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SPACES OF σ-FINITE LINEAR MEASURE

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Abstract. Spaces of finite n-dimensional Hausdorff measure are an important generalization of n-dimensional polyhedra. Continua of finite linear measure (also called continua of finite length) were first characterized by Eilenberg in 1938. It is well-known that the property of having finite linear measure is not preserved under finite unions of closed sets. Mauldin proved that if X is a compact metric space which is the union of finitely many closed sets each of which admits a σ-finite linear measure then X admits a σ-finite linear measure. We answer in the strongest possible way a 1989 question (private communication) of Mauldin. We prove that if a separable metric space is a countable union of closed subspaces each of which admits finite linear measure then it admits σ-finite linear measure. In particular, it can be embedded in the 1-dimensional Nöbeling space ν_1^3 so that the image has σ-finite linear measure with respect to the usual metric on ν_1^3.

1. Introduction. Mauldin in 1990 [9] proved that if a compact metric space may be expressed as a finite union of closed subsets each admitting σ-finite linear Hausdorff measure, then the whole space admits σ-finite linear Hausdorff measure. He asked for a characterization of spaces that admit σ-finite linear Hausdorff measure. We answer in the strongest possible way a 1989 question of Mauldin. We prove that if a separable metric space is a countable union of closed subspaces each of which admits finite linear measure then it can be embedded in the 1-dimensional Nöbeling space ν_1^3 so that the image has σ-finite linear measure with respect to the usual metric on ν_1^3. The proof relies significantly on the construction of Buskirk, Nikiel and Tymchatyn [2].

Eilenberg and Harrold [6] asked for a characterization of continua admitting finite n-dimensional Hausdorff measure. They obtained a number of characterizations of continua of finite linear measure. Most useful for us they proved that a space X admits a finite linear Hausdorff measure if and only if it is totally regular, i.e. for each $x \in X$ and for each neighbourhood U of x there exist uncountably many nested neighbourhoods $\{U_\alpha\}$ of x with $U_\alpha \subset U$ such that $\mathrm{Bd}(U_\alpha) \cap \mathrm{Bd}(U_\beta) = \emptyset$ for $\alpha \neq \beta$ and with $\mathrm{Bd}(U_\alpha)$ finite.

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In particular, X is hereditarily locally connected, i.e. each connected subset of X is locally connected.

All spaces in this paper are separable and metric. We let $(\mathbb{R}^3, d)$ denote the Euclidean 3-space with its usual metric.

2. Preliminaries

DEFINITION 2.1. Let (X, ρ) be a separable metric space and $\alpha \geq 0$. Then the α-dimensional Hausdorff measure H^α_ρ on X is defined by

$$H^\alpha_\rho(A) = \sup_{\delta>0} \inf \Big\{ \sum_{i=1}^{\infty} (\mathrm{diam}_\rho(U_i))^\alpha \;\Big|\; A \subset \bigcup_{i=1}^{\infty} U_i \subseteq X, \\ \mathrm{diam}_\rho(U_i) < \delta \text{ for every } i \in \mathbb{N} \Big\}$$

for any $A \subset X$. We call H^1_ρ the *linear Hausdorff measure* on (X, ρ).

DEFINITION 2.2. The *n-dimensional Nöbeling space* ν^{2n+1}_n is the subspace of the Euclidean space $\mathbb{R}^{2n+1}$ which consists of all points with at most n rational coordinates.

The space ν^{2n+1}_n is universal for separable metric spaces of dimension at most n.

Fremlin [7, Theorem 5H] proved that a space of finite linear measure embeds in $(\mathbb{R}^3, d)$ so its image has finite linear measure with respect to the metric d. We shall need the following strengthening of Fremlin's result:

THEOREM 2.3. *Let C' be a space which admits a metric ρ such that $H^1_\rho(C') < \infty$. Then*

(i) *C' embeds in a continuum $C \subset \nu^3_1$ with $H^1_d(C) < \infty$.*
(ii) *If $K \subset C$ is a discrete set such that each point $x \in K$ has an uncountable local basis in C of open sets with two-point boundaries in C and if K is contained in ν^3_1, then the embedding may be taken to be the identity on K.*

Proof. The proof essentially depends on the ideas from [2, Theorems 3 and 4].

