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THE NUMBER OF COMPLETE EXCEPTIONAL SEQUENCES FOR A DYNKIN ALGEBRA

BY

MUSTAFA OBAID, KHALID NAUMAN, WAFA S. M. AL-SHAMMAKH, WAFAA FAKIEH and CLAUS MICHAEL RINGEL (Jeddah)

Abstract. The Dynkin algebras are the hereditary artin algebras of finite representation type. The paper determines the number of complete exceptional sequences for any Dynkin algebra. Since the complete exceptional sequences for a Dynkin algebra of Dynkin type Δ correspond bijectively to the maximal chains in the lattice of non-crossing partitions of type Δ, the calculations presented here may also be considered as a categorification of the corresponding result for non-crossing partitions.

1. Introduction. We consider Dynkin algebras Λ, that is, hereditary artin algebras of finite representation type. Note that the indecomposable Λ-modules correspond bijectively to the positive roots of a Dynkin diagram $\Delta(\Lambda)$; such a diagram is the disjoint union of connected diagrams and the connected Dynkin diagrams are of the form $\mathbb{A}_n, \mathbb{B}_n, \ldots, \mathbb{G}_2$. Let us remark that the vertices i of $\Delta(\Lambda)$ correspond bijectively to the simple Λ-modules, there is an edge between two vertices if and only if there is a non-trivial extension between the corresponding simple modules (in one of the two possible directions), and the lacing (in the cases $\mathbb{B}_n, \mathbb{C}_n, \mathbb{F}_4, \mathbb{G}_2$) records the relative size of the endomorphism rings of the simple modules (see [DR1] or [DR2]). We call Λ a *Dynkin algebra* of type $\Delta(\Lambda)$, the number of simple Λ-modules will be called the *rank* of Λ (let us stress that when we refer to the number of modules of some kind or the number of sequences of modules, we mean of course the number of isomorphism classes).

Given a Dynkin algebra Λ, an *exceptional sequence* for Λ is a sequence $(M_1, \ldots, M_t)$ of indecomposable Λ-modules such that $\mathrm{Hom}(M_i, M_j) = 0 = \mathrm{Ext}^1(M_i, M_j)$ for $i > j$. The cardinality of an exceptional sequence is bounded by the rank n of Λ, and exceptional sequences of cardinality n are said to be *complete*. Any exceptional sequence $(M_1, \ldots, M_t)$ can be extended to a

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complete exceptional sequence $(M_1, \dots, M_n)$; in case $t = n-1$, the extension is unique (for all these assertions, see [CB] and [R2]).

Let $e(\Lambda)$ be the number of complete exceptional sequences for the Dynkin algebra Λ. In case Λ is the path algebra of a quiver, the number $e(\Lambda)$ has been determined by Seidel [Se] in 2001. The aim of this note is to finalize these investigations by dealing also with the Dynkin diagrams which are not simply laced. There are direct connections between the representation theory of a Dynkin algebra Λ and the lattice L of non-crossing partitions of type $\Delta(\Lambda)$ which we will outline at the end of the introduction. In particular, the complete exceptional sequences for Λ correspond bijectively to the maximal chains in L. Thus, our calculations may also be considered as a categorification of the corresponding result for L.

As we will see, the number $e(\Lambda)$ only depends on $\Delta = \Delta(\Lambda)$, thus we may write $e(\Delta)$ instead of $e(\Lambda)$. Also, the shuffle lemma presented in Section 2 shows that it is sufficient to look at connected Dynkin diagrams Δ.

The following table exhibits the numbers $e(\Delta)$ for any connected Dynkin diagram Δ:

Δ	$\mathbb{A}_n$	$\mathbb{B}_n, \mathbb{C}_n$	$\mathbb{D}_n$	$\mathbb{E}_6$	$\mathbb{E}_7$	$\mathbb{E}_8$	$\mathbb{F}_4$	$\mathbb{G}_2$
$e(\Delta)$	$(n+1)^{n-1}$	n^n	$2(n-1)^n$	$2^9 \cdot 3^4$	$2 \cdot 3^{12}$	$2 \cdot 3^5 \cdot 5^7$	$2^4 \cdot 3^3$	$2 \cdot 3$

It seems to be of interest that the numbers $e(\Delta)$ have few different prime factors, all of them being rather small. Using the table, one easily verifies the following remarkable formula:

$$\boxed{e(\Delta) = \frac{n!\, h(\Delta)^n}{|W(\Delta)|}}$$

where $W(\Delta)$ is the Weyl group of type Δ, and $h(\Delta)$ the corresponding Coxeter number. Here are the numbers in question, as given, for example, in the appendix of [B]:

Δ	$\mathbb{A}_n$	$\mathbb{B}_n, \mathbb{C}_n$	$\mathbb{D}_n$	$\mathbb{E}_6$	$\mathbb{E}_7$	$\mathbb{E}_8$	$\mathbb{F}_4$	$\mathbb{G}_2$
$h(\Delta)$	$n+1$	$2n$	$2(n-1)$	$2^2 \cdot 3$	$2 \cdot 3^2$	$2 \cdot 3 \cdot 5$	$2^2 \cdot 3$	$2 \cdot 3$
$\lvert W(\Delta)\rvert$	$(n+1)!$	$2^n n!$	$2^{n-1} n!$	$2^7 \cdot 3^4 \cdot 5$	$2^{10} \cdot 3^4 \cdot 5 \cdot 7$	$2^{14} \cdot 3^5 \cdot 5^2 \cdot 7$	$2^7 \cdot 3^2$	$2^2 \cdot 3$

Unfortunately, our proof does not provide any illumination of the formula (and we should admit that the observation that the formula holds is stolen from Chapoton [Ch], see the end of the introduction).

As already mentioned, for Λ the path algebra of a quiver (thus for typical Dynkin algebras of type $\mathbb{A}_n, \mathbb{D}_n, \mathbb{E}_n$), the numbers $e(\Lambda)$ have been deter-

mined by Seidel [Se]. The essential cases which were missing are the Dynkin algebras of type $\mathbb{B}_n$. The inductive strategy of proof works for all types. However, we will also show a direct relationship between the cases $\mathbb{B}_n$ and $\mathbb{A}_{n-1}$, and this could be used directly in order to complete Seidel's considerations. Clearly, for $n = 2$, the number $e(\Lambda)$ is just the number of indecomposable modules, in particular $e(\mathbb{G}_2) = 6$.

Here is an outline of the proof. We will use induction on the rank n of Λ. If M is an indecomposable Λ-module, let $M^\perp$ be the full subcategory of mod Λ consisting of all modules N such that $\mathrm{Hom}(M, N) = 0 = \mathrm{Ext}^1(M, N)$. Since M is exceptional, one knows that $M^\perp$ is (equivalent to) the module category of a hereditary artin algebra of rank $n - 1$ (see [GL] or [S2]), thus by induction we may assume to know $e(M^\perp)$. Obviously, the complete exceptional sequences $(M_1, \dots, M_n)$ with $M_n = M$ correspond bijectively to the complete exceptional sequences in $M^\perp$, thus $e(M^\perp)$ is the number of complete exceptional sequences in mod Λ whose last entry is M. In Section 3 we will see that there is a vertex i_M of Δ such that $e(M^\perp) = e(\Delta(i_M))$, where $\Delta(i)$ is obtained from $\Delta = \Delta(\Lambda)$ by deleting the vertex i and all the edges involving i. Thus

$$e(\Delta) = \sum_M e(\Delta(i_M)),$$

and therefore, for Δ connected, we have the reduction formula

$$e(\Delta) = \frac{h}{2} \sum_{i \in \Delta_0} e(\Delta(i))$$

where h is the Coxeter number for Δ (see Section 4). In Section 5 we will use the reduction formula to obtain the entries of the table, proceeding case by case. The proof of cases $\mathbb{A}_n, \mathbb{B}_n, \mathbb{C}_n, \mathbb{D}_n$ relies on some well-known recursion formulas which go back to Abel [Ab] (see the Appendix). Conversely, one may observe that the interpretation using complete exceptional sequences provides a categorification of these formulas.

Since we deal with artin algebras (and not more generally with artinian rings), the diagrams which arise are the Dynkin diagrams $\mathbb{A}_n, \dots, \mathbb{G}_2$. If one is interested in all the finite Coxeter diagrams (thus also in $\mathbb{I}_2(m), \mathbb{H}_3, \mathbb{H}_4$), one may consider in the same way corresponding artinian rings (they are known to exist for $\mathbb{I}_2(5), \mathbb{H}_3, \mathbb{H}_4$, see [S1] as well [DRS] and [O]); this will be done in [FR].

The general frame. The calculations presented here can be seen in a broader frame, since the representation theory of hereditary artinian rings has turned out to be an intriguing tool for dealing with various questions in different parts of mathematics. In particular, there is a strong relationship

to the theory of (generalized) non-crossing partitions (see for example [Ag]) as observed first by Fomin and Zelevinsky.

As Ingalls and Thomas [IT] have shown, given the path algebra Λ of a finite directed quiver of type Δ, there is a poset isomorphism between the poset of thick subcategories of $\operatorname{mod}\Lambda$ with generators and the poset $\mathrm{NC}(\Delta)$ of non-crossing partitions of type Δ (and this result can easily be extended to arbitrary hereditary artin algebras Λ); we recall that a full subcategory is said to be *thick* (or "wide") provided it is closed under kernels, cokernels and extensions. Of course, in case Λ is of finite representation type, any thick subcategory has a generator.

