COLLOQUIUM MATHEMATICUM

VOL. 133 2013 NO. 2

BANACH SPACES WIDELY COMPLEMENTED IN EACH OTHER

BY

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Abstract. Suppose that X and Y are Banach spaces that embed complementably into each other. Are X and Y necessarily isomorphic? In this generality, the answer is no, as proved by W. T. Gowers in 1996. However, if X contains a complemented copy of its square X^2, then X is isomorphic to Y whenever there exists $p \in \mathbb{N}$ such that X^p can be decomposed into a direct sum of X^{p-1} and Y. Motivated by this fact, we introduce the concept of (p, q, r) *widely complemented* subspaces in Banach spaces, where p, q and $r \in \mathbb{N}$. Then, we completely determine when X is isomorphic to Y whenever X is (p, q, r) widely complemented in Y and Y is (t, u, v) widely complemented in X. This new notion of complementability leads naturally to an extension of the Square-cube Problem for Banach spaces, the p-q-r Problem.

1. Introduction.

1. Introduction. If X and Y are Banach spaces, then $X \sim Y$ means that X is isomorphic to Y, and $Y \overset{c}{\hookrightarrow} X$ means that X contains a complemented copy of Y, that is, X contains a subspace isomorphic to Y which is complemented in X. If $n \in \mathbb{N} = \{1, 2, 3, \ldots\}$ and X is a Banach space, then X^n denotes the sum of n copies of X, $X \oplus \cdots \oplus X$. It will be useful to define $X^0 = \{0\}$.

Pełczyński [17] proved that if two Banach spaces X and Y satisfy

$$X \sim X^2 \quad \text{and} \quad Y \sim Y^2, \tag{1.1}$$

then X is isomorphic to Y whenever

$$Y \overset{c}{\hookrightarrow} X \quad \text{and} \quad X \overset{c}{\hookrightarrow} Y. \tag{1.2}$$

The problem whether this holds without the hypothesis (1.1) has been known as Schroeder–Bernstein Problem for Banach spaces. In 1996, W. T. Gowers [14] gave a first negative solution to this problem (see also [1]–[6] and [15]. More recently [16] a $C(K)$ space was introduced which is also a solution to this problem.

In the present paper we turn our attention to Banach spaces X and Y satisfying (1.2) and the following condition weaker than (1.1):

$$X^2 \overset{c}{\hookrightarrow} X. \tag{1.3}$$

2010 *Mathematics Subject Classification*: Primary 46B03; Secondary 46B25.
Key words and phrases: Schroeder–Bernstein problem, Banach spaces complemented in each other.

DOI: 10.4064/cm133-2-14

The motivation of this work is the fact that if (1.2) and (1.3) hold, then X is isomorphic to Y if and only if there exists $p \in \mathbb{N}$ such that

$$X^p \sim X^{p-1} \oplus Y. \tag{1.4}$$

Indeed, let us show the non-trivial implication. Since (1.3) holds, it follows that $X^p \overset{c}{\hookrightarrow} X$. Let A, B and C be Banach spaces satisfying

$$X \sim Y \oplus A, \quad Y \sim X \oplus B \quad \text{and} \quad X \sim X^p \oplus C. \tag{1.5}$$

Adding A to both sides of the third condition of (1.5) we deduce

$$\begin{aligned} X \oplus A &\sim X^p \oplus A \oplus C \sim X^{p-1} \oplus Y \oplus A \oplus C \sim X^{p-1} \oplus X \oplus C \\ &\sim X^p \oplus C \sim X. \end{aligned} \tag{1.6}$$

Now adding B to both sides of (1.6) we conclude

$$X \sim Y \oplus A \sim X \oplus A \oplus B \sim X \oplus B \sim Y,$$

and we are done.

This remark leads us to strengthening the classical concept of complemented subspaces in Banach spaces X which contain a complemented copy of X^2.

Definition 1.1. Let X be a Banach space containing a complemented copy of its square X^2. A Banach space Y is *widely complemented* in X if there exist p, q and $r \in \mathbb{N}$ such that

$$X^p \sim X^q \oplus Y^r. \tag{1.7}$$

In this case, we say that Y is (p, q, r) *widely complemented* in X.