Let us prove part (i) first. By [5], the space C' is totally regular. Let C'' be the Freudenthal compactification of C' (see [8, p. 109]). Then C'' is a totally regular, metric compactum because finite separators of C' separate the distinct points of C''. The components of C'' form a null family of locally connected continua. By a standard argument one can adjoin to C'' a countable null sequence of arcs to obtain a totally regular metric continuum C which contains C''. By [2, Theorem 3] the space C is the inverse limit of an inverse sequence (C_n, f^{n+1}_n) of finite connected graphs and monotone, surjective bonding maps so that each $f_n^{n+1}\colon C_{n+1} \to C_n$ has at most one non-degenerate fibre.

Represent the 1-dimensional Nöbeling space ν_1^3 as $\mathbb{R}^3 \setminus \bigcup_{i=1}^{\infty} A_i$ where each A_i is a straight line in $\mathbb{R}^3$ with each point of A_i having at least two rational coordinates. We equip ν_1^3 with the restriction of the usual metric d from $\mathbb{R}^3$.

We may assume that C and $\bigcup_{n=1}^{\infty} C_n$ are embedded in ν_1^3 so that C is also the limit of the sequence $\{C_n\}$ in the Hausdorff metric generated by d. Indeed, suppose that C_1 is embedded as a polygonal graph in ν_1^3. Let ε_1 be less than half the distance from the compact set C_1 to A_1. Assume that n is a positive integer and that $C_1, \ldots, C_n$ are embedded as polygonal graphs in ν_1^3 and $\varepsilon_1 > 2\varepsilon_2 > \cdots > 2^{n-1}\varepsilon_n$ are positive numbers such that for all $1 \le i \le n-1$,

(1) $|H_d^1(C_{i+1}) - H_d^1(C_i)| < 2^{-i-1}$,

(2) the Hausdorff distance from C_{i+1} to C_i is less than 2^{-i-2},

(3) the non-degenerate fiber of f_i^{i+1} has length less than 2^{-i-1},

(4) f_i^{i+1} is the identity off a sufficiently small neighbourhood of the non-degenerate element of f_i^{i+1},

(5) the distance from C_i to A_j is greater than $2\varepsilon_j$ for $j \le i \le n$.

Let $\varepsilon_{n+1} > 0$ be smaller than $\frac{1}{2}\min\{\varepsilon_n, d(C_1 \cup \cdots \cup C_n, A_1 \cup \cdots \cup A_{n+1})\}$. We may take C_{n+1} to be a polygonal graph in ν_1^3 so that conditions (1)–(5) are satisfied for $1 \le i \le n$.

It follows that the sequence $\{C_n\}$ converges to C in the Hausdorff metric.

The limit $\lim H_d^1(C_n)$ exists as the limit of a Cauchy sequence of real numbers. In particular, $\lim H_d^1(C_n) = H_d^1(C)$ by (1) and (4) because for each n we have $C \subset C_n \cup \bigcup_{m=n+1}^{\infty} B_m$ where B_m is a ball and $\sum_{m=n+1}^{\infty} \operatorname{diam}_d(B_m) < 2^{-n}\varepsilon_1$.

By (5), $C \subset \nu_1^3$.

Now we show that part (ii) of the theorem is true. For each $x \in K$ let V_x be an open neighbourhood of x with two-point boundary $\{a_x, b_x\}$. Let

$$E(a_x, b_x, V_x) = \{y \in V_x \mid y \text{ separates } a_x \text{ and } b_x \text{ in } V_x\} \cup \{a_x, b_x\}.$$

By [10, III, 4.2], $E(a_x, b_x, V_x)$ is compact. By [10, III, 1.31], $E(a_x, b_x, V_x)$ is naturally ordered. Let $F^0(a_x, b_x, V_x)$ be the set of condensation points of $E(a_x, b_x, V_x)$. Then in the decomposition $\mathcal{G}_x$ of $\overline{V_x}$ to an arc it is easy to see that we may take the equivalence class of x in $\mathcal{G}_x$ to be $\{x\}$. With this additional observation the proof of (ii) goes through as in [2, Theorem 3]. ∎

It follows trivially from Theorem 2.3 that every space of finite length has a compactification of finite length in (ν_1^3, d).

