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ON WEAKLY GIBSON F_σ-MEASURABLE MAPPINGS

BY

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Abstract. A function $f : X \to Y$ between topological spaces is said to be a *weakly Gibson function* if $f(\overline{U}) \subseteq \overline{f(U)}$ for any open connected set $U \subseteq X$. We prove that if X is a locally connected hereditarily Baire space and Y is a T_1-space then an F_σ-measurable mapping $f : X \to Y$ is weakly Gibson if and only if for any connected set $C \subseteq X$ with dense connected interior the image $f(C)$ is connected. Moreover, we show that each weakly Gibson F_σ-measurable mapping $f : \mathbb{R}^n \to Y$, where Y is a T_1-space, has a connected graph.

1. Introduction. The classical theorem of Kuratowski and Sierpiński [8] states that any Darboux Baire-one function $f : \mathbb{R} \to \mathbb{R}$ has a connected graph.

In 2010 K. Kellum [6] introduced Gibson and weak Gibson properties for a mapping f between topological spaces X and Y. He calls f [weakly] *Gibson* if $f(\overline{U}) \subseteq \overline{f(U)}$ for an arbitrary open [and connected] set $U \subseteq X$. Since every Darboux function has the weak Gibson property [5], it is natural to ask whether the theorem of Kuratowski–Sierpiński remains valid if we replace the Darboux property by the weak Gibson property. It was shown in [5] that any weakly Gibson barely continuous mapping (in the sense that for each non-empty closed subspace $F \subseteq X$ the restriction $f|_F$ has a continuity point) defined on a connected and locally connected space X and with values in a topological space Y has a connected graph. It turns out that the condition of bare continuity in the above mentioned result from [5] is not necessary (see Example 4.4).

In this paper we consider weakly Gibson mappings $f : X \to Y$ which are F_σ-*measurable*, i.e. the preimage $f^{-1}(V)$ of an open set $V \subseteq Y$ is an F_σ-set in X. Note that in the case when Y is a perfectly normal space, every Baire-one mapping $f : X \to Y$ is F_σ-measurable (see for instance [7, p. 394]). In Section 2 we introduce the notions of $\mathcal{G}$-closed and $\mathcal{W}$-closed sets and prove that the Euclidean space $\mathbb{R}^n$ cannot be written as a union of two non-empty disjoint F_σ and G_δ $\mathcal{W}$-closed subsets; moreover, a connected and locally connected hereditarily Baire space cannot be written as a union

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of two non-empty disjoint F_σ and G_δ $\mathcal{G}$-closed subsets. Using these facts, we prove in Section 3 that each F_σ-measurable mapping f between a locally connected hereditarily Baire space X and a T_1-space Y is weakly Gibson if and only if for any connected set $C \subseteq X$ with dense connected interior the image $f(C)$ is connected. This generalizes a result of M. Evans and P. Humke [3] who proved similar theorem for $X = \mathbb{R}^n$ and $Y = \mathbb{R}$. We also prove that each weakly Gibson F_σ-measurable mapping $f : \mathbb{R}^n \to Y$, where Y is a T_1-space, has a connected graph.

2. $\mathcal{A}$-closed sets and their properties. Let X be a topological space and let

- $\mathcal{T}(X)$ be the collection of all open subsets of X,
- $\mathcal{C}(X)$ be the collection of all connected subsets of X,
- $\mathcal{G}(X)$ be the collection of all connected open subsets of X.

Let X be a topological vector space and let

- $\mathcal{W}(X)$ be the collection of all open convex subsets of X.

Let $\mathcal{A}(X)$ be a collection of subsets of X. A subset $E \subseteq X$ is called *closed with respect to* $\mathcal{A}(X)$, or briefly $\mathcal{A}$-*closed*, if for any $A \in \mathcal{A}(X)$ with $A \subseteq E$ we have $\overline{A} \subseteq E$.

Proposition 2.1. *Let X be a connected and locally connected space and U be an open $\mathcal{G}$-closed subset of X. Then $U = \emptyset$ or $U = X$.*

Proof. Consider a component C of U. The local connectedness of U implies that C is clopen in U, and consequently C is open in X. Since U is $\mathcal{G}$-closed, $\overline{C} \subseteq U$. Therefore, $\overline{C} = C$ because C is a component. Hence, C is clopen in a connected space X. Therefore, $C = \emptyset$ or $C = X$. Since U is the union of all of its components, $U = \emptyset$ or $U = X$. ∎

We need the following auxiliary fact.

Lemma 2.2 ([7, p. 136]). *Let A and B be subsets of a topological space X such that A is connected and $A \cap B \neq \emptyset \neq A \setminus B$. Then $A \cap \operatorname{fr} B \neq \emptyset$.*

For a point x_0 of a normed space X and for $\varepsilon > 0$ we denote by $B(x_0, \varepsilon)$ (resp. $B[x_0, \varepsilon]$) the open (resp. closed) ball with center at x_0 and radius ε.

If a subset of a topological space is simultaneously F_σ and G_δ, then it is said to be *ambiguous*.

Theorem 2.3. *Let X be a hereditarily Baire space, and let X_1 and X_2 be ambiguous disjoint $\mathcal{A}$-closed subsets of X such that $X = X_1 \cup X_2$. If*

(1) *X is a connected and locally connected space and $\mathcal{A}(X) = \mathcal{G}(X)$, or*
(2) *$X = \mathbb{R}^n$, $n \geq 1$, and $\mathcal{A}(X) = \mathcal{W}(X)$,*

then $X_1 = X$ or $X_2 = X$.

Proof. To obtain a contradiction, suppose that $X_1 \neq X$ and $X_2 \neq X$. Let $F = \overline{X}_1 \cap \overline{X}_2$. Since X is connected, $F \neq \emptyset$. We show that $X_1 \cap F$ is dense in F. Supposing otherwise, choose $x_0 \in F$ and an open neighborhood U of x_0 in X such that

$$U \cap F \subseteq X_2.$$

Then $x_0 \in \overline{X}_1 \cap X_2$.

(1) Since X is locally connected, we may assume that U is connected. Note that $U \cap X_1 \neq \emptyset$ and select $a \in U \cap X_1$. Then $a \notin \overline{X}_2$. Let G be the component of $X \setminus \overline{X}_2$ which contains a. Then G is open in X. Note that $U \cap G \neq \emptyset \neq U \setminus G$. Lemma 2.2 implies that $U \cap \operatorname{fr} G \neq \emptyset$. Since G is closed in $X \setminus \overline{X}_2$, $\operatorname{fr} G \subseteq \overline{X}_2$. Moreover, $G \subseteq X_1$. Therefore, $\operatorname{fr} G \subseteq F$. Choose $b \in U \cap \operatorname{fr} G$. Then $b \in X_2$. Since X_1 is $\mathcal{G}$-closed, $b \in \overline{G} \subseteq X_1$, which is impossible.

(2) We may suppose that $U = B(x_0, \varepsilon)$. Take $a \in B(x_0, \varepsilon/2) \cap X_1$. Let

$$R = \sup\{r : B(a,r) \subseteq X_1\}.$$

Note that $R \leq \varepsilon/2$, since $x_0 \in X_2$. We have

$$d(x, x_0) \leq d(x,a) + d(a, x_0) < R + \varepsilon/2 < \varepsilon/2 + \varepsilon/2 = \varepsilon$$

for all $x \in B[a,R]$. Hence, $B[a,R] \subseteq U$. It is not hard to verify that $B[a,R] \cap \overline{X}_2 \neq \emptyset$, as $B[a,R]$ is compact. Therefore, there is $b \in B[a,R] \cap \overline{X}_2$. Since $B(a,R)$ is open and convex and X_1 is $\mathcal{W}$-closed, $b \in X_1$. But $b \in U \cap F$, which implies that $b \in X_2$. Thus, $b \in X_1 \cap X_2$, which is impossible.

Hence, $X_1 \cap F$ is dense in F. It can be proved similarly that $X_2 \cap F$ is dense in F. Thus $X_1 \cap F$ and $X_2 \cap F$ are disjoint dense G_δ-subsets of a Baire space F, which is a contradiction. Therefore, $X_1 = X$ or $X_2 = X$. ∎

3. Applications of $\mathcal{A}$-closed sets. We say that a mapping $f : X \to Y$ has the *Gibson property with respect to a collection* $\mathcal{A}(X)$, or f is $\mathcal{A}$-*Gibson*, if for any $A \in \mathcal{A}(X)$ we have

$$f(\overline{A}) \subseteq \overline{f(A)}.$$

If $\mathcal{A}(X) = \mathcal{T}(X)$ then f is said to be a *Gibson mapping*, and if $\mathcal{A}(X) = \mathcal{G}(X)$ then f is a *weakly Gibson mapping* (see [6]).

A mapping $f : X \to Y$ is *strongly Gibson with respect to* $\mathcal{A}(X)$, or *strongly* $\mathcal{A}$-*Gibson*, if for any $x \in X$ and $A \in \mathcal{A}(X)$ such that $x \in \overline{A}$ we have

$$f(x) \in \overline{f(A \cap U)}$$

for every neighborhood U of x in X.

THEOREM 3.1. *Let X be a topological space, Y a T_1-space, and $f : X \to Y$ a mapping such that for any connected set $C \subseteq X$ with dense connected interior the set $f(C)$ is connected. Then f is a weakly Gibson mapping.*

If, moreover, X is a locally convex space then f has the strong Gibson property with respect to the collection $\mathcal{W}(X)$.

Proof. Fix an arbitrary open connected set $U \subseteq X$, a point $x_0 \in \overline{U}$ and an open neighborhood V of $f(x_0)$ in Y. Denote $C = U \cup \{x_0\}$. Then the inclusions $U \subseteq C \subseteq \overline{U}$ imply that $f(C)$ is a connected set. Assume $f(U) \cap V = \emptyset$. Then

$$f(C) = f(U \cup \{x_0\}) = f(U) \cup \{f(x_0)\} \subseteq (Y \setminus V) \cup \{f(x_0)\},$$

which contradicts the connectedness of $f(C)$.

Now let X be a locally convex space. Fix a set $G \in \mathcal{W}(X)$, a point $x_0 \in \overline{G}$, an open convex neighborhood W of x_0 in X and an open neighborhood V of $f(x_0)$ in Y. Denote $U = W \cap G$. Clearly, $U \in \mathcal{G}(X)$. The rest of the proof runs as before. ∎

The converse is true for F_σ-measurable mappings defined on a locally connected hereditarily Baire space.

THEOREM 3.2. *Let X be a locally connected hereditarily Baire space, Y a topological space, and $f : X \to Y$ a weakly Gibson F_σ-measurable mapping. Then for any connected set $C \subseteq X$ with dense connected interior the set $f(C)$ is connected.*

Proof. Let $C \in \mathcal{C}(X)$, $U = \operatorname{int} C$ and $C \subseteq \overline{U}$.

We first prove that $f(U)$ is a connected set. Suppose, contrary to our claim, that $f(U) = W_1 \cup W_2$, where W_1 and W_2 are non-empty disjoint open subsets of $f(U)$. Set $g = f|_U$. Evidently, $g : U \to f(U)$ is a weakly Gibson F_σ-measurable mapping. Let $A_i = g^{-1}(W_i)$ for $i = 1, 2$. Then every set A_i is $\mathcal{G}$-closed in U, as g is weakly Gibson. Moreover, every A_i is ambiguous in U, $U = A_1 \cup A_2$ and $A_1 \cap A_2 = \emptyset$. Taking into account that U is a hereditarily Baire connected and locally connected space, we obtain $A_1 = U$ or $A_2 = U$ according to Theorem 2.3(1). Then $W_1 = \emptyset$ or $W_2 = \emptyset$, a contradiction. Therefore, $f(U)$ is a connected set.

Since f is weakly Gibson, $f(U) \subseteq f(C) \subseteq f(\overline{U}) \subseteq \overline{f(U)}$. Consequently, the set $f(C)$ is connected. ∎

For a mapping $f : X \to Y$ we define $\gamma_f : X \to X \times Y$ by

$$\gamma_f(x) = (x, f(x)).$$

Note that if X is a connected and locally connected hereditarily Baire space and γ_f is an F_σ-measurable weakly Gibson mapping then Theorem 3.2 implies that f has a connected graph Γ, since $\Gamma = \gamma_f(X)$. It is not hard to prove that γ_f remains weakly Gibson for any weakly Gibson mapping $f : \mathbb{R} \to \mathbb{R}$. But Example 4.2 shows that γ_f need not be weakly Gibson for a weakly Gibson F_σ-measurable mapping $f : \mathbb{R}^2 \to \mathbb{R}$.

THEOREM 3.3. *Let $X = \mathbb{R}^n$ with $n \geq 1$ and let Y be a T_1-space. If $f : X \to Y$ is a weakly Gibson F_σ-measurable mapping then f has a connected graph.*

Proof. We first observe that by Theorem 3.2 for any $U \in \mathcal{G}(X)$ and for any C with $U \subseteq C \subseteq \overline{U}$ the set $f(C)$ is connected. Hence f has the strong Gibson property with respect to the collection $\mathcal{W}(X)$ according to Theorem 3.1. It is easy to see that γ_f is also $\mathcal{W}$-strongly Gibson.

We show that $\gamma_f : X \to X \times Y$ is F_σ-measurable. Let $\{B_k : k \in \mathbb{N}\}$ be a base of open sets in X and W be an arbitrary open set in $X \times Y$. Put

$$V_k = \bigcup\{V : V \text{ is open in } Y \text{ and } B_k \times V \subseteq W\}.$$

Then $W = \bigcup_{k=1}^{\infty}(B_k \times V_k)$. Since $\gamma_f^{-1}(W) = \bigcup_{k=1}^{\infty}(B_k \cap f^{-1}(V_k))$, $\gamma_f^{-1}(W)$ is an F_σ-subset of X.

Now assume that $Y_0 = \gamma_f(X)$ is not connected and choose open disjoint non-empty subsets W_1 and W_2 of Y_0 such that $Y_0 = W_1 \cup W_2$. Let $X_i = \gamma_f^{-1}(W_i)$ for $i = 1, 2$. It is easy to check that X_1 and X_2 are $\mathcal{W}$-closed ambiguous subsets of X. Moreover, $X_1 \cap X_2 = \emptyset$ and $X = X_1 \cup X_2$. Hence $X_1 = X$ or $X_2 = X$ by Theorem 2.3(2). Consequently, $W_1 = \emptyset$ or $W_2 = \emptyset$, a contradiction. ■

The following question is open.

QUESTION 3.4. *Let X be a normed space, Y a T_1-space, and $f : X \to Y$ a weakly Gibson F_σ-measurable mapping. Is the graph of f a connected set?*

4. Examples. Our first example shows that the class of all F_σ-measurable Darboux mappings is strictly wider than the class of all Baire-one Darboux mappings.

EXAMPLE 4.1. *There exist a connected subset $Y \subseteq \mathbb{R}^2$ and an F_σ-measurable Darboux function $f : \mathbb{R} \to Y$ which is not Baire-one.*

Proof. Let $\mathbb{Q} = \{r_n : n \in \mathbb{N}\}$ be the set of all rational numbers. For every $n \in \mathbb{N}$ we consider the function $\varphi_n : \mathbb{R} \to \mathbb{R}$ defined by

$$\varphi_n(x) = \begin{cases} \sin \dfrac{1}{x - r_n}, & x \neq r_n, \\ 0, & x = r_n. \end{cases}$$

Define $g : \mathbb{R} \to \mathbb{R}$ by

$$g(x) = \sum_{n=1}^{\infty} \frac{1}{2^n} \varphi_n(x).$$

Let

$$Y = \{(x, y) \in \mathbb{R}^2 : y = g(x)\} \quad \text{and} \quad f = \gamma_g.$$

Observe that for every n the function

$$g_n(x) = \sum_{k=1}^{n} \frac{1}{2^k}\varphi_k(x)$$

is a Baire-one Darboux function. Since the sequence $(g_n)_{n=1}^{\infty}$ is uniformly convergent to g on $\mathbb{R}$, g is a Baire-one Darboux function [1, Theorem 3.4]. Consequently, the graph of $g|_C$ is connected for every connected subset $C \subseteq \mathbb{R}$ according to [1, Theorem 1.1]. Therefore, $f : \mathbb{R} \to Y$ is a Darboux function. Moreover, $f : \mathbb{R} \to \mathbb{R}^2$ is a Baire-one mapping, which implies that $f : \mathbb{R} \to Y$ is F_σ-measurable.

Note that the space Y is *punctiform* (i.e., Y does not contain any continuum of cardinality larger than one), since g is discontinuous on everywhere dense set $\mathbb{Q}$ (see [8]). Hence each continuous mapping between $\mathbb{R}$ and Y is constant. Therefore, $f : \mathbb{R} \to Y$ is not a Baire-one mapping. ∎

EXAMPLE 4.2. *For all $(x, y) \in \mathbb{R}^2$ define*

$$f(x,y) = \begin{cases} \sin(1/x), & x > 0, \\ 1, & x \le 0. \end{cases}$$

Then $f : \mathbb{R}^2 \to \mathbb{R}$ is an F_σ-measurable weakly Gibson function, but γ_f is not weakly Gibson.

Proof. We show that f is weakly Gibson. It is sufficient to check that f is weakly Gibson at each point of the set $\{0\} \times \mathbb{R}$. Fix $y_0 \in \mathbb{R}$ and an open connected set $U \subseteq \mathbb{R}^2$ such that $p_0 = (0, y_0) \in \overline{U} \setminus U$. Take an arbitrary neighborhood V of $f(p_0)$ in $\mathbb{R}$. Clearly, $f(p) \in V$ for all $p \in U \cap ((-\infty, 0] \times \mathbb{R})$. Consider the case $U \subseteq (0, \infty) \times \mathbb{R}$. Since $p_0 \in \overline{U}$ and U is connected, there exists $n \in \mathbb{N}$ such that $U \cap \left(\{\frac{1}{\pi/2+2\pi n}\} \times \mathbb{R}\right) \neq \emptyset$. Let $y \in \mathbb{R}$ with $p = \left(\frac{1}{\pi/2+2\pi n}, y\right) \in U$. Then $f(p) = 1$ and $f(p) \in V$. Hence, f is weakly Gibson.

Consider the open connected set

$$U = \{(x, y) \in \mathbb{R}^2 : x > 0 \text{ and } |y - \sin(1/x)| < x\}$$

and let $C = U \cup \{(0, 0)\}$. Then $U \subseteq C \subseteq \overline{U}$. Note that $\gamma_f : \mathbb{R}^2 \to \mathbb{R}^3$ is F_σ-measurable. One easily checks that $\gamma_f(C)$ is not connected. Therefore, γ_f is not weakly Gibson by Theorem 3.2. ∎

Finally, we give an example of a space Y and an F_σ-measurable Darboux mapping $f : \mathbb{R} \to Y$ which is not barely continuous.

We first need some definitions and auxiliary facts. For a topological space Y we denote by $\mathcal{F}(Y)$ the space of all non-empty closed subsets of Y equipped with the Vietoris topology. A multivalued mapping $F : X \to Y$ is said to be *upper* (resp. *lower*) *continuous at* $x_0 \in X$ if for any open set V in Y such that $F(x_0) \subseteq V$ (resp. $F(x_0) \cap V \neq \emptyset$) there exists a neighborhood U

of x_0 in X such that for every $x \in U$ we have $F(x) \subseteq V$ $(F(x) \cap V \neq \emptyset)$. A multivalued mapping f which is upper and lower continuous at x_0 is called *continuous at* x_0.

LEMMA 4.3. *There exists a continuous mapping $f_0 : \mathbb{R} \to \mathcal{F}(\mathbb{R})$ such that for all $x \in [0,1]$ and $p \in P = \{1/n : n \in \mathbb{N}\} \cup \{0\}$ there are $n_p \in \mathbb{N}$, a strictly increasing unbounded sequence $(v_n)_{n \geq n_p}$ of reals $v_n > 0$ and a strictly decreasing unbounded sequence $(u_n)_{n \geq n_p}$ of reals $u_n < 0$ such that*

$$f_0(u_n) = f_0(v_n) = \{p\} \cup \bigcup_{k=1}^{n} [k, k+x] \cup \bigcup_{k>n} \{k\}$$

for all $n \geq n_p$.

Proof. Let $P = \{p_n : n \in \mathbb{N}\}$. Choose a continuous function $\varphi_0 : \mathbb{R} \to [0,1]$ with $\varphi_0(x) = p_k$ if $|x| \in \left[n + \frac{2k-1}{2n}, n + \frac{k}{n}\right]$, where $n \in \mathbb{N}$ and $k = 1, \ldots, n$. For every $n \in \mathbb{N}$ define a continuous function $\varphi_n : \mathbb{R} \to [n, n+1]$ by

$$\varphi_n(x) = \begin{cases} n, & |x| \leq n, \\ n + \sin(4\pi k|x|), & |x| \in (k, k+1],\ k \geq n. \end{cases}$$

Let $f_0(x) = \{\varphi_0(x)\} \cup \bigcup_{n=1}^{\infty} [n, \varphi_n(x)]$. Since all the functions φ_n are continuous and $\varphi_k(x) = k$ for $x \in [-n, n]$ and $k \geq n$, f_0 is continuous.

Fix $p = p_n \in P$ and $x \in [0,1]$. Denote $n_p = n$. For all $k \geq n$ choose $v_k \in \left[k + \frac{2n-1}{2k}, k + \frac{n}{k}\right]$ such that $\sin(4\pi k v_k) = x$. Then for every $k \geq n$ we have $\varphi_0(v_k) = p$, $\varphi_1(v_k) = 1 + x, \ldots, \varphi_k(v_k) = k + x$ and $\varphi_i(v_k) = i$ for $i > k$, i.e., the sequence $(v_k)_{k \geq n}$ satisfies the condition of the lemma. It remains to set $u_k = -v_k$ for all $k \in \mathbb{N}$. ∎

EXAMPLE 4.4. *There exists a Baire-one F_σ-measurable Darboux mapping $f : \mathbb{R} \to \mathcal{F}(\mathbb{R})$ such that the restriction $f|_C$ of f to the Cantor set $C \subseteq \mathbb{R}$ is everywhere discontinuous and $f(\mathbb{R})$ is hereditarily Lindelöf* (*in particular, $f(\mathbb{R})$ is perfectly normal*).

Proof. Let $\mathbb{R} \setminus C = \bigcup_{n=1}^{\infty} I_n$, where $I_n = (a_n, b_n)$. Set $A = \{a_n : n \in \mathbb{N}\}$ and $B = C \setminus A$. For every $n \in \mathbb{N}$ we choose a homeomorphism $\psi_n : I_n \to \mathbb{R}$. Define

$$f(x) = \begin{cases} f_0(\psi_n(x)), & x \in I_n, \\ \{1/n\} \cup \bigcup_{k=1}^{\infty} [k, k+x], & n \in \mathbb{N},\ x = a_n, \\ \{0\} \cup \bigcup_{k=1}^{\infty} [k, k+x], & x \in B, \end{cases}$$

where f_0 is the function from Lemma 4.3.

We show that f is a Baire-one mapping. For every $n \in \mathbb{N}$, applying Lemma 4.3, we find a number m_n, a strictly increasing sequence $(v_k^{(n)})_{k \geq m_n}$

of $v_k^{(n)} \in ((a_n+b_n)/2, b_n)$ and a strictly decreasing sequence $(u_k^{(n)})_{k\geq m_n}$ of $u_k^{(n)} \in (a_n, (a_n+b_n)/2)$ such that

$$f_0(\psi_n(u_k^{(n)})) = \{1/n\} \cup \bigcup_{i=1}^{k} [i, i+a_n] \cup \bigcup_{i>k} \{i\}$$

and

$$f_0(\psi_n(v_k^{(n)})) = \{0\} \cup \bigcup_{i=1}^{k} [i, i+a_n] \cup \bigcup_{i>k} \{i\}$$

for all $i \geq m_n$.

For every $n \in \mathbb{N}$ denote $M_n = \{k \leq n : m_k \leq n\}$. Clearly, $M_n \subseteq M_{n+1}$ for all n and $\mathbb{N} = \bigcup_{n=1}^{\infty} M_n$. Choose a sequence of continuous functions $g_n : \mathbb{R} \to [0,1]$ which is pointwise convergent to the function

$$g(x) = \begin{cases} 0, & x \in \mathbb{R} \setminus A, \\ 1/n, & n \in \mathbb{N},\, x = a_n. \end{cases}$$

Without loss of generality, we assume that $g_n(u_k^{(n)}) = 1/n$ and $g_n(v_k^{(n)}) = 0$ if $n \in M_k$. Now for every $k \in \mathbb{N}$ define

$$f_k(x) = \begin{cases} f_0(\psi_n(x)), & x \in [u_k^{(n)}, v_k^{(n)}],\, n \in M_k, \\ \{g_k(x)\} \cup \bigcup_{i=1}^{k} [i, i+x] \cup \bigcup_{i>k} \{i\}, & x \in \mathbb{R} \setminus \bigcup_{n\in M_k} [u_k^{(n)}, v_k^{(n)}]. \end{cases}$$

It is easy to see that each f_k is continuous and $\lim_{k\to\infty} f_k(x) = f(x)$ for all $x \in \mathbb{R}$.

We now prove that f has the Darboux property. Let $I \subseteq \mathbb{R}$ be a connected set of cardinality larger than one. If $I \subseteq I_n$ for some $n \in \mathbb{N}$ then $f(I)$ is connected, provided the restriction $f|_{I_n}$ is continuous. Suppose $I \not\subseteq I_n$ for every $n \in \mathbb{N}$. Let $M = \{n \in \mathbb{N} : J_n = I_n \cap I \neq \emptyset\}$. Note that the set $G = \bigcup_{n\in M} J_n$ is dense in I. Set $f(I) = U \cup V$, where U and V are disjoint clopen sets in $f(\mathbb{R})$. Denote $K = \{n \in M : f(J_n) \subseteq U\}$ and $L = \{n \in M : f(J_n) \subseteq V\}$. Since the restriction of f to each J_n is continuous, we have $G = G_1 \cup G_2$, where $G_1 = \bigcup_{n\in K} J_n$, $G_2 = \bigcup_{n\in L} J_n$ and $G_1 \cap G_2 = \emptyset$. Lemma 4.3 implies that $f(\overline{G}_i) \subseteq \overline{f(G_i)}$ for $i = 1, 2$. Hence, $f(\overline{G}_1) \subseteq U$ and $f(\overline{G}_2) \subseteq V$. Therefore, $I = \overline{G}_1 \cup \overline{G}_2$ and $\overline{G}_1 \cap \overline{G}_2 = \emptyset$. Consequently, $G_1 = \emptyset$ or $G_2 = \emptyset$. Thus, $U = \emptyset$ or $V = \emptyset$.

To show that $Y = f(\mathbb{R})$ is hereditarily Lindelöf it is sufficient to prove that so are $Y_1 = f(\mathbb{R}\setminus C)$ and $Y_2 = f(C)$. Note that $Y_1 = f_0(\mathbb{R})$ is hereditarily Lindelöf, since Y_1 is the continuous image of $\mathbb{R}$ under the continuous mapping f_0 with values in the Hausdorff space $\mathcal{F}(\mathbb{R})$. Since $f(a_n) \cap [0,1] = \{1/n\}$ for every $n \in \mathbb{N}$, and $f(b) \cap [0,1] = \{0\}$ for each $b \in B$, the space $f(A)$ is a countable discrete subspace of Y_2. Moreover, for each $b \in B$ the sets $f((b-\varepsilon, b] \cap C)$, where $\varepsilon > 0$, form a base of neighborhoods of $f(x_0)$ in Y_2. Since an arbitrary union of sets of the form $(u, v]$ is a union of a sequence $(u_n, v_n]$, Y_2 is hereditarily Lindelöf. Hence, X is hereditarily Lindelöf, so X is perfectly normal.

Since Y is perfectly normal and f is a Baire-one function, f is F_σ-measurable [7, p. 394]. It remains to prove that the restriction $f|_C$ of f to the Cantor set C is everywhere discontinuous. Note that $f|_C$ is discontinuous everywhere on A, since $f(A)$ is discrete in Y_2. Moreover, for every $b \in B$ no set of the form $(b-\varepsilon, b] \cap C$ is a neighborhood of b in C. Therefore, $f|_C$ is discontinuous at each $b \in B$. ∎

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