Hubery and Krause [HK] have pointed out that the Ingalls–Thomas bijection yields a bijection between the complete exceptional sequences for Λ and the maximal chains in the poset $\mathrm{NC}(\Delta)$. Namely, given a complete exceptional sequence $(M_1, \dots, M_n)$ for Λ let $\mathcal{U}_i = (M_{i+1} \oplus \cdots \oplus M_n)^{\perp}$ for $0 \le i \le n$. Then $0 = \mathcal{U}_0 \subset \mathcal{U}_1 \subset \cdots \subset \mathcal{U}_n = \operatorname{mod}\Lambda$ is a maximal chain of thick subcategories of $\operatorname{mod}\Lambda$ with generators. Conversely, let us assume that $0 = \mathcal{U}_0 \subset \mathcal{U}_1 \subset \cdots \subset \mathcal{U}_n = \operatorname{mod}\Lambda$ is a maximal chain of thick subcategories of $\operatorname{mod}\Lambda$ with generators. Then $\mathcal{U}_{n-1}$ is the module category of a hereditary artin algebra of rank $n-1$, thus by induction the chain $0 = \mathcal{U}_0 \subset \mathcal{U}_1 \subset \cdots \subset \mathcal{U}_{n-1}$ corresponds to a complete exceptional sequence $(M_1, \dots, M_{n-1})$ in $\mathcal{U}_{n-1}$, and this is an exceptional sequence for Λ of cardinality $n-1$. As already mentioned, there is a uniquely determined Λ-module M_n such that $(M_1, \dots, M_n)$ is a complete exceptional sequence for Λ. We see that there is a canonical bijection between the complete exceptional sequences for Λ and the set of maximal chains of thick subcategories of $\operatorname{mod}\Lambda$ with generators, thus with the maximal chains in $\mathrm{NC}(\Delta)$.

This shows that the numbers $e(\Delta)$ calculated here for the Dynkin diagrams Δ via representation theory are nothing else than the numbers of maximal chains in $\mathrm{NC}(\Delta)$ (in the Dynkin case, this poset is even a lattice) or, equivalently, the numbers of factorizations of a fixed Coxeter element as a product of n reflections. The latter numbers for $\Delta = \mathbb{A}_n, \mathbb{B}_n, \mathbb{D}_n$ have been determined in a famous letter [D] of Deligne to Looijenga. The numbers of maximal chains in $\mathrm{NC}(\Delta)$ have been calculated for the cases $\mathbb{A}_n$, $\mathbb{B}_n$ and $\mathbb{D}_n$ by Kreweras [K], Reiner [Rn] and Athanasiadis–Reiner [AR], respectively, and in general by Chapoton [Ch] and Reading [Rd] (see also Chapuy–Stump [CS]). It seems that the term $n!h^n/|W|$ is first mentioned by Chapoton [Ch].

The present paper only relies on well-known properties of the module category of an artin algebra. On the other hand, the result presented here, and indeed also the main steps of our proof, may be considered as a categorification of the considerations of Deligne and Reading.

2. The shuffle lemma

LEMMA 1 (Shuffle Lemma). *Let Λ, Λ' be representation-finite hereditary artin algebras of ranks n, n' respectively. Then*

$$e(\Lambda \times \Lambda') = \binom{n+n'}{n} e(\Lambda) e(\Lambda').$$

Proof. Let $(E_1, \dots, E_n)$ be a (complete) exceptional sequence in $\operatorname{mod} \Lambda$ and let $(E'_1, \dots, E'_{n'})$ be a (complete) exceptional sequence in $\operatorname{mod} \Lambda'$. Let I be a subset of $\{1, \dots, n+n'\}$ of cardinality n, say $I = \{i_1 < \cdots < i_n\}$, and let $\{j_1 < \cdots < j_{n'}\}$ be its complement. Let $(M_1, \dots, M_{n+n'})$ be defined by $M_{i_t} = E_t$ for $1 \le t \le n$ and $M_{j_t} = E'_t$ for $1 \le t \le n'$. Then clearly $(M_1, \dots, M_{n+n'})$ is a complete exceptional sequence in $\operatorname{mod}(\Lambda \times \Lambda')$, and every complete exceptional sequence in $\operatorname{mod}(\Lambda \times \Lambda')$ is obtained in this way. Thus, fixing a subset I of cardinality n, the number of complete exceptional sequences $(M_1, \dots, M_{n+n'})$ in $\operatorname{mod}(\Lambda \times \Lambda')$ with M_i in $\operatorname{mod} \Lambda$ for all $i \in I$ is equal to $e(\Lambda)e(\Lambda')$, and the number of such subsets I is just $\binom{n+n'}{n}$. This completes the proof.

3. The category $M^\perp$. Let Λ be a representation-finite hereditary artin algebra of rank n. Let $\Delta = \Delta(\Lambda)$. Given a vertex i of Δ, let $\Delta(i)$ be obtained from Δ by deleting the vertex i and the edges involving i (it is of course again a Dynkin diagram).

Let τ be the Auslander–Reiten translation for Λ. For every indecomposable Λ-module M, there is a natural number t such that $\tau^t M$ is indecomposable projective, thus $\tau^t M = P(i_M)$ for a (uniquely determined) vertex i_M of Δ.

Let M be an indecomposable module. It is known that the category $M^\perp$ is equivalent to a module category $\operatorname{mod} \Lambda'$ where Λ' is a representation-finite hereditary artin algebra of rank $n-1$.

LEMMA 2. *Let M be an indecomposable module and assume that $M^\perp$ is equivalent to the module category* $\operatorname{mod} \Lambda'$. *Then Λ' has type $\Delta(i_M)$.*

Proof. First, assume that $M = P(i)$ is indecomposable projective, thus $i = i_M$. Let ϵ_i be an idempotent of Λ such that $P(i) = \Lambda\epsilon_i$. Then $M^\perp$ is the set of Λ-modules N with $\operatorname{Hom}(P(i), N) = 0$, thus the set of $\Lambda/\Lambda\epsilon_i\Lambda$-modules. On the other hand, we have $\Delta(\Lambda/\Lambda\epsilon_i\Lambda) = \Delta(i)$.

Now assume that M is indecomposable and not projective. There is a slice $\mathcal{S}$ (in the sense of [R2]) in the Auslander–Reiten quiver of Λ such that M is a sink for $\mathcal{S}$. Let $M_1, \dots, M_n$ be the indecomposable modules in $\mathcal{S}$, one from each isomorphism class, and assume that $M_n = M$. Since M is a sink of $\mathcal{S}$, we know that $\operatorname{Hom}(M, M_i) = 0$ for $1 \le i \le n-1$, thus the modules

$M_1, \dots, M_{n-1}$ belong to $M^\perp$. Let

$$T = \bigoplus_{i=1}^{n-1} M_i;$$

then T is a tilting module for $M^\perp = \operatorname{mod} \Lambda'$ (it has no self-extensions and enough indecomposable direct summands). Since $\mathcal{S}$ is a slice, we know that the endomorphism ring of $\bigoplus_{i=1}^{n} M_i$ is hereditary, thus also $\operatorname{End}(T)^{\mathrm{op}}$ is hereditary and the Dynkin diagram $\Delta(\operatorname{End}(T)^{\mathrm{op}})$ is just $\Delta(i_M)$. A tilting module with hereditary endomorphism ring is a slice module (see for example [R3, Section 1.2]). Thus T is a slice module for $\operatorname{mod} \Lambda'$ and therefore Λ' and $\operatorname{End}(T)^{\mathrm{op}}$ have the same Dynkin type. This shows that the Dynkin type of Λ' is $\Delta(i_M)$.

4. The reduction formula. We assume by induction that $e(\Lambda')$ only depends on $\Delta(\Lambda')$ for any representation-finite hereditary artin algebra Λ' of rank $n' < n$.

PROPOSITION. *Let Λ be a connected representation-finite hereditary artin algebra of rank n and type Δ. Then*

$$e(\Lambda) = \frac{h}{2} \sum_{i \in \Delta_0} e(\Delta(i))$$

where h is the Coxeter number for Δ.

This reduction formula shows that $e(\Lambda)$ only depends on $\Delta = \Delta(\Lambda)$.

Proof. If M is an indecomposable Λ-module, then we have seen in Section 3 that $M^\perp$ is equivalent to the module category $\operatorname{mod} \Lambda'$, where Λ' is of type $\Delta(i_M)$. Thus

$$e(M^\perp) = e(\Delta(i_M)).$$

For any vertex i of Δ, let $m(i)$ be the length of the τ-orbit of $P(i)$, thus there are precisely $m(i)$ indecomposable modules M such that $i_M = i$. Therefore

$$e(\Lambda) = \sum_M e(M^\perp) = \sum_M e(\Delta(i_M)) = \sum_i m(i) e(\Delta(i)).$$

We have to distinguish two cases. First, assume that Δ is not of the form $\mathbb{A}_n$, $\mathbb{D}_{2m+1}$ or $\mathbb{E}_6$. In this case, we have $m(i) = h/2$ for any vertex i of Δ. Therefore

$$\sum_i m(i) e(\Delta(i)) = \sum_i \frac{h}{2} e(\Delta(i)).$$

Second, assume that Δ is equal to $\mathbb{A}_n$, $\mathbb{D}_{2m+1}$ or $\mathbb{E}_6$. Thus, there is a (unique) automorphism ρ of Δ of order 2. One knows that $m(i) + m(\rho(i)) = h$ for all vertices i of Δ. The automorphism ρ shows that $e(\Delta(\rho(i))) = e(\Delta(i))$,

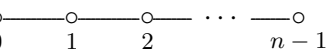
thus

$$2\sum_i m(i)e(\Delta(i)) = \sum_i m(i)e(\Delta(i)) + \sum_i m(\rho(i))e(\Delta(\rho(i)))$$
$$= \sum_i \big(m(i) + m(\rho(i))\big)e(\Delta(i)) = \sum_i h \cdot e(\Delta(i)).$$

Dividing by 2 we obtain the required formula.

5. The different cases

Type $\mathbb{A}_n$. This concerns the diagram



We have $\Delta(i) = \mathbb{A}_i \sqcup \mathbb{A}_{n-i-1}$, therefore, by the shuffle lemma and induction,

$$e(\Delta(i)) = \binom{n-1}{i} e(A_i)e(\mathbb{A}_{n-i-1}) = \binom{n-1}{i}(i+1)^{i-1}(n-i)^{n-i-2}.$$

Thus we have to calculate

$$\sum_{i=0}^{n-1} e(\Delta(i)) = \sum_{i=0}^{n-1} \binom{n-1}{i}(i+1)^{i-1}(n-i)^{n-i-2},$$

but this is the coefficient $F(n-1)$ of the power series $F = A * A$ (see the Appendix), and the formula (1) asserts that $F(n-1) = 2(n+1)^{n-2}$.

Now $h = n+1$, thus

$$\frac{h}{2}\sum_{i=1}^{n} e(\Delta(i)) = \frac{n+1}{2} 2(n+1)^{n-2} = (n+1)^{n-1}.$$

Type $\mathbb{B}_n$: The relationship between $\mathbb{B}_n$ and $\mathbb{A}_{n-1}$. Let us directly show the relationship

$$e(\mathbb{B}_n) = n^2 \cdot e(\mathbb{A}_{n-1}).$$

Proof. Let Λ be a hereditary artin algebra of type $\mathbb{B}_n$. Let P be the indecomposable projective Λ-module such that $\mathbf{dim}\, P$ is a short root. If $(M_1, \dots, M_n)$ is an exceptional sequence in $\operatorname{mod}\Lambda$, then there is precisely one index i such that $\mathbf{dim}\, M_i$ is a short root (see [R2]). Thus, let $\mathcal{E}_i(\operatorname{mod}\Lambda)$ be the set of exceptional sequences in $\operatorname{mod}\Lambda$ such that $\mathbf{dim}\, M_i$ is a short root, and let $e_i(\operatorname{mod}\Lambda)$ be the cardinality of $\mathcal{E}_i(\operatorname{mod}\Lambda)$. If $i < n$, and $(M_1, \dots, M_n)$ belongs to $\mathcal{E}_i(\operatorname{mod}\Lambda)$, then there is a uniquely determined element $(M_1, \dots, M_{i-1}, M_{i+1}, M_i^*, M_{i+2}, \dots, M_n)$ in $\mathcal{E}_{i+1}(\operatorname{mod}\Lambda)$, and every element of $\mathcal{E}_{i+1}(\operatorname{mod}\Lambda)$ is obtained in this way (again, see [R2]). This shows that $e_i(\operatorname{mod}\Lambda) = e_{i+1}(\operatorname{mod}\Lambda)$ and therefore

$$e(\Lambda) = \sum_{i=1}^{n} e_i(\Lambda) = n \cdot e_n(\Lambda).$$

There are precisely n indecomposable modules M such that $\mathbf{dim}\, M$ is a short root, namely the modules in the τ-orbit $\mathcal{O}(P)$ of P. For any module M in $\mathcal{O}(P)$, the exceptional sequences $(M_1, \dots, M_n)$ with $M_n = M$ correspond bijectively to the exceptional sequences in $M^\perp$, and $M^\perp$ is equivalent to the module category $\operatorname{mod} \Lambda_M$ with Λ_M a hereditary artin algebra of type $\mathbb{A}_{n-1}$. This shows that

$$e_n \pmod{\Lambda} = \sum_{M \in \mathcal{O}(P)} e(M^\perp) = n \cdot e(\mathbb{A}_{n-1}).$$

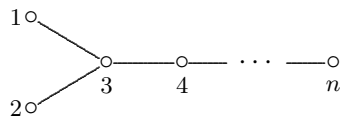
This completes the proof.

Type $\mathbb{C}_n$. There is the corresponding formula

$$e(\mathbb{C}_n) = n^2 \cdot e(\mathbb{A}_{n-1})$$

(with a similar proof).

Type $\mathbb{D}_n$. This concerns the diagram



with $n \geq 4$. Actually, also the cases $n = 3$ and $n = 2$ are of interest: for $n = 3$, we have $\mathbb{D}_3 = \mathbb{A}_3$, for $n = 2$ we deal with $\mathbb{D}_2 = \mathbb{A}_1 \sqcup \mathbb{A}_1$.

Before we proceed, let us mention the following notation (see the Appendix): For any $n \geq 0$, let $A(n) = (n+1)^{n-1}$ and $D(n) = (n-1)^n$.

For $k \geq 4$, we have $\Delta(k) = \mathbb{D}_{k-1} \sqcup \mathbb{A}_{n-k}$, thus the shuffling lemma yields

$$\begin{aligned} e(\Delta(k)) &= \binom{n-1}{k-1} e(\mathbb{D}_{k-1}) \cdot e(\mathbb{A}_{n-k}) \\ &= \binom{n-1}{k-1} 2(k-1)^k \cdot (n-k+1)^{n-k-1} \\ &= \binom{n-1}{k-1} 2D(k-1)A(n-k). \end{aligned}$$

For $k = 3$, we have $\Delta(3) = \mathbb{A}_1 \sqcup \mathbb{A}_1 \sqcup \mathbb{A}_{n-3}$, and $D(2) = 1$, thus

$$\begin{aligned} e(\Delta(3)) &= \frac{(n-1)!}{1!1!(n-3)!} e(\mathbb{A}_{n-3}) = \binom{n-1}{2} \cdot 2 \cdot (n-2)^{n-4} \\ &= \binom{n-1}{2} \cdot 2D(2)A(n-3). \end{aligned}$$

For $k = 1$ and $k = 2$, we have $\Delta(k) = \mathbb{A}_{n-1}$, therefore

$$e(\Delta(k)) = e(\mathbb{A}_{n-1}) = n^{n-2} = A(n-1),$$

thus

$$e(\Delta(1)) + e(\Delta(2)) = \binom{n-1}{0} 2D(0)A(n-1)$$

(since $D(0) = 1$).

Taking into account that $D(1) = 0$, we see that

$$\sum_{k=1}^{n} e(\Delta(k)) = e(\Delta(1)) + e(\Delta(2)) + \sum_{k=3}^{n} e(\Delta(k)) = \sum_{k=1}^{n} \binom{n-1}{k-1} 2D(k-1)A(n-k)$$

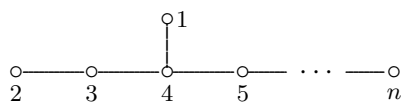
but this is the coefficient $G(n-1)$ of the power series $G = D * A$ (see the Appendix). The formula (3) in the Appendix asserts that $G(n-1) = (n-1)^{n-1}$.

Since the Coxeter number for $\mathbb{D}_n$ is $h = 2(n-1)$, we have

$$\frac{h}{2} \sum_{k=1}^{n} e(\Delta(k)) = (n-1) \cdot 2 \cdot (n-1)^{n-1} = 2(n-1)^n,$$

as we wanted to show.

Type $\mathbb{E}_n$. This concerns the diagrams



and we will deal with the cases $n = 6, 7, 8$.

Type $\mathbb{E}_6$:

i	$\Delta(i)$	$e(\Delta(i))$
1	$\mathbb{A}_5$	1296
2	$\mathbb{D}_5$	2048
3	$\mathbb{A}_1 \sqcup \mathbb{A}_4$	$\frac{5!}{1!4!}\, 1 \cdot 125$
4	$\mathbb{A}_2 \sqcup \mathbb{A}_1 \sqcup \mathbb{A}_2$	$\frac{5!}{2!1!2!}\, 3 \cdot 1 \cdot 3$

We see that

$$e(\mathbb{E}_6) = \frac{h}{2}\big(e(\mathbb{A}_5) + 2e(\mathbb{D}_5) + 2e(\mathbb{A}_1 \sqcup \mathbb{A}_4) + e(\mathbb{A}_2 \sqcup \mathbb{A}_1 \sqcup \mathbb{A}_2)\big) = 41\,472 = 2^9 3^4.$$

Type $\mathbb{E}_7$:

i	$\Delta(i)$	$e(\Delta(i))$
1	$\mathbb{A}_6$	16807
2	$\mathbb{D}_6$	46656
3	$\mathbb{A}_1 \sqcup \mathbb{A}_5$	$\frac{6!}{1!5!}\, 1 \cdot 1296$
4	$\mathbb{A}_2 \sqcup \mathbb{A}_1 \sqcup \mathbb{A}_3$	$\frac{6!}{2!1!3!}\, 3 \cdot 1 \cdot 16$
5	$\mathbb{A}_4 \sqcup \mathbb{A}_2$	$\frac{6!}{4!2!}\, 125 \cdot 3$
6	$\mathbb{D}_5 \sqcup \mathbb{A}_1$	$\frac{6!}{5!1!}\, 2048 \cdot 1$
7	$\mathbb{E}_6$	41472

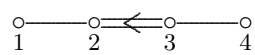
$$e(\mathbb{E}_7) = 1\,062\,882 = 2 \cdot 3^{12}.$$

Type $\mathbb{E}_8$:

i	$\Delta(i)$	$e(\Delta(i))$
1	$\mathbb{A}_7$	262144
2	$\mathbb{D}_7$	559872
3	$\mathbb{A}_1 \sqcup \mathbb{A}_6$	$\frac{7!}{1!6!}\, 1 \cdot 16807$
4	$\mathbb{A}_2 \sqcup \mathbb{A}_1 \sqcup \mathbb{A}_4$	$\frac{7!}{2!1!4!}\, 3 \cdot 1 \cdot 125$
5	$\mathbb{A}_4 \sqcup \mathbb{A}_3$	$\frac{7!}{4!3!}\, 125 \cdot 16$
6	$\mathbb{D}_5 \sqcup \mathbb{A}_2$	$\frac{7!}{5!2!}\, 2048 \cdot 3$
7	$\mathbb{E}_6 \sqcup \mathbb{A}_1$	$\frac{7!}{6!1!}\, 41472 \cdot 1$
8	$\mathbb{E}_7$	1062882

$$e(\mathbb{E}_8) = 37\,968\,750 = 2 \cdot 3^5 \cdot 5^7.$$

Type $\mathbb{F}_4$. This concerns the diagram



i	$\Delta(i)$	$e(\Delta(i))$
1	$\mathbb{B}_3$	27
2	$\mathbb{A}_1 \sqcup \mathbb{A}_2$	$\frac{3!}{1!2!}\, 1 \cdot 3$
3	$\mathbb{A}_2 \sqcup \mathbb{A}_1$	$\frac{3!}{2!1!}\, 3 \cdot 1$
4	$\mathbb{C}_3$	27

$$e(\mathbb{F}_4) = 432 = 2^4 \cdot 3^3.$$

6. Appendix: The binomial convolution of some power series. Let $\mathbb{Z}[T]$ be the set of formal power series $F = \sum_{n\geq 0} F(n)T^n$ in one variable T with integer coefficients $F(n)$. Given power series $F = \sum_n F(n)T^n$ and $G = \sum_n G(n)T^n$, the *binomial convolution* $F * G$ is by definition the power series $\sum_n H(n)T^n$ with $H(n) = \sum_k \binom{n}{k} F(k)G(n-k)$ (see [GKP]).

We are interested in the power series A, B, D with coefficients $A(n) = (n+1)^{n-1}$, $B(n) = n^n$ and $D(n) = (n-1)^n$, thus

$$A = \sum_{n\geq 0} (n+1)^{n-1}T^n = 1 + T + 3T^2 + 16T^3 + 125T^4 + \cdots,$$

$$B = \sum_{n\geq 0} n^n T^n = 1 + T + 4T^2 + 27T^3 + 256T^4 + \cdots,$$

$$D = \sum_{n\geq 0} (n-1)^n T^n = 1 + T^2 + 8T^3 + 81T^4 + \cdots.$$

The main result of the paper asserts that $e(\mathbb{A}_n) = A(n)$ and $e(\mathbb{B}_n) = e(\mathbb{C}_n) = B(n)$ for $n \geq 1$ and that $e(\mathbb{D}_n) = 2D(n)$ for $n \geq 2$. Our proofs in Section 5

use two of the following identities, namely (1) and (3) (and we could use (2) in order to deal with the cases $\mathbb{B}_n$):

PROPOSITION.

$$A * A = \sum_{n\geq 0} 2(n+2)^{n-1}T^n, \tag{1}$$

$$A * B = \sum_{n\geq 0} (n+1)^n T^n, \tag{2}$$

$$A * D = \sum_{n\geq 0} n^n T^n = B. \tag{3}$$

Proof. Let us recall Abel's identity [Ab]

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x(x-kz)^{k-1}(y+kz)^{n-k},$$

which is valid in any commutative ring with x being invertible. Several proofs can be found in Comptet [Co]. We need Abel's identity for $x = 1$ and $z = -1$, thus the identity

$$(1+y)^n = \sum_{k=0}^{n} \binom{n}{k} (1+k)^{k-1}(y-k)^{n-k}.$$

Let us start with the proof of (2), using Abel's identity for $y = n$ (and $x = 1,\ z = -1$):

$$(1+n)^n = \sum_{k=0}^{n} \binom{n}{k} (1+k)^{k-1}(n-k)^{n-k} = \sum_{k=0}^{n} \binom{n}{k} A(k)B(n-k) = (A*B)(n).$$

For the proof of (3), we use Abel's identity for $y = n-1$ (and $x = 1,\ z = -1$):

$$n^n = (1+(n-1))^n = \sum_{k=0}^{n} \binom{n}{k} (1+k)^{k-1}(n-1-k)^{n-k}$$

$$= \sum_{k=0}^{n} \binom{n}{k} A(k)D(n-k) = (A*D)(n).$$

For the proof of (1), we expand $(n+2)^{n-1}$ with $y = n+1$ (and again $x = 1$, $z = -1$):

$$(1+(n+1))^{n-1} = \sum_{k=0}^{n} \binom{n-1}{k} (1+k)^{k-1}(n+1-k)^{n-1-k}; \tag{$*$}$$

note that we have added the summand with index $k = n$; there is no harm, since by definition $\binom{n-1}{n} = 0$. Replacing the summation index k by $n-k$,

and using the equality $\binom{n-1}{n-k} = \binom{n-1}{k-1}$, we see that we also have

$$(**) \qquad (1+(n+1))^{n-1} = \sum_{k=0}^{n} \binom{n-1}{k-1} (1+n-k)^{n-k-1}(k+1)^{k-1}.$$

Since $\binom{n-1}{k} + \binom{n-1}{k-1} = \binom{n}{k}$, the summation of $(*)$ and $(**)$ yields

$$\begin{aligned} 2(n+2)^{n-1} &= \sum_{k=0}^{n} \binom{n}{k} (k+1)^{k-1}(n-k+1)^{n-k-1} \\ &= \sum_{k=0}^{n} \binom{n}{k} A(k)A(n-k) = (A*A)(n). \end{aligned}$$

This completes the proof of the Proposition.

It seems to us that these binomial convolution formulas are very pretty; as an example, let us exhibit the coefficients of T^4 in $A*A$, $A*B$, $A*D$:

$$\begin{aligned} (A*A)(4) &\quad 1\cdot 125\cdot 1 + 4\cdot 1\cdot 16 + 6\cdot 3\cdot 3 + 4\cdot 1\cdot 16 + 1\cdot 125\cdot 1 = 2\cdot 6^3 \\ (A*B)(4) &\quad 1\cdot 256\cdot 1 + 4\cdot 1\cdot 27 + 6\cdot 3\cdot 4 + 4\cdot 1\cdot 16 + 1\cdot 125\cdot 1 = 5^4 \\ (A*D)(4) &\quad 1\cdot\ 82\cdot 1 + 4\cdot 1\cdot\ 8 + 6\cdot 3\cdot 1 + 4\cdot 0\cdot 16 + 1\cdot 125\cdot 1 = 4^4 \end{aligned}$$

Finally, let us add some general information concerning the sequences A, B, D as provided by Sloane's On-Line Encyclopedia of Integer Sequences [Sl]. The sequence $A(n) = (n+1)^{n-1}$ is the Sloane sequence A000272, but shifted by 1, thus $A(n)$ is the *number of trees on $n+1$ labeled nodes.* The sequence $B(n) = n^n$ is the Sloane sequence A000312, so $B(n)$ is the *number of functions from $\{1,\dots,n\}$ to itself.* The sequence $D(n) = (n-1)^n$ with $e(\mathbb{D}_n) = 2D(n)$ for $n \geq 2$ is the Sloane sequence A065440, so $D(n)$ is the *number of functions from $\{1,\dots,n\}$ to itself without fixed points.*

n	$A(n)$	$B(n)$	$2D(n)$
0	1	1	2
1	1	1	0
2	3	4	2
3	16	27	16
4	125	256	162
5	1 296	3 125	2048
6	16 807	46 656	31 250
7	262 144	823 543	559 872
8	4 782 969	12 777 216	11 529 602
9	100 000 000	387 420 489	268 435 456
10	2 357 947 691	10 000 000 000	6 973 568 802

The table gives the first terms of the sequences $A, B, 2D$, namely $A(n)$, $B(n), 2D(n)$, with $n \leq 10$; note that $A(n) = e(\mathbb{A}_n)$ and $B(n) = e(\mathbb{B}_n)$ for $n \geq 1$, and $2D(n) = e(\mathbb{D}_n)$ for $n \geq 2$.

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Mustafa Obaid, Khalid Nauman, Wafa S. M. Al-Shammakh, Wafaa Fakieh,
Claus Michael Ringel
King Abdulaziz University
Faculty of Science
PO Box 80203
Jeddah 21589, Saudi Arabia
E-mail: drmobaid@yahoo.com
snauman@kau.edu.sa
wafasalem25@hotmail.com
wafaa.fakieh@hotmail.com
ringel@math.uni-bielefeld.de

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