Definition 2.4. A closed subset A of a complete metric space Y is called a *Z-set* if for each open cover $\mathcal{U}$ of Y there is a function $f\colon Y \to Y \setminus A$ which is $\mathcal{U}$-close to Id_Y, i.e. for every $y \in Y$ there is $U \in \mathcal{U}$ with $y, f(y) \in U$. If the map f can be chosen in such a way that $\overline{f(Y)} \cap A = \emptyset$ then A is called a *strong Z-set*.

DEFINITION 2.5. For a space A and a complete metric space Y an embedding $g\colon A \to Y$ is called a *Z-embedding* if its image is a Z-set in Y.

DEFINITION 2.6. Let Y and Z be topological spaces and let $C(Y, Z)$ denote the set of all continuous functions from Y to Z. For each map $f\colon Y \to Z$ and for each open cover $\mathcal{S}$ of Z we let $B(f, \mathcal{S})$ denote the set of all maps in $C(Y, Z)$ that are $\mathcal{S}$-close to f. Define a collection $\mathcal{T}$ of subsets of $C(Y, Z)$ by the rule: a subset $U \subset C(Y, Z)$ is an element of $\mathcal{T}$ if for every $f \in U$, there exists an open cover $\mathcal{U}$ of Z such that $B(f, \mathcal{U}) \subset U$. If U and V are elements of $\mathcal{T}$ such that $B(f, \mathcal{U}) \subset U$ and $B(f, \mathcal{V}) \subset V$ for open covers $\mathcal{U}$ and $\mathcal{V}$ of Z, then $B(f, \mathcal{W}) \subset U \cap V$ for any open cover $\mathcal{W}$ which refines both $\mathcal{U}$ and $\mathcal{V}$. The collection $\mathcal{T}$ is called the *limitation topology* on $C(Y, Z)$.

It is known that the limitation topology coincides with the topology of uniform convergence with respect to all compatible metrics on Y and Z (see [3, Lemma 2.1.4]).

DEFINITION 2.7. Let n be a positive integer. A Polish space Y is called an *absolute [neighbourhood] extensor in dimension n*, or briefly, an $A[N]E(n)$-space, if any map $f\colon A \to Y$, defined on a closed subspace A of a Polish space B with $\dim B \leq n$, can be extended to a map of the space B [respectively, of a neighbourhood of A in B] into Y.

DEFINITION 2.8. A Polish space Y is called *strongly $\mathcal{A}_{\omega,n}$-universal* if any map of any at most n-dimensional Polish space into Y can be arbitrarily closely approximated by closed embeddings.

We will need the following result (see [3, Proposition 5.1.7]).

PROPOSITION 2.9. *Let Y be an at most n-dimensional strongly $\mathcal{A}_{\omega,n}$-universal Polish* ANE(n)*-space, and A a closed subspace of an at most n-dimensional Polish space B. Then each map $f\colon B \to Y$ such that the restriction $f|_A$ is a Z-embedding can be arbitrarily closely approximated by Z-embeddings coinciding with f on A. In particular, the set of all Z-embeddings of B into Y is a dense G_δ subset of $C(B, Y)$.*

It is known that the n-dimensional Nöbeling space ν_n^{2n+1} is a strongly $\mathcal{A}_{\omega,n}$-universal, ANE(n)-space. The following two statements are proved in [4] as Proposition 3.6 and Lemma 3.2, respectively.

PROPOSITION 2.10. *Let P be an at most n-dimensional Polish space and let $C(P, \nu_n^{2n+1})$ denote the set of all continuous functions from P into ν_n^{2n+1} with the limitation topology. Then the set of all Z-embeddings of P into ν_n^{2n+1} is a dense G_δ subset of $C(P, \nu_n^{2n+1})$.*

Proposition 2.11. *Each compact subset of ν_n^{2n+1} is a strong Z-set.*

Definition 2.12. A point x of a connected space X is a *local cut point* of X if it disconnects some connected neighbourhood of x. The local cut point x is said to be *of order* 2 *in* X if it has a basis of neighbourhoods with two-point boundaries.

Theorem 2.13. *If X is a connected and totally regular space then X has at each point an uncountable local basis $\{U_\alpha\}$ of open sets with finite boundaries and such that each boundary point of U_α is a point of order* 2 *in X.*

Proof. Let Y be a totally regular continuum containing X and constructed as in the proof of Theorem 2.3. Each local cut point of Y is a local cut point of X. By [10, III, 9.2] all but at most countably many local cut points of Y are of order 2 in Y. ∎

Definition 2.14. We say that a space Y *admits σ-finite linear measure* if there is a metric ρ on Y and a family $\{A_i\}_{i=1}^{\infty}$ of closed subsets of Y with $Y = \bigcup_{i=1}^{\infty} A_i$ and $H_\rho^1(A_i) < \infty$ for each i.

3. Main result

Theorem 3.1. *Let $X = \bigcup_{i=1}^{\infty} X_i$ where each X_i is totally regular and closed in X. Then the space X can be embedded in ν_1^3 so that the image of X has σ-finite linear measure with respect to the usual metric d on ν_1^3.*

Proof. Let $h_1' \colon X_1 \to \tilde{X}_1 \subset \nu_1^3$ be a compactification of X_1 where $\tilde{X}_1$ has finite length with respect to the metric d by Theorem 2.3. Let $\pi \colon X \to X \cup_{h_1'} \tilde{X}_1$ be the natural projection of X into the adjunction space $X \cup_{h_1'} \tilde{X}_1$. Since $\tilde{X}_1$ is compact, it is a Z-set in ν_1^3 by Proposition 2.11. Since ν_1^3 is an ANE(1) and $X \cup_{h_1'} \tilde{X}_1$ is one-dimensional, separable metric, $\mathrm{id}_{\tilde{X}_1}$ extends to a continuous map $\varphi \colon X \cup_{h_1'} \tilde{X}_1 \to \nu_1^3$. By Proposition 2.9, φ can be approximated by a homeomorphism $\tilde{\varphi} \colon X \cup_{h_1'} \tilde{X}_1 \to \nu_1^3$. Let $h_1 \colon X \to \nu_1^3$ be the embedding $\tilde{\varphi} \circ \pi$.

Let $\mathcal{U}_1'$ be a locally finite cover of $\mathbb{R}^3 \setminus \tilde{X}_1$ by open topological 3-balls such that $\mathrm{diam}_d(U') < \min\{1/4, d(\tilde{X}_1, U')/4\}$ for each $U' \in \mathcal{U}_1'$. We denote by $\mathcal{U}_1$ the cover of $\nu_1^3 \setminus \tilde{X}_1$ which is induced by $\mathcal{U}_1'$, i.e. $\mathcal{U}_1 = \{U' \cap \nu_1^3 \mid U' \in \mathcal{U}_1'\}$. Since $h_1(X_2) \setminus \tilde{X}_1$ is totally regular, let $\mathcal{V}_2 = \{X_{2,1}, X_{2,2}, \dots\}$ be a locally finite in $\mathbb{R}^3 \setminus \tilde{X}_1$ closed cover of $h_1(X_2) \setminus \tilde{X}_1$ and let $\{I_{2,1}, I_{2,2}, \dots\}$ be finite sets of local cut points of order 2 in $h_1(X_2)$ such that

$$I_{2,i} \subset X_{2,i}, \qquad X_{2,i} \cap X_{2,j} \subset I_{2,i} \cap I_{2,j} \quad \text{for } i \neq j,$$

$$\bigcup_{i=1}^{\infty} I_{2,i} \text{ is discrete in } \mathbb{R}^3 \setminus \tilde{X}_1$$

and such that $\mathcal{V}_2$ refines $\mathcal{U}_1$. For each i let $U_{2,i} \in \mathcal{U}_1$ satisfy $X_{2,i} \subset U_{2,i}$. Also for each i let $T_{2,i} \subset U_{2,i}$ be a polygonal tree in ν_1^3 with set of endpoints $I_{2,i}$ such that $T_{2,i} \cap T_{2,j} = I_{2,i} \cap I_{2,j}$. For each i let $W_{2,i} = W'_{2,i} \cap \nu_1^3$ where $W'_{2,i}$ is a closed polyhedral 3-ball in $\mathbb{R}^3$ such that

$$T_{2,i} \subset W_{2,i} \subset U_{2,i}, \quad T_{2,i} \cap \mathrm{Bd}(W_{2,i}) = I_{2,i}$$

and

$$W_{2,i} \cap W_{2,j} = I_{2,i} \cap I_{2,j} \quad \text{for } i \neq j.$$

For each i let $h_{2,i} \colon X_{2,i} \to \tilde{X}_{2,i} \subset \mathrm{Int}_{\nu_1^3}(W_{2,i}) \cup I_{2,i}$ be a compactification where each $\tilde{X}_{2,i}$ has finite length with respect to the metric d and $h_{2,i}|_{I_{2,i}} = \mathrm{Id}_{I_{2,i}}$. Let $\tilde{X}_2 = \bigcup_{i=1}^{\infty} \tilde{X}_{2,i}$. Note that $\tilde{X}_1 \cup \tilde{X}_2$ is compact. Let $h'_2 \colon \tilde{X}_1 \cup h_1(X_2) \to \tilde{X}_1 \cup \tilde{X}_2$ be an embedding such that $h'_2|_{\tilde{X}_1} = \mathrm{Id}_{\tilde{X}_1}$ and $h'_2|_{X_{2,i}} = h_{2,i}$ for all i. Note that $h'_2|_{h_1(X_2 \setminus X_1)}$ is $\mathcal{U}_1$-close to $h_1|_{X_2 \setminus X_1}$.

Since $\tilde{X}_1 \cup \tilde{X}_2$ is compact in ν_1^3, it is a strong Z-set, and so h'_2 can be extended to an embedding h_2 of $\tilde{X}_1 \cup h_1(X)$ such that $h_2|_{h_1(X) \setminus \tilde{X}_1}$ is $\mathcal{U}_1$-close to $h_1|_{X \setminus X_1}$.

Let $\mathcal{U}'_2$ be a locally finite cover of $\mathbb{R}^3 \setminus (\tilde{X}_1 \cup \tilde{X}_2)$ by open topological 3-balls such that $\mathrm{diam}_d(U') < \min\{1/8, d(\tilde{X}_1 \cup \tilde{X}_2, U')/8\}$ for $U' \in \mathcal{U}'_2$. We denote by $\mathcal{U}_2$ the cover of $\nu_1^3 \setminus (\tilde{X}_1 \cup \tilde{X}_2)$ which is induced by $\mathcal{U}'_2$, i.e. $\mathcal{U}_2 = \{U' \cap \nu_1^3 \mid U' \in \mathcal{U}'_2\}$.

Suppose now that for $1 \leq n \leq k-1$ the covers $\mathcal{U}_n$, the spaces $\tilde{X}_n$ and the embeddings h_n are defined so that the following conditions are satisfied:

(1) $\tilde{X}_1 \cup \cdots \cup \tilde{X}_n$ is compact and of σ-finite linear measure in (ν_1^3, d),

(2) $\mathcal{U}_n$ is a cover of $\nu_1^3 \setminus (\tilde{X}_1 \cup \cdots \cup \tilde{X}_n)$ induced by a locally finite cover of $\mathbb{R}^3 \setminus (\tilde{X}_1 \cup \cdots \cup \tilde{X}_n)$ by open topological 3-balls such that

$$\mathrm{diam}_d(U) < \min\{2^{-n-1}, 2^{-n-1} d(\tilde{X}_1 \cup \cdots \cup \tilde{X}_n, U)\}$$

for each $U \in \mathcal{U}_n$,

(3) the map

$$h_n \colon h_{n-1} \circ \cdots \circ h_1(X) \cup \tilde{X}_1 \cup \cdots \cup \tilde{X}_{n-1} \to \nu_1^3$$

is an embedding such that

$$h_n|_{\tilde{X}_1 \cup \cdots \cup \tilde{X}_{n-1}} = \mathrm{Id}_{\tilde{X}_1 \cup \cdots \cup \tilde{X}_{n-1}}$$

and

$$h_n|_{h_{n-1} \circ \cdots \circ h_1(X) \setminus (\tilde{X}_1 \cup \cdots \cup \tilde{X}_{n-1})} \text{ is } \mathcal{U}_{n-1}\text{-close to}$$
$$h_{n-1}|_{h_{n-2} \circ \cdots \circ h_1(X) \setminus (\tilde{X}_1 \cup \cdots \cup \tilde{X}_{n-1})}.$$

Let $\mathcal{V}_k = \{X_{k,1}, X_{k,2}, \dots\}$ be a locally finite in $\mathbb{R}^3 \setminus (\tilde{X}_1 \cup \cdots \cup \tilde{X}_{k-1})$ closed cover of $h_{k-1} \circ \cdots \circ h_1(X_k) \setminus (\tilde{X}_1 \cup \cdots \cup \tilde{X}_{k-1})$ and let $\{I_{k,1}, I_{k,2}, \dots\}$

be finite sets of local cut points of order 2 in $h_{k-1}\circ\cdots\circ h_1(X_k)$ such that

$$I_{k,i}\subset X_{k,i},\qquad X_{k,i}\cap X_{k,j}=I_{k,i}\cap I_{k,j}\quad\text{for } i\neq j,$$
$$\bigcup_{i=1}^{\infty} I_{k,i}\text{ is discrete in }\mathbb{R}^3\setminus(\tilde X_1\cup\cdots\cup\tilde X_{k-1})$$

and such that $\mathcal{V}_k$ refines $\mathcal{U}_{k-1}$. For each i let $U_{k,i}\in\mathcal{U}_{k-1}$ satisfy $X_{k,i}\subset U_{k,i}$ and let $T_{k,i}$ be a polygonal tree in $U_{k,i}$ with set of endpoints $I_{k,i}$ with $T_{k,i}\cap T_{k,j}=I_{k,i}\cap I_{k,j}$. For each i let $W_{k,i}=W'_{k,i}\cap\nu_1^3$ where $W'_{k,i}$ is a closed polyhedral 3-ball in $\mathbb{R}^3$ such that

$$T_{k,i}\subset W_{k,i}\subset U_{k,i},\qquad T_{k,i}\cap\operatorname{Bd}(W_{k,i})=I_{k,i}$$

and

$$W_{k,i}\cap W_{k,j}=I_{k,i}\cap I_{k,j}\qquad\text{for } i\neq j.$$

For each i let $h_{k,i}\colon X_{k,i}\to\tilde X_{k,i}\subset\operatorname{Int}_{\nu_1^3}(W_{k,i})\cup I_{k,i}$ be a compactification where $\tilde X_{k,i}$ has finite length with respect to d and $h_{k,i}|_{I_{k,i}}=\mathrm{Id}_{I_{k,i}}$. Let $\tilde X_k=\bigcup_{i=1}^{\infty}\tilde X_{k,i}$ and let

$$h'_k\colon \tilde X_1\cup\cdots\cup\tilde X_{k-1}\cup h_{k-1}\circ\cdots\circ h_1(X_k)\to\tilde X_1\cup\cdots\cup\tilde X_k$$

be a compactification such that

$$h'_k|_{\tilde X_1\cup\cdots\cup\tilde X_{k-1}}=\mathrm{Id}_{\tilde X_1\cup\cdots\cup\tilde X_{k-1}}\quad\text{and}\quad h'_k|_{X_{k,i}}=h_{k,i}\quad\text{for each } i.$$

Note that

$$h'_k|_{h_{k-1}\circ\cdots\circ h_1(X_k\setminus(X_1\cup\cdots\cup X_{k-1}))}\text{ is }\mathcal{U}_{k-1}\text{-close to }h_{k-1}\circ\cdots\circ h_1|_{X_k\setminus(X_1\cup\cdots\cup X_{k-1})}.$$

Since $\tilde X_1\cup\cdots\cup\tilde X_k$ is compact in ν_1^3, h'_k can be extended to an embedding h_k of $\tilde X_1\cup\cdots\cup\tilde X_{k-1}\cup h_{k-1}\circ\cdots\circ h_1(X)$ such that

$$h_k|_{h_{k-1}\circ\cdots\circ h_1(X)\setminus(\tilde X_1\cup\cdots\cup\tilde X_{k-1})}\text{ is }\mathcal{U}_{k-1}\text{-close to }h_{k-1}|_{h_{k-2}\circ\cdots\circ h_1(X)\setminus(\tilde X_1\cup\cdots\cup\tilde X_{k-1})}.$$

Let $\mathcal{U}'_k$ be a locally finite cover of $\mathbb{R}^3\setminus(\tilde X_1\cup\cdots\tilde X_k)$ by open polyhedral 3-balls with $\operatorname{diam}_d(U)<\min\{2^{-k-1},2^{-k-1}d(\tilde X_1\cup\cdots\cup\tilde X_k,U)\}$ for each $U\in\mathcal{U}'_k$ and let $\mathcal{U}_k$ be the corresponding induced cover of $\nu_1^3\setminus(\tilde X_1\cup\cdots\cup\tilde X_k)$.

Then by induction h_k is defined for each positive integer k. Let $h=\lim_{k\to\infty}h_k\circ\cdots\circ h_1$. Since the sequence $\{h_k\circ\cdots\circ h_1\}_{k=1}^{\infty}$ is uniformly convergent, h is a continuous function. Since every function $h_k\circ\cdots\circ h_1$ is one-to-one, for each $x\in X$ there exists a positive integer n such that $x\in X_n$ and $h_k\circ\cdots\circ h_n\circ\cdots\circ h_1(x)=h_n\circ\cdots\circ h_1(x)$ for $k\geq n$. It follows that h is one-to-one. If $x\in X\setminus(X_1\cup\cdots\cup X_k)$ and $h_k\circ\cdots\circ h_1(x)\in U\in\mathcal{U}_{k-1}$ then

$$h(x)\in\operatorname{St}^2(U,\mathcal{U}_{k-1})\subset\overline{\operatorname{St}^2(U,\mathcal{U}_{k-1})}\subset h(X)\setminus(\tilde X_1\cup\cdots\cup\tilde X_k)$$

as in [1, Theorem 4.2]. Hence, h is open. Thus, h is an embedding of X into $\bigcup_{i=1}^{\infty}\tilde X_i$. The space $\bigcup_{i=1}^{\infty}\tilde X_i$ is σ-compact and of σ-finite linear measure. ∎

Note. Theorem 3.1 is sharp in the following sense. It is not true that a space of σ-finite linear measure embeds in a compact space of σ-finite linear measure. For if $X = \mathbb{Q} \times [0, 1]$ where $\mathbb{Q}$ is the space of rational numbers then X has σ-finite linear measure. It is easy to see that if $\tilde{X}$ is a metric compactification of X then each separation of $\tilde{X}$ between $(0, 0)$ and $(0, 1)$ contains a perfect set. However, Mauldin has shown that a space with σ-finite linear measure has a basis of open sets with countable boundaries.

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