds in any topological space X. Moreover, one can see from the proof that the necessity part is valid for any set $\mathcal{A}$.

Proposition 2.2. *Let X be a topological space, $\mathcal{A}$ a set with $|\mathcal{A}|\geq 2$ and for each $\alpha\in\mathcal{A}$, X_α a non-meager subset of X. Assume also that*

(i) *$X_{\alpha_1}\cap X_{\alpha_2}=\emptyset$ iff $\alpha_1\neq\alpha_2$, and*
(ii) *$\bigcup_{\alpha\in\mathcal{A}} O_{X''_\alpha}$ is connected.*

Then for any non-empty proper subset $\mathcal{A}'$ of $\mathcal{A}$ the set $Y = \bigcup_{\alpha\in\mathcal{A}'} X_\alpha$ does not have the Baire property.

Proof. We follow the idea of proof of [Ch, Theorem 4.1]. Assume that Y has the Baire property. Then by Remark 2.1 the set $O_{X''_\alpha} \setminus Y$ is meager for each $\alpha \in \mathcal{A}'$. Further we will need the following statement.

Claim. *For any $\alpha_1 \in \mathcal{A}'$ and $\alpha_2 \in \mathcal{A} \setminus \mathcal{A}'$ we have $O_{X''_{\alpha_1}} \cap O_{X''_{\alpha_2}} = \emptyset$.*

Proof. Let $V = O_{X''_{\alpha_1}} \cap O_{X''_{\alpha_2}} \neq \emptyset$. It follows from [Ch, Corollary 2.1(iii)] that the set $V \cap X''_{\alpha_2}$ is non-meager. Since $X''_{\alpha_2} \subset X_{\alpha_2}$ and $X_{\alpha_2} \cap Y = \emptyset$ by the condition (i), we have $V \cap X''_{\alpha_2} \subset O_{X''_{\alpha_1}} \setminus Y$, where $O_{X''_{\alpha_1}} \setminus Y$ is supposed to be meager. We have a contradiction which proves the Claim. ∎

Put $U_1 = \bigcup_{\alpha\in\mathcal{A}'} O_{X''_\alpha}$ and $U_2 = \bigcup_{\alpha\in\mathcal{A}\setminus\mathcal{A}'} O_{X''_\alpha}$. Note that the sets U_1, U_2 are non-empty, open and by the Claim they are disjoint. So the set $\bigcup_{\alpha\in\mathcal{A}} O_{X''_\alpha} = U_1 \cup U_2$ is disconnected. We have a contradiction with (ii). ∎

Remark 2.3. We notice that the condition (ii) from Proposition 2.2 cannot be erased. On the other hand the condition is not necessary.

(i) Let X be the subspace $\{0, 1\}$ of the real line $\mathbb{R}$ and $\mathcal{A} = \{1, 2\}$. Set $X_1 = \{0\}$ and $X_2 = \{1\}$. Note that $O_{X''_1} = X_1$ and $O_{X''_2} = X_2$. Hence $O_{X''_1} \cap O_{X''_2} = \emptyset$ and the sets X_1, X_2 are open in X.
(ii) Let X be the real line $\mathbb{R}$, $\mathcal{A} = \{1, 2\}$ and S a Vitali set of $\mathbb{R}$ such that $S \subset (0, 1)$. Set $X_1 = S$ and $X_2 = 2 + S$. Note that $O_{X''_1} \cap O_{X''_2} = \emptyset$ and the sets X_1, X_2 do not have the Baire property.

Let $\mathcal{H}(X)$ be the group of homeomorphisms of the space X. The following statement is trivial.

Lemma 2.4. *Let $h \in \mathcal{H}(X)$ and $A \subset X$. Then*

(i) *A is meager iff $h(A)$ is meager;*
(ii) *if A is non-meager then $h(O_A) = O_{h(A)}$, $h(A') = (h(A))'$ and $h(A'') = (h(A))''$.*

Proposition 2.5. *Let X be a topological space, $\mathcal{H}^*$ a non-empty subset of $\mathcal{H}(X)$ with $|\mathcal{H}^*| \geq 2$ and A a non-meager subset of X. Assume also that*

(i) *for any elements $h_1 \neq h_2$ of $\mathcal{H}^*$, $h_1(A) \cap h_2(A) = \emptyset$, and*
(ii) *$\bigcup_{h\in\mathcal{H}^*} h(O_{A''})$ is connected.*

Then for any non-empty proper subset $\mathcal{H}'$ of $\mathcal{H}^*$ the set $\bigcup_{h\in\mathcal{H}'} h(A)$ does not have the Baire property.

Proof. Since for each $h \in \mathcal{H}*$ the set $h(A)$ is non-meager and $h(O_{A''}) = O_{(h(A))''}$ by Lemma 2.4, the statement follows from Proposition 2.2. ∎

PROPOSITION 2.6. *Let X be a topological space of the second cathegory, $\mathcal{H}^*$ a non-empty countable subset of $\mathcal{H}(X)$ with $|\mathcal{H}^*| \geq 2$ and A a subset of X. Assume also that*

(i) *for any elements $h_1 \neq h_2$ of $\mathcal{H}^*$, $h_1(A) \cap h_2(A) = \emptyset$,*
(ii) *$X \setminus \bigcup_{h\in\mathcal{H}^*} h(A)$ is meager, and*
(iii) *$\bigcup_{h\in\mathcal{H}^*} h(O_{A''})$ is connected.*

Then for any non-empty proper subset $\mathcal{H}'$ of $\mathcal{H}^*$ the set $\bigcup_{h\in\mathcal{H}'} h(A)$ does not have the Baire property.

Proof. Since $X \setminus \bigcup_{h\in\mathcal{H}^*} h(A)$ is meager and the space X is of second category, the set A is non-meager. Applying Proposition 2.5 we get the statement. ∎

Let Q be a countable dense subgroup of the additive group of the real numbers. One can consider the Vitali construction (see [V]) with the group Q instead of the group $\mathbb{Q}$ of rational numbers (cf. [K]). The analogue of a Vitali set with the respect to Q will be called a *Vitali Q-selector of* $\mathbb{R}$.

EXAMPLE 2.7 ([Ch, Theorem 4.1 for $Q = \mathbb{Q}$]). Let $X = \mathbb{R}$, H^* be the group of translations of $\mathbb{R}$ by numbers from Q and A a Vitali Q-selector. Note that

(i) for any elements $h_1 \neq h_2$ of $\mathcal{H}^*$, $h_1(A) \cap h_2(A) = \emptyset$;
(ii) $\mathbb{R} \setminus \bigcup_{h\in\mathcal{H}^*} h(A) = \emptyset$;
(iii) $\bigcup_{h\in\mathcal{H}^*} h(O_{A''}) = \mathbb{R}$ is connected. ∎

EXAMPLE 2.8 ([Ch, Remark 4.2 for $Q = \mathbb{Q}$]). Let $X = \mathbb{R}^n$, H^* be the group of translations of $\mathbb{R}^n$ by vectors with all coordinates from Q and A a *Vitali Q-selector of* $\mathbb{R}^n$, that is, $A = \prod_{i=1}^n A_i$, where A_i is a Vitali Q-selector of $\mathbb{R}$ for each $i \leq n$. Note that

(i) for any elements $h_1 \neq h_2$ of $\mathcal{H}^*$, $h_1(A) \cap h_2(A) = \emptyset$;
(ii) $\mathbb{R}^n \setminus \bigcup_{h\in\mathcal{H}^*} h(A) = \emptyset$;
(iii) $\bigcup_{h\in\mathcal{H}^*} h(O_{A''}) = \mathbb{R}^n$ is connected. ∎

3. A method of constructing countable families of sets without the Baire property. Let τ_1 be a topology on a set X.

Recall [ChN, Definition 3.1] that a topology τ_2 on X is said to be an *admissible extension of* τ_1 if

(i) $\tau_1 \subset \tau_2$, and
(ii) τ_1 is a π-base for τ_2, i.e. for each non-empty element O of τ_2 there is a non-empty element V of τ_1 which is a subset of O.

Let us denote the closure (resp. the interior or the boundary) of a subset A of the set X in the space (X, τ_i) by $\mathrm{Cl}_{\tau_i} A$ (resp. $\mathrm{Int}_{\tau_i} A$ or $\mathrm{Bd}_{\tau_i} A$), where $i = 1, 2$.

Lemma 3.1. *Let X be a set, τ_1 and τ_2 topologies on X such that τ_2 is an admissible extension of τ_1, and O a non-empty element of τ_2. Then the set $A = O \setminus \operatorname{Int}_{\tau_1} O$ is nowhere dense in the space (X, τ_1) (in particular, A is a meager set in (X, τ_1)).*

Proof. Put $V = \operatorname{Int}_{\tau_1} O$ and note that $V \neq \emptyset$.

Claim. $\operatorname{Cl}_{\tau_1} V \supset O$.

Proof. Assume that $W = O \setminus \operatorname{Cl}_{\tau_1} V \neq \emptyset$. Since τ_2 is an admissible extension of τ_1 and $W \in \tau_2$, there is $\emptyset \neq U \in \tau_1$ such that $U \subset W \subset O$. It is easy to see that U must be a subset of V. We have a contradiction which proves the Claim. ∎

It follows from the Claim that $\operatorname{Bd}_{\tau_1} V \supset A$. Hence, the set A is nowhere dense in the space (X, τ_1). ∎

Lemma 3.2. *Let X be a set, τ_1 and τ_2 topologies on X such that τ_2 is an admissible extension of τ_1, and $A \subset X$. Assume also that A has the Baire property in the space (X, τ_2). Then A has the Baire property in (X, τ_1).*

Proof. Suppose that $A = (O \setminus M) \cup N$, where O is open in (X, τ_2) and M, N are meager in (X, τ_2). Note that by [ChN, Proposition 3.4] the sets M, N are also meager in (X, τ_1). Moreover, by Lemma 3.1 the set $O \setminus \operatorname{Int}_{\tau_1} O$ is meager in (X, τ_1). Observe that

$$A = (\operatorname{Int}_{\tau_1} O \setminus M) \cup (((O \setminus \operatorname{Int}_{\tau_1} O) \setminus M) \cup N).$$

Hence, A has the Baire property in the space (X, τ_1). ∎

Let n be a positive integer. Denote by τ_S (resp. τ_0) the Sorgenfrey topology (resp. the Euclidean topology) on the set $\mathbb{R}$ of real numbers and by τ_S^n (resp. τ_0^n) the product $\prod_{i=1}^n (\tau_S)_i$ (resp. $\prod_{i=1}^n (\tau_0)_i$), where $(\tau_S)_i = \tau_S$ (resp. $(\tau_0)_i = \tau_0$) for each $i \leq n$.

Let now k, m be non-negative integers. Note that the topology $\tau_S^k \times \tau_0^m$ on the set $\mathbb{R}^{k+m}$ is an admissible extension of the Euclidean topology τ_0^{k+m} on $\mathbb{R}^{k+m}$. Let us also observe that if $k \geq 1$ then $(\mathbb{R}^{k+m}, \tau_S^k \times \tau_0^m)$ is disconnected and if $k \geq 2$ then $(\mathbb{R}^{k+m}, \tau_S^k \times \tau_0^m)$ is not normal.

Applying Example 2.8 and Lemma 3.2 we get the following statement.

Proposition 3.3. *Let Q be a countable dense subgroup of the additive group of real numbers. If S is a Vitali Q-selector of $\mathbb{R}^n$ for some integer $n \geq 1$ and $\mathcal{A}$ is any non-empty proper subset of Q^n then $\bigcup\{\overline{r} + S : \overline{r} \in \mathcal{A}\} \in \mathcal{B}_p^C((\mathbb{R}^n, \tau))$, where τ is any admissible extension of τ_0^n.*

Remark 3.4. For the case $n = 1$, $\tau = \tau_S$ and $Q = \mathbb{Q}$ the statement was proved in [Ch, Corollary 4.1].

LEMMA 3.5. *Let Y be a non-empty open subset of a space X. Then*

(i) *if M is a nowhere dense subset of X then $M \cap Y$ is a nowhere dense subset of Y;*
(ii) *if M is a meager subset of X then $M \cap Y$ is a meager subset of Y.*

Proof. (i) Let $V = \mathrm{Int}_Y \mathrm{Cl}_Y(M \cap Y) \neq \emptyset$. Note that $V \subset \mathrm{Int}_Y(\mathrm{Cl}_X M \cap Y) \subset \mathrm{Int}_X(\mathrm{Cl}_X M \cap Y) \subset \mathrm{Int}_X \mathrm{Cl}_X M$. Hence, $\mathrm{Int}_X \mathrm{Cl}_X M \neq \emptyset$.

(ii) follows evidently from (i). ∎

LEMMA 3.6. *Let X be a space, Y a non-empty open subset of X and $A \subset X$. Assume also that A has the Baire property in X. Then $A \cap Y$ has the Baire property in Y.*

Proof. Let $A = (O \setminus M) \cup N$, where O is open in X and M, N are meager in X. Note that $A \cap Y = ((O \cap Y) \setminus (M \cap Y)) \cup (N \cap Y)$ and $O \cap Y$ is open in Y and $M \cap Y, N \cap Y$ are meager in Y (by Lemma 3.5). ∎

REMARK 3.7. Let us note that the openness of the set Y in the space X in the lemmas is essential. Indeed, let X be the Euclidean plane with the x, y-axes, Y the x-axis and A a Vitali set of Y. Note that Y is nowhere dense in X. This implies that A is also nowhere dense in X. But A does not have the Baire property in Y.

THEOREM 3.8. *Let X be a space and Y an open subset of X which is homeomorphic to the space $(\mathbb{R}^n, \tau)$ for some admissible extension τ of the Euclidean topology τ_0^n, where n is a positive integer. Then there is an infinite disjoint countable family $\mathcal{F}$ of sets in X such that*

(i) *$\bigcup \mathcal{F} = X$, and*
(ii) *for each non-empty proper subfamily $\mathcal{F}'$ of $\mathcal{F}$ the set $\bigcup \mathcal{F}'$ does not have the Baire property in X.*

Moreover:

(a) *if the set Y is dense in the space X or $Z = X \setminus \mathrm{Cl}_X Y \neq \emptyset$ and there is a countable infinite disjoint family $\mathcal{H} = \{H_i\}_{i=1}^{\infty}$ of sets dense in Z then each element of $\mathcal{F}$ can be chosen dense in X;*
(b) *if the space X is separable metrizable, $\tau = \tau_0^n$ and the set $X \setminus Y$ is countable-dimensional then each element of $\mathcal{F}$ can be chosen zero-dimensional.*

Proof. By Proposition 3.3 there exists an infinite disjoint countable family $\mathcal{G} = \{Y_i\}_{i=1}^{\infty}$ of sets such that $\bigcup \mathcal{G} = Y$, and for each non-empty proper subfamily $\mathcal{G}'$ of $\mathcal{G}$ the set $\bigcup \mathcal{G}'$ does not have the Baire property in Y. Then put $X_1 = Y_1 \cup (X \setminus Y)$ and $X_i = Y_i$, $i \geq 2$. Let us notice that the countable family $\mathcal{F} = \{X_i\}_{i=1}^{\infty}$ of sets is also disjoint and $\bigcup \mathcal{F} = X$. Since for each non-empty proper subfamily $\mathcal{F}'$ of $\mathcal{F}$ there is a non-empty proper subfamily

$\mathcal{G}'$ of $\mathcal{G}$ such that $(\bigcup \mathcal{F}') \cap Y = \bigcup \mathcal{G}'$, it follows from Lemma 3.6 that the set $\bigcup \mathcal{F}'$ does not have the Baire property in X.

Let us now prove (a). Observe that there is a Vitali set of $\mathbb{R}$ which is dense in $\mathbb{R}$ (see [Ch, Proposition 3.3]), which implies the existense of a Vitali set of $\mathbb{R}^n$ (see Example 2.8 for the definition) which is dense in $\mathbb{R}^n$. So each set of the family $\mathcal{G}$ can be chosen dense in the subspace Y of X. Furthermore, if Y is dense in X then each element of the family $\mathcal{F}$ defined above will be dense in X. Assume now that $Z = X \setminus \mathrm{Cl}_X Y \neq \emptyset$ and there is a countable infinite disjoint family $\mathcal{H} = \{H_i\}_{i=1}^{\infty}$ of sets dense in Z. In this case put $X_1 = Y_1 \cup ((X \setminus Y) \setminus \bigcup_{i=2}^{\infty} H_i)$ and $X_i = Y_i \cup H_i$, $i \geq 2$. Let us notice that the countable family $\mathcal{F} = \{X_i\}_{i=1}^{\infty}$ of sets is disjoint and dense in X. Moreover, $\bigcup \mathcal{F} = X$. Since for each non-empty proper subfamily $\mathcal{F}'$ of $\mathcal{F}$ there is a non-empty proper subfamily $\mathcal{G}'$ of $\mathcal{G}$ such that $(\bigcup \mathcal{F}') \cap Y = \bigcup \mathcal{G}'$, it follows from Lemma 3.6 that $\bigcup \mathcal{F}'$ does not have the Baire property in X.

To prove (b), recall (cf. [E2]) that $X \setminus Y = \bigcup_{i=1}^{\infty} Z_i$, where the sets $Z_1, Z_2, \ldots$ are disjoint and $\dim Z_i = 0$ for each i. Put $X_i = Y_i \cup Z_i$ for each $i \geq 1$. Since every set of the family $\mathcal{G}$ is also zero-dimensional it follows from the sum theorem for the dimension dim that the sets $X_1, X_2, \ldots$ are zero-dimensional. ∎

Remark 3.9. Note that in each separable metrizable space each of whose open non-empty subsets is uncountable there is an infinite countable disjoint family consisting of dense sets. In fact, let $\mathcal{B} = \{U_i\}_{i=1}^{\infty}$ be a countable base for the space. For each integer $i \geq 1$ choose in U_i a countable infinite set $A_i = \{a_j^i : j \geq 1\}$ such that $A_i \cap \bigcup_{j<i} A_j = \emptyset$. Put now $H_j = \{a_j^i : i \geq 1\}$, $j \geq 1$, and note that the sets H_j, $j \geq 1$, are dense in the space and disjoint.

Corollary 3.10. *Let X be an n-dimensional separable metrizable manifold for some positive integer n. Then there exists an infinite disjoint countable family $\mathcal{F}$ of zero-dimensional, dense in X sets such that*

(i) $\bigcup \mathcal{F} = X$, *and*
(ii) *for each non-empty proper subfamily $\mathcal{F}'$ of $\mathcal{F}$ the set $\bigcup \mathcal{F}'$ does not have the Baire property in X.*

Proof. Let Y be a subset of X which is homeomorphic to $\mathbb{R}^n$, and let Y_i, $i \geq 1$, be a family of dense zero-dimensional subsets of Y, which can be obtained from Example 2.8 (see also the proof of Theorem 3.8(a)). Set $B = \mathrm{Bd}_X Y$ and $V = X \setminus \mathrm{Cl}_X Y$.

Assume first that $V = \emptyset$, and note that $\dim B = k < n$. Let us decompose the set B into $k+1$ disjoint zero-dimensional subsets B_i, $i \leq k+1$. Put now $X_i = Y_i \cup B_i$, $i \leq k+1$, and $X_i = Y_i$, $i \geq k+2$. Observe that the sets X_i, $i \geq 1$, satisfy the corollary.

Assume now that $V = X \setminus \mathrm{Cl}_X Y \neq \emptyset$. Observe that for each point $x \in V$ there is an open neighborhood O_x of x which is homeomorphic to $\mathbb{R}^n$. Choose in V a countable base $\mathcal{B} = \{B_i : i \geq 1\}$ for open sets with $(n-1)$-dimensional boundaries. Set $V_0 = V \setminus \bigcup_{i=1}^{\infty} \mathrm{Bd}_V(B_i)$. Notice that $\dim V_0 = 0$ and each open non-empty subset of V_0 is uncountable. Decompose the set $\bigcup_{i=1}^{\infty} \mathrm{Bd}_V(B_i)$ into n zero-dimensional disjoint sets $V_1, \ldots, V_n$. Note that $V_0, \ldots, V_n$ are dense in V. By Remark 3.9 we can decompose V_0 into sets V_0^i, $i \geq 1$, dense in V_0. Observe that V_0^i, $i \geq 1$, are dense in V and zero-dimensional. Now the argument can be finished as in the first case. ∎

4. Concluding remarks. It is known that no Bernstein set of $\mathbb{R}$ has the Baire property. We will show this fact with the help of Proposition 2.2.

Recall that a subset A of the real line $\mathbb{R}$ is called a *Bernstein set* if for each Cantor set $C \subset \mathbb{R}$ we have $A \cap C \neq \emptyset$ and $(\mathbb{R} \setminus A) \cap C \neq \emptyset$.

LEMMA 4.1. *For each meager subset M of $\mathbb{R}$ there exists a Cantor set C such that $M \cap C = \emptyset$. In particular, each Bernstein set B of $\mathbb{R}$ is non-meager and $O_{B''} = \mathbb{R}$.*

Proof. We can suppose that $M = \bigcup_{i=1} M_i$, where M_i is nowhere dense and closed in $\mathbb{R}$ for each i. Hence, the set $N = \mathbb{R} \setminus M$ is topologically complete.

If N contains a non-degenerate interval $[a, b]$ then N contains a Cantor set. Otherwise, N is zero-dimensional and nowhere locally compact. This implies that N is homeomorphic to the space $\mathbb{P}$ of irrational numbers (cf. [vM]). Hence, N must also contain a Cantor set.

Since $B \setminus B''$ is meager (see [Ch, Proposition 2.1]) and $B'' \subset B$, the set B'' must be dense in $\mathbb{R}$ by the main statement of this lemma. Hence $O_{B''} = \mathbb{R}$. ∎

PROPOSITION 4.2. *No Bernstein set of $\mathbb{R}$ has the Baire property.*

Proof. Let M be a Bernstein set of $\mathbb{R}$. Note that $\mathbb{R} \setminus M$ is also a Bernstein set and the sets M, $\mathbb{R} \setminus M$ are disjoint and non-meager. Moreover, $O_{M''} = O_{(\mathbb{R} \setminus M)''} = \mathbb{R}$. By Proposition 2.2 the set M does not have the Baire property. ∎

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