It is also useful to denote

$$Y \overset{(p,q,r)}{\hookrightarrow} X \quad \text{whenever} \quad X^2 \overset{c}{\hookrightarrow} X \text{ and } X^p \sim X^q \oplus Y^r, \tag{1.8}$$

that is, whenever Y is (p, q, r) widely complemented in X.

Remark 1.2. By (1.2)–(1.4) we see that X is isomorphic to Y whenever there exists $p \in \mathbb{N}$ such that

$$Y \overset{(p,p-1,1)}{\hookrightarrow} X \quad \text{and} \quad X \overset{c}{\hookrightarrow} Y.$$

Nevertheless, thanks to some Banach spaces constructed by W. T. Gowers and B. Maurey in 1997 (see the third section below and the proof of [8, Proposition 7.2]), for every $p, q, r \in \mathbb{N}$ with $q \neq p-1$ or $r \neq 1$, there are non-isomorphic Banach spaces X and Y such that

$$Y \overset{(p,q,r)}{\hookrightarrow} X \quad \text{and} \quad X \overset{c}{\hookrightarrow} Y.$$

On the other hand, we can easily check that X is isomorphic to Y whenever

$$Y \overset{(3,1,2)}{\hookrightarrow} X \quad \text{and} \quad X \overset{(4,1,3)}{\hookrightarrow} Y,$$

or still

$$Y \xhookrightarrow{(2,3,1)} X \quad \text{and} \quad X \xhookrightarrow{(2,4,1)} Y.$$

It is then natural, in the spirit of [10]–[12] to consider the Schroeder–Bernstein type problem for widely complemented subspaces in Banach spaces; that is, we are led to the following problem.

PROBLEM 1.3. Determine the set of pairs of triples $\{(p,q,r),(s,t,u)\}$ such that X is isomorphic to Y whenever these Banach spaces satisfy

$$Y \xhookrightarrow{(p,q,r)} X \quad \text{and} \quad X \xhookrightarrow{(s,t,u)} Y.$$

The main aim of this paper is to solve this problem. In order to do this it is convenient to do define:

DEFINITION 1.4. We say that $\{(p,q,r),(s,t,u)\}$ is a *Schroeder–Bernstein pair of triples for Banach spaces* (for short, SBpt) when X is isomorphic to Y whenever these Banach spaces satisfy

$$Y \xhookrightarrow{(p,q,r)} X \quad \text{and} \quad X \xhookrightarrow{(s,t,u)} Y.$$

Equivalently, $\{(p,q,r),(s,t,u)\}$ is a SBpt when X is isomorphic to Y whenever X and Y satisfy (1.1), X contains a complemented copy of X^2 and the following Decomposition Scheme holds:

$$\begin{cases} X^p \sim X^q \oplus Y^r \\ Y^s \sim Y^t \oplus X^u. \end{cases} \tag{1.9}$$

We also say that $\mathcal{N} = (p-q)(s-t) - ru$ is the *w-number* of the pair of triples $\{(p,q,r),(s,t,u)\}$.

From now on our purpose is to prove the following characterization of the pairs of triples which are SBpt. This is an immediate consequence of Propositions 2.1, 2.2, 3.1, 3.3, 3.4 and 3.5.

THEOREM 1.5. *A pair of triples $\{(p,q,r),(s,t,u)\}$ with w-number $\mathcal{N}$ is a SBpt if and only if one of the following conditions holds:*

(a) $\mathcal{N} \neq 0$ *and* $\mathcal{N}$ *divides* $q-p+r$ *and* $t-s+u$*;*
(b) $\mathcal{N} = 0$*,* $p = q+r$ *and* $gcd(r,u) = 1$*.*

REMARK 1.6. Observe that in virtue of (1.8), the hypothesis "X contains a complemented copy of X^2" is implicit in Definition 1.4. Moreover, this hypothesis is essential to obtain Theorem 1.5. Indeed, it is an open problem to characterize the sextuples (p,q,r,s,t,u) such that X is isomorphic to Y whenever these Banach spaces satisfy (1.1) and the Decomposition Scheme (1.9) holds (see [7, Conjecture 4.3]).

REMARK 1.7. The hypothesis "X contains a complemented copy of X^2" in Definition 1.1 does not imply that (1.7) is true for some p, q and $r \in \mathbb{N}$, even in the case where (1.1) holds. Indeed, we recall that two Banach spaces X and Y are said to be *totally incomparable* if no infinite-dimensional subspace of X is isomorphic to a subspace of Y. Fix two totally incomparable Banach spaces X and Y from the class of spaces constructed in [13]. Then by [5] there exists a Banach spaces Z satisfying

(a) $Z \sim Z^2$ [5, p. 31];
(b) $Z \sim Z \oplus X^m \oplus Y^m$ for all $m \in \mathbb{N}$ [5, p. 31];
(c) $Z \not\sim Z \oplus X^m$ for all $m \in \mathbb{N}$ [5, Theorem 3.4].

Define $E = Z \oplus X$. Then by (b), $Z \overset{c}{\hookrightarrow} E$ and $E \overset{c}{\hookrightarrow} Z$. Suppose that there exist $p, q, r \in \mathbb{N}$ such that $Z^p \sim Z^q \oplus E^r$. According to (a) we see that $Z \sim Z \oplus X^r$, which is absurd by (c).

However, we do not know whether the hypothesis "X contains a complemented copy of X^2" in Definition 1.1 is a consequence of (1.7) when $1 < p \neq q + r$ and (1.1) holds, that is:

Problem 1.8 (The p-q-r Problem for Banach spaces). Let $p, q, r \in \mathbb{N}$ with $1 < p \neq q + r$, and X and let Y Banach spaces satisfying (1.1). Is it true that

$$X^p \sim X^q \oplus Y^r \;\Rightarrow\; X^2 \overset{c}{\hookrightarrow} X?$$

Remark 1.9. Observe that the p-q-r Problem in the case where $p = 3$, $q = r = 1$ and $X = Y$ is the Square-Cube Problem for Banach spaces (see [18, p. 367]).

2. Sufficient conditions to a pair of triples to be a SBpt. We start this section by recalling that in [7] a quintuple (p, q, r, s, t) in $\mathbb{N} \cup \{0\}$ with $p + q \geq 2$, $r + s + t \geq 3$, $(r, s) \neq (0, 0)$ and $t \geq 1$ was said to be a *Schroeder–Bernstein quintuple* (for short, SBq) if X is isomorphic to Y whenever these Banach spaces satisfy (1.1) and the following Decomposition Scheme holds:

$$\begin{cases} X \sim X^p \oplus Y^q, \\ Y^t \sim X^r \oplus Y^s. \end{cases}$$

The number $\nabla = (p-1)(s-t) - rq$ is the discriminant of the quintuple (p, q, r, s, t). We recall the following characterization of SBq (see [7, Theorem 1.2]). A quintuple (p, q, r, s, t) in $\mathbb{N} \cup \{0\}$ with $p + q \geq 2$, $r + s + t \geq 3$, $(r, s) \neq (0, 0)$ and $t \geq 1$ is a SBq if and only if its discriminant ∇ is different from zero and ∇ divides $p + q - 1$ and $r + s - t$.

Notice that the "if" part of Theorem 1.5 is Propositions 2.1 and 2.2 below.

Proposition 2.1. *Every pair of triples $\{(p, q, r), (s, t, u)\}$ with w-number $\mathcal{N}$ different from zero such that $\mathcal{N}$ divides $q - p + r$ and $t - s + u$ is a SBpt.*

Proof. Let X and Y be Banach spaces satisfying (1.1) and (1.3). Suppose that the Decomposition Scheme (1.9) holds for some pair of triples $\{(p,q,r),(s,t,u)\}$ with w-number $\mathcal{N}$ different from zero such that $\mathcal{N}$ divides $q-p+r$ and $t-s+u$. We will show that X is isomorphic to Y. By the symmetry of Definition 1.3, we only consider two cases: $p \le q$; $p > q$ and $s > t$.

CASE 1: $p \le q$. Let C be a Banach space satisfying $X \sim X^p \oplus C$. Adding C to both sides of the first condition of the Decomposition Scheme (1.9) we have

$$\begin{aligned} X &\sim X^p \oplus C \sim X^q \oplus Y^r \oplus C \sim X^{q-p} \oplus X^p \oplus C \oplus Y^r \sim X^{q-p} \oplus X \oplus Y^r \\ &\sim X^{q-p+1} \oplus Y^r. \end{aligned}$$

Therefore by the second condition of (1.9) we see that

$$\begin{cases} X \sim X^{q-p+1} \oplus Y^r, \\ Y^s \sim X^u \oplus Y^t. \end{cases} \tag{3.1}$$

Since the discriminant ∇ of the quintuple $(q-p+1, r, u, t, s)$ is equal to $(q-p)(t-s) - ru = \mathcal{N}$, it follows that ∇ is different from zero and ∇ divides $(q-p+1)+r-1 = q-p+r$ and $t-s+u$. By the characterization of the Schroeder–Bernstein quintuples mentioned above we conclude that $X \sim Y$.

CASE 2: $p > q$ and $s > t$. Let C be a Banach space satisfying

$$X \sim X^q \oplus C. \tag{3.2}$$

Adding C to both sides of the first condition of Decomposition Scheme (1.9) we infer

$$X^{p-q} \oplus X^q \oplus C \sim X^q \oplus C \oplus Y^r.$$

Hence by (3.2),

$$X^{p-q+1} \sim X \oplus Y^r. \tag{3.3}$$

Now adding Y^r to both sides of (3.3) we have

$$X^{p-q} \oplus X \oplus Y^r \sim X \oplus Y^{2r}. \tag{3.4}$$

Next by using (3.3) in (3.4) we deduce

$$X^{2(p-q)+1} \sim X^{p-q} \oplus X^{p-q+1} \sim X \oplus Y^{2r}.$$

So by induction we conclude

$$X^{m(p-q)+1} \sim X \oplus Y^{mr}, \qquad \forall m \in \mathbb{N}. \tag{3.5}$$

Analogously,

$$Y^{n(s-t)+1} \sim Y \oplus X^{nu}, \qquad \forall n \in \mathbb{N}. \tag{3.6}$$

Now there are two subcases:

SUBCASE 2.1: $\mathcal{N} > 0$. So $(p-q)(s-t) > ur$ and we cannot have $p-q \leq r$ and $s-t \leq u$. Moreover, since

$$\mathcal{N} = (p-q-r)(s-t) + r(s-t-u), \tag{3.7}$$

we also cannot have $p-q > r$ and $s-t > u$, otherwise $\mathcal{N} > p-q-r$ and $\mathcal{N}$ would not divide $p-q-r$, which is absurd. So by symmetry, we suppose that $p-q \leq r$ and $s-t > u$. Let $m, n \in \mathbb{N}$ be such that $q-p+r = n\mathcal{N}$ and $t-s+u = -m\mathcal{N}$. By (3.7) we obtain

$$mr = n(s-t) + 1. \tag{3.8}$$

Furthermore, we also have

$$\mathcal{N} = (q-p)(t-s+u) - u(q-p+r). \tag{3.9}$$

Consequently,

$$m(p-q) = nu + 1. \tag{3.10}$$

Finally by using (3.5), (3.6), (3.8) and (3.10) we conclude

$$X^{m(p-q)+1} \sim X \oplus Y^{n(s-t)+1} \sim X \oplus Y \oplus X^{nu} \sim X^{nu+1} \oplus Y = X^{m(p-q)} \oplus Y.$$

Thus by Remark 1.1, $X \sim Y$.

SUBCASE 2.2: $\mathcal{N} < 0$. Hence $(p-q)(s-t) < ur$ and we cannot have $p-q \geq r$ and $s-t \geq u$. Moreover, we also cannot have $p-q < r$ and $s-t < u$, otherwise by using (3.7), $\mathcal{N} < p-q-r$ and $\mathcal{N}$ would not divide $p-q-r$, which is absurd. Again by symmetry, we assume that $p-q \leq r$ and $s-t > u$. Let $m, n \in \mathbb{N}$ be such that $q-p+r = -n\mathcal{N}$ and $t-s+u = m\mathcal{N}$. By (3.7) and (3.9) we deduce

$$n(s-t) = mr + 1 \quad \text{and} \quad nu = m(p-q) + 1. \tag{3.11}$$

Next by using (3.5), (3.6) and (3.11) we have

$$\begin{aligned} Y^{n(s-t)+1} &\sim Y \oplus X^{nu} = Y \oplus X^{m(p-q)+1} \sim Y \oplus X \oplus Y^{mr} \sim Y^{mr+1} \oplus X \\ &= Y^{m(s-t)} \oplus X. \end{aligned}$$

Therefore according to Remark 1.1, $X \sim Y$. ∎

PROPOSITION 2.2. *Every pair of triples $\{(p,q,r),(s,t,u)\}$ with $p = q+r$, $\mathcal{N} = 0$ and $\gcd(r,u) = 1$ is a SBpt.*

Proof. Let X and Y be Banach spaces satisfying (1.1) and (1.3). Assume that the Decomposition Scheme (1.9) holds for some pair of triples $\{(p,q,r),(s,t,u)\}$ with $p = q+r$, $\mathcal{N} = 0$ and $\gcd(r,u) = 1$. We will show that X is isomorphic to Y.

According to the well-known Bézout theorem there exist m and n in $\mathbb{N}$ such that $mr = nu + 1$ or $nu = mr + 1$. Without loss of generality, we may suppose that $mr = nu + 1$. Since $\mathcal{N} = 0$, it follows that $s = t + r$. So, by using (3.5) and (3.6) we conclude

$$X^{mr+1} \sim X \oplus Y^{mr} = X \oplus Y^{nu+1} \sim X \oplus Y \oplus X^{nu} \sim X^{nu+1} \oplus Y = X^{mr} \oplus Y.$$

Once again by Remark 1.1, $X \sim Y$. ∎

3. Necessary conditions for a pair of triples to be a SBpt. The goal of this section is to complete the proof of Theorem 1.5. Notice that this is a direct consequence of Propositions 3.1, 3.3, 3.4 and 3.5 below. In order to prove these propositions we recall that in [15, p. 563], for every $v \in \mathbb{N}$, $v \geq 2$ Banach spaces X_v were constructed having the following property:

$$X_v^m \sim X_v^n, \tag{3.12}$$

with $m, n \in \mathbb{N}$, if and only if m is equal to n modulo v. In particular observe that for all $m \in \mathbb{N}$ we have

$$X_v^{2m} \overset{c}{\hookrightarrow} X_v^m. \tag{3.13}$$

Proposition 3.1. *If a pair of triples $\{(p, q, r), (s, t, u)\}$ with $p \leq q$ or $s \leq t$ and w-number $\mathcal{N}$ is a SBpt, then $\mathcal{N} \neq 0$ and $\mathcal{N}$ divides $q - p + r$ and $t - s + u$.*

Proof. Without loss of generality we may suppose that $p \leq q$. Assume that a pair of triples $\{(p, q, r), (s, t, u)\}$ with $p \leq q$ and with w-number $\mathcal{N}$ does not satisfy: $\mathcal{N}$ is different from zero and $\mathcal{N}$ divides $q - p + r$ and $t - s + u$. Since the discriminant ∇ of the quintuple $(q - p + 1, r, u, t, s)$ is equal to $\mathcal{N}$, it does not satisfy: ∇ is different from zero and ∇ divides $(q - p + 1) + r - 1 = q - p + r$ and $t - s + u$. So by the proof of [7, Theorem 2.1] there are non-isomorphic Banach spaces X and Y such that the Decomposition Scheme (3.1) holds. Moreover, $X^2 \overset{c}{\hookrightarrow} X$ because X is taken from the Banach spaces X_v^m, $v \in \mathbb{N}$, $v \geq 2$ and $m \in \mathbb{N}$, and (3.13) holds.

Now adding X^{p-1} to both sides of the first condition of (3.1) we infer that the Decomposition Scheme (1.9) holds and hence $\{(p, q, r), (s, t, u)\}$ is not a SBpt. ∎

We will need the following lemma.

Lemma 3.2. *Let $p, q, r, s, t, u \in \mathbb{N}$. Suppose that there exist $i, j, v \in \mathbb{N}$ with $v \geq 2$ satisfying*

(a) *v divides $i(q - p) + jr$;*
(b) *v divides $iu + j(t - s)$;*
(c) *v does not divide $j - i$.*

Then $\{(p, q, r), (s, t, u)\}$ is not a SBpt.

Proof. Let $n \in \mathbb{N}$ be such that $nv - j + i > 0$ and $nv - i + j > 0$. Since $j + (nv - j + i) - i = nv$ and $i + (nv - i + j) - j = nv$, by the property of Banach spaces X_v mentioned above in (3.12) and by the conditions (a) and (b) we have

$$\begin{cases} X_v^i \sim X_v^j \oplus X_v^{nv-j+i}, \\ X_v^j \sim X_v^i \oplus X_v^{nv-i+j}, \end{cases}$$

and

$$\begin{cases} X_v^{ip} \sim X_v^{iq} \oplus X_v^{jr}, \\ X_v^{js} \sim X_