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ON LÉVY'S BROWNIAN MOTION INDEXED BY ELEMENTS OF COMPACT GROUPS

BY

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Abstract. We investigate positive definiteness of the Brownian kernel $K(x,y) = \frac{1}{2}(d(x,x_0)+d(y,x_0)-d(x,y))$ on a compact group G and in particular for $G = SO(n)$.

1. Introduction. In 1959 P. Lévy [8] asked about the existence of a process X indexed by the points of a metric space $(\mathscr{X}, d)$ and generalizing the Brownian motion, i.e. of a real Gaussian process which would be centered, vanishing at some point $x_0 \in \mathscr{X}$ and such that $\mathbb{E}(|X_x - X_y|^2) = d(x,y)$. By polarization, the covariance function of such a process would be

$$K(x,y) = \tfrac{1}{2}(d(x,x_0)+d(y,x_0)-d(x,y)) \tag{1.1}$$

so that the above mentioned existence is equivalent to the kernel K being positive definite. Positive definiteness of K for $\mathscr{X} = \mathbb{R}^m$ and d the Euclidean metric was proved by Schoenberg [14] in 1938, and P. Lévy himself constructed the Brownian motion on $\mathscr{X} = \mathbb{S}^{m-1}$, the euclidean sphere of $\mathbb{R}^m$, d being the distance along the geodesics. Later Gangolli [5] gave an analytical proof of the positive definiteness of the kernel (1.1) for the same metric space $(\mathbb{S}^{m-1}, d)$, in a paper that dealt with this question for a large class of homogeneous spaces.

Finally Kubo et al. [6] proved the positive definiteness of the kernel (1.1) for the Riemannian metric spaces of constant sectional curvature equal to $-1, 0$ or 1, thus adding the hyperbolic disk to the list. To be precise, in the case of the hyperbolic space $\mathcal{H}_m = \{(x_0, x_1, \dots, x_m) \in \mathbb{R}^{m+1} : x_1^2 + \dots + x_m^2 - x_0^2 = 1\}$, the distance under consideration is the unique, up to multiplicative constants, Riemannian distance that is invariant with respect to the action of $G = L_m$, the Lorentz group.

In this short note we investigate this question for $\mathscr{X} = SO(n)$. The answer is that the kernel (1.1) is not positive definite on $SO(n)$ for $n > 2$. This is somehow surprising as, in particular, $SO(3)$ is locally isometric to $SU(2)$, where positive definiteness of the kernel K is immediate, as shown below.

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We have been led to the question of the existence of the Brownian motion indexed by elements of these groups—in particular of $SO(3)$—in connection with the analysis and modeling of the Cosmic Microwave Background which has recently become an active research field (see [7], [9], [10], [11] e.g.) and which has attracted attention to the study of random fields ([1], [2], [13] e.g.). More precisely, in modern cosmological models the CMB is seen as a realization of an invariant random field in a vector bundle over the sphere $\mathbb{S}^2$ and the analysis of its components (polarization e.g.) requires the *spin* random fields theory. This leads naturally to the investigation of invariant random fields on $SO(3)$ enjoying particular properties and therefore to the question of the existence of a privileged random field, i.e. Lévy's Brownian random field on $SO(3)$.

In §2 we recall some elementary facts about invariant distances and positive definite kernels. In §3 we treat the case $G = SU(2)$, recalling well known facts about the invariant distance and Haar measure of this group. Positive definiteness of K for $SU(2)$ is just a simple remark, but these facts are needed in §4 where we treat the case $SO(3)$ and deduce from it the case $SO(n)$, $n \geq 3$.

2. Some elementary facts. In this section we recall some well known facts about Lie groups (see mainly [3] and also [4], [15]).

2.1. Invariant distance of a compact Lie group. From now on we denote by G a compact Lie group. It is well known that G admits a bi-invariant Riemannian metric (see [4, p. 66] e.g.), which we shall denote by $\{\langle\cdot,\cdot\rangle_g\}_{g\in G}$, where of course $\langle\cdot,\cdot\rangle_g$ is an inner product on the tangent space T_gG to the manifold G at g and the family $\{\langle\cdot,\cdot\rangle_g\}_{g\in G}$ smoothly depends on g. By bi-invariance, for $g \in G$ the diffeomorphisms L_g and R_g (resp. left multiplication and right multiplication of the group) are isometries. Since the tangent space T_gG at any point g can be translated to the tangent space T_eG at the identity element e of the group, the metric $\{\langle\cdot,\cdot\rangle_g\}_{g\in G}$ is completely characterized by $\langle\cdot,\cdot\rangle_e$. Moreover, T_eG being the Lie algebra $\mathfrak{g}$ of G, the bi-invariant metric corresponds to an inner product $\langle\cdot,\cdot\rangle$ on $\mathfrak{g}$ which is invariant under the adjoint representation Ad of G. Indeed there is a one-to-one correspondence between bi-invariant Riemannian metrics on G and Ad-invariant inner products on $\mathfrak{g}$. If in addition $\mathfrak{g}$ is semisimple, then the negative Killing form of G is an Ad-invariant inner product on $\mathfrak{g}$ itself.

If there exists a unique (up to a multiplicative factor) bi-invariant metric on G (for a sufficient condition see [4, Th. 2.43]) and $\mathfrak{g}$ is semisimple, then this metric is necessarily proportional to the negative Killing form of $\mathfrak{g}$. It is well known that this is the case for $SO(n)$ ($n \neq 4$) and $SU(n)$; furthermore, the (natural) Riemannian metric on $SO(n)$ induced by the embedding $SO(n) \hookrightarrow \mathbb{R}^{n^2}$ corresponds to the negative Killing form of $so(n)$.

Endowed with this bi-invariant Riemannian metric, G becomes a metric space, with a distance d which is bi-invariant. Therefore the function $g \in G \mapsto d(g,e)$ is a class function, because

$$d(g,e) = d(hg,h) = d(hgh^{-1}, hh^{-1}) = d(hgh^{-1}, e), \qquad g,h \in G. \tag{2.1}$$

It is well known that geodesics on G through the identity e are exactly the one-parameter subgroups of G (see [12, p. 113] e.g.), thus a geodesic from e is the curve on G given by

$$\gamma_X(t) : t \in [0,1] \mapsto \exp(tX)$$

for some $X \in \mathfrak{g}$. The length of this geodesic is

$$L(\gamma_X) = \|X\| = \sqrt{\langle X, X\rangle}.$$

Therefore

$$d(g,e) = \inf_{X \in \mathfrak{g} \,:\, \exp X = g} \|X\|.$$

2.2. Brownian kernels on a metric space. Let $(\mathscr{X}, d)$ be a metric space.

LEMMA 2.1. *The kernel K in* (1.1) *is positive definite on $\mathscr{X}$ if and only if d is a restricted negative definite kernel, i.e., for every choice of elements $x_1, \dots, x_n \in \mathscr{X}$ and of complex numbers $\xi_1, \dots, \xi_n$ with $\sum_{i=1}^n \xi_i = 0$,*

$$\sum_{i,j=1}^n d(x_i, x_j)\xi_i\overline{\xi_j} \le 0. \tag{2.2}$$

Proof. For every $x_1, \dots, x_n \in \mathscr{X}$ and complex numbers $\xi_1, \dots, \xi_n$,

$$\sum_{i,j} K(x_i, x_j)\xi_i\overline{\xi_j} = \frac{1}{2}\Big(\overline{a}\sum_i d(x_i, x_0)\xi_i + a\sum_j d(x_j, x_0)\overline{\xi_j} - \sum_{i,j} d(x_i, x_j)\xi_i\overline{\xi_j}\Big) \tag{2.3}$$

where $a := \sum_i \xi_i$. If $a = 0$ then it is immediate that in (2.3) the l.h.s. is ≥ 0 if and only if the r.h.s. is ≤ 0. Otherwise set $\xi_0 := -a$ so that $\sum_{i=0}^n \xi_i = 0$. The equality

$$\sum_{i,j=0}^n K(x_i, x_j)\xi_i\overline{\xi_j} = \sum_{i,j=1}^n K(x_i, x_j)\xi_i\overline{\xi_j} \tag{2.4}$$

is then easy to check, keeping in mind that $K(x_i, x_0) = K(x_0, x_j) = 0$, which finishes the proof. ∎

For a more general proof see [5, p. 127, proof of Lemma 2.5].

If $\mathscr{X}$ is the homogeneous space of some topological group G, and d is a G-invariant distance, then (2.2) is satisfied if and only if for every choice of elements $g_1, \dots, g_n \in G$ and of complex numbers $\xi_1, \dots, \xi_n$ with $\sum_{i=1}^n \xi_i = 0$,

$$\sum_{i,j=1}^n d(g_i g_j^{-1} x_0, x_0)\xi_i \overline{\xi_j} \le 0 \tag{2.5}$$

where $x_0 \in \mathscr{X}$ is a fixed point. We shall say that the function $g \in G \mapsto d(gx_0, x_0)$ is *restricted negative definite* on G if it satisfies (2.5).

In our case of interest, $\mathscr{X} = G$ is a compact (Lie) group and d is a bi-invariant distance as in §2.1. The Peter–Weyl development (see [3] e.g.) for the class function $d(\cdot, e)$ on G is

$$d(g, e) = \sum_{\ell \in \widehat{G}} \alpha_\ell \chi_\ell(g) \tag{2.6}$$

where $\widehat{G}$ denotes the family of equivalence classes of irreducible representations of G, and χ_ℓ the character of the ℓth irreducible representation of G.

Remark 2.2. A function ϕ with a development as in (2.6) is restricted negative definite if and only if $\alpha_\ell \le 0$ but for the trivial representation.

Actually note first that, by standard approximation arguments, ϕ is restricted negative definite if and only if for every continuous function $f : G \to \mathbb{C}$ with 0-mean (i.e. orthogonal to the constants),

$$\int_G \int_G \phi(gh^{-1}) f(g) \overline{f(h)}\, dg\, dh \le 0, \tag{2.7}$$

dg denoting the Haar measure of G. Choosing $f = \chi_\ell$ on the l.h.s. of (2.7) and denoting by d_ℓ the dimension of the corresponding representation, we find by a straightforward computation that

$$\int_G \int_G \phi(gh^{-1}) \chi_\ell(g) \overline{\chi_\ell(h)}\, dg\, dh = \frac{\alpha_\ell}{d_\ell}. \tag{2.8}$$

so that if ϕ is restricted negative definite, then necessarily $\alpha_\ell \le 0$.

Conversely, if $\alpha_\ell \le 0$ but for the trivial representation, then ϕ is restricted negative definite, as the characters χ_ℓ are positive definite and orthogonal to the constants.

3. $SU(2)$.

The special unitary group $SU(2)$ consists of the complex unitary 2×2-matrices g such that $\det(g) = 1$. Every $g \in SU(2)$ has the form

$$g = \begin{pmatrix} a & b \\ -\overline{b} & \overline{a} \end{pmatrix}, \qquad a, b \in \mathbb{C},\ |a|^2 + |b|^2 = 1. \tag{3.1}$$

If $a = a_1 + ia_2$ and $b = b_1 + ib_2$, then the map

$$\Phi(g) = (a_1, a_2, b_1, b_2) \tag{3.2}$$

is a homeomorphism (see [3], [15] e.g.) between $SU(2)$ and the unit sphere $\mathbb{S}^3$ of $\mathbb{R}^4$. Moreover the right translation

$$R_g : h \mapsto hg, \qquad h, g \in SU(2),$$

of $SU(2)$ is a rotation (an element of $SO(4)$) of $\mathbb{S}^3$ (identified with $SU(2)$). The homeomorphism (3.2) preserves the invariant measure, i.e., if dg is the normalized Haar measure on $SU(2)$, then $\Phi(dg)$ is the normalized Lebesgue measure on $\mathbb{S}^3$. As the 3-dimensional polar coordinates on $\mathbb{S}^3$ are

$$\begin{aligned} a_1 &= \cos\theta, \\ a_2 &= \sin\theta\cos\varphi, \\ b_1 &= \sin\theta\sin\varphi\cos\psi, \\ b_2 &= \sin\theta\sin\varphi\sin\psi, \end{aligned} \tag{3.3}$$

with $(\theta, \varphi, \psi) \in [0, \pi] \times [0, \pi] \times [0, 2\pi]$, the normalized Haar integral of $SU(2)$ for an integrable function f is

$$\int_{SU(2)} f(g)\, dg = \frac{1}{2\pi^2} \int_0^{\pi} \sin\varphi\, d\varphi \int_0^{\pi} \sin^2\theta\, d\theta \int_0^{2\pi} f(\theta, \varphi, \psi)\, d\psi. \tag{3.4}$$

The bi-invariant Riemannian metric on $SU(2)$ is necessarily proportional to the negative Killing form of its Lie algebra $su(2)$ (the real vector space of anti-hermitian complex 2×2 matrices). We consider the bi-invariant metric corresponding to the Ad-invariant inner product on $su(2)$,

$$\langle X, Y \rangle = -\tfrac{1}{2}\operatorname{tr}(XY), \qquad X, Y \in su(2).$$

Therefore as an orthonormal basis of $su(2)$ we can take the matrices

$$X_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad X_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

The homeomorphism (3.2) is actually an isometry between $SU(2)$ endowed with this distance and $\mathbb{S}^3$. Hence the restricted negative definiteness of the kernel d on $SU(2)$ is an immediate consequence of this property on $\mathbb{S}^3$, which is known to be true as mentioned in the introduction ([5], [8], [6]). In order to develop a comparison with $SO(3)$, we shall give a different proof of this fact in §5.

4. $SO(n)$**.** We first investigate the case $n = 3$. The group $SO(3)$ can also be realized as a quotient of $SU(2)$. Actually the adjoint representation Ad of $SU(2)$ is a surjective morphism from $SU(2)$ onto $SO(3)$ with kernel $\{\pm e\}$ (see [3] e.g.). Hence the well known result

$$SO(3) \cong SU(2)/\{\pm e\}. \tag{4.1}$$

Let us explicitly recall this morphism: if $a = a_1 + ia_2, b = b_1 + ib_2$ with $|a|^2 + |b|^2 = 1$ and

$$\widetilde{g} = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}$$

then the orthogonal matrix $\mathrm{Ad}(\widetilde{g})$ is given by

$$g = \begin{pmatrix} a_1^2 - a_2^2 - (b_1^2 - b_2^2) & -2a_1a_2 - 2b_1b_2 & -2(a_1b_1 - a_2b_2) \\ 2a_1a_2 - 2b_1b_2 & (a_1^2 - a_2^2) + (b_1^2 - b_2^2) & -2(a_1b_2 + a_2b_1) \\ 2(a_1b_1 + a_2b_2) & -2(-a_1b_2 + a_2b_1) & |a|^2 - |b|^2 \end{pmatrix}. \tag{4.2}$$

The isomorphism in (4.1) might suggest that the positive definiteness of the Brownian kernel on $SU(2)$ implies a similar result for $SO(3)$. This is not true and actually it turns out that the distance $(g, h) \mapsto d(g, h)$ on $SO(3)$ induced by its bi-invariant Riemannian metric *is not a restricted negative definite kernel* (see Lemma 2.1).

As for $SU(2)$, the bi-invariant Riemannian metric on $SO(3)$ is proportional to the negative Killing form of its Lie algebra $so(3)$ (the real antisymmetric 3×3 matrices). We shall consider the Ad-invariant inner product on $so(3)$ defined as

$$\langle A, B \rangle = -\tfrac{1}{2}\,\mathrm{tr}(AB), \qquad A, B \in so(3).$$

An orthonormal basis for $so(3)$ is then given by the matrices

$$A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Similarly to the case of $SU(2)$, it is easy to compute the distance from $g \in SO(3)$ to the identity. Actually g is conjugate to the matrix

$$\Delta(t) = \begin{pmatrix} \cos t & \sin t & 0 \\ -\sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix} = \exp(tA_1)$$

where $t \in [0, \pi]$ is the rotation angle of g. Therefore if d still denotes the distance induced by the bi-invariant metric, then

$$d(g, e) = d(\Delta(t), e) = t,$$

i.e. the distance from g to e is the rotation angle of g.

Let us denote by $\{\chi_\ell\}_{\ell \geq 0}$ the set of characters for $SO(3)$. It is easy to compute the Peter–Weyl development (2.6) for $d(\cdot, e)$, as the characters χ_ℓ are also simple functions of the rotation angle. More precisely, if t is the rotation angle of g (see [10] e.g.), then

$$\chi_\ell(g) = \frac{\sin\frac{(2\ell+1)t}{2}}{\sin\frac{t}{2}} = 1 + 2\sum_{m=1}^{\ell}\cos(mt).$$

We shall prove that the coefficient

$$\alpha_\ell = \int\limits_{SO(3)} d(g,e)\chi_\ell(g)\,dg$$

is positive for some $\ell \geq 1$. As both $d(\cdot,e)$ and χ_ℓ are functions of the rotation angle t, we have

$$\alpha_\ell = \int\limits_0^\pi t\Big(1 + 2\sum_{j=1}^{\ell}\cos(jt)\Big)p_T(t)\,dt$$

where p_T is the density of $t = t(g)$, considered as a r.v. on the probability space $(SO(3), dg)$. The next statements are devoted to the computation of the density p_T. This is certainly well known but we were unable to find a reference in the literature. We first compute the density of the trace of g.

PROPOSITION 4.1. *The distribution of the trace of a matrix in $SO(3)$ with respect to the normalized Haar measure is given by the density*

$$f(y) = \frac{1}{2\pi}(3-y)^{1/2}(y+1)^{-1/2}1_{[-1,3]}(y). \tag{4.3}$$

Proof. The trace of the matrix (4.2) is equal to

$$\mathrm{tr}(g) = 3a_1^2 - a_2^2 - b_1^2 - b_2^2.$$

Under the normalized Haar measure of $SU(2)$ the vector (a_1, a_2, b_1, b_2) is uniformly distributed on the sphere $\mathbb{S}^3$. Recall the normalized Haar integral (3.4) so that, taking the corresponding marginal, θ has density

$$f_1(\theta) = \frac{2}{\pi}\sin^2(\theta)\,d\theta. \tag{4.4}$$

Now

$$3a_1^2 - a_2^2 - b_1^2 - b_2^2 = 4\cos^2\theta - 1.$$

Let us first compute the density of $Y = \cos^2 X$, where X is distributed according to the density (4.4). This is elementary as

$$\begin{aligned} F_Y(t) &= \mathbb{P}(\cos^2 X \leq t) = \mathbb{P}(\arccos(\sqrt{t}) \leq X \leq \arccos(-\sqrt{t})) \\ &= \frac{2}{\pi}\int\limits_{\arccos(\sqrt{t})}^{\arccos(-\sqrt{t})} \sin^2(\theta)\,d\theta. \end{aligned}$$

Taking the derivative it is easily found that the density of Y is, for $0 < t < 1$,

$$F'_Y(t) = \frac{2}{\pi}(1-t)^{1/2}t^{-1/2}.$$

By an elementary change of variable the distribution of the trace $4Y-1$ is therefore given by (4.3). ∎

COROLLARY 4.2. *The distribution of the rotation angle of a matrix in* $SO(3)$ *is*

$$p_T(t) = \frac{1}{\pi}(1-\cos t)\,1_{[0,\pi]}(t).$$

Proof. It suffices to remark that if t is the rotation angle of g, then its trace is equal to $2\cos t + 1$. Therefore p_T is the distribution of $W = \arccos\big(\frac{Y-1}{2}\big)$, Y being distributed as (4.3). The elementary details are left to the reader. ∎

Now it is easy to compute the Fourier development of the function $d(\cdot, e)$.

PROPOSITION 4.3. *The kernel* d *on* $SO(3)$ *is not restricted negative definite.*

Proof. It is enough to show that in the Fourier development

$$d(g,e) = \sum_{\ell\ge 0} \alpha_\ell \chi_\ell(g),$$

$\alpha_\ell > 0$ for some $\ell \ge 1$ (see Remark 2.2). We have

$$\begin{aligned}\alpha_\ell &= \int_{SO(3)} d(g,e)\chi_\ell(g)\,dg = \frac{1}{\pi}\int_0^\pi t\Big(1 + 2\sum_{m=1}^{\ell}\cos(mt)\Big)(1-\cos t)\,dt \\ &= \underbrace{\frac{1}{\pi}\int_0^\pi t(1-\cos t)\,dt}_{:=I_1} + \frac{2}{\pi}\sum_{m=1}^{\ell}\underbrace{\int_0^\pi t\cos(mt)\,dt}_{:=I_2} - \frac{2}{\pi}\sum_{m=1}^{\ell}\underbrace{\int_0^\pi t\cos(mt)\cos t\,dt}_{:=I_3}.\end{aligned}$$

Now integration by parts gives

$$I_1 = \frac{\pi^2}{2} + 2, \qquad I_2 = \frac{(-1)^m - 1}{m^2},$$

whereas, if $m \ne 1$, we have

$$I_3 = \int_0^\pi t\cos(mt)\cos t\,dt = \frac{m^2+1}{(m^2-1)^2}((-1)^m + 1),$$

and for $m = 1$,

$$I_3 = \int_0^\pi t\cos^2 t\,dt = \frac{\pi^2}{4}.$$

Putting things together we find

$$\alpha_\ell = \frac{2}{\pi}\bigg(1 + \sum_{m=1}^{\ell}\frac{(-1)^m - 1}{m^2} + \sum_{m=2}^{\ell}\frac{m^2+1}{(m^2-1)^2}((-1)^m + 1)\bigg).$$

If $\ell = 2$, for instance, we find $\alpha_2 = \frac{2}{9\pi} > 0$, but it is easy to see that $\alpha_\ell > 0$ for every ℓ even. ∎

Consider now the case $n > 3$. Then $SO(n)$ contains a closed subgroup H that is isomorphic to $SO(3)$, and the restriction to H of any bi-invariant distance d on $SO(n)$ is a bi-invariant distance $\widetilde{d}$ on $SO(3)$. By Proposition 4.3, $\widetilde{d}$ is not restricted negative definite, therefore there exist $g_1, \dots, g_m \in H$ and $\xi_1, \dots, \xi_m \in \mathbb{R}$ with $\sum_{i=1}^m \xi_i = 0$ such that

$$\sum_{i,j} d(g_i, g_j)\xi_i\xi_j = \sum_{i,j} \widetilde{d}(g_i, g_j)\xi_i\xi_j > 0. \tag{4.5}$$

We have therefore

Corollary 4.4. *No bi-invariant distance d on $SO(n)$, $n \geq 3$, is a restricted negative definite kernel.*

Note that a bi-invariant Riemannian metric on $SO(4)$ is not unique, meaning that it is not necessarily proportional to the negative Killing form of $so(4)$. In this case Corollary 4.4 states that no such bi-invariant distance can be restricted negative definite.

5. Final remarks. We were intrigued by the different behavior of the invariant distance of $SU(2)$ and $SO(3)$ despite these groups being locally isometric, and decided to compute also for $SU(2)$ the development

$$d(g, e) = \sum_\ell \alpha_\ell \chi_\ell(g). \tag{5.1}$$

This is not difficult, since if we denote by t the distance of g from e, the characters of $SU(2)$ are

$$\chi_\ell(g) = \frac{\sin((\ell+1)t)}{\sin t}, \qquad t \neq k\pi,$$

and $\chi_\ell(e) = \ell+1$ if $t = 0$, $\chi_\ell(g) = (-1)^\ell(\ell+1)$ if $t = \pi$. Then it is elementary to compute, for $\ell > 0$,

$$\alpha_\ell = \frac{1}{\pi}\int_0^\pi t \sin((\ell+1)t) \sin t \, dt = \begin{cases} -\dfrac{8}{\pi}\,\dfrac{m+1}{m^2(m+2)^2}, & \ell \text{ odd}, \\ 0, & \ell \text{ even}, \end{cases}$$

thus confirming the restricted negative definiteness of d (see Remark 2.2). Note also that the coefficients corresponding to the even numbered representations, which are also representations of $SO(3)$, here vanish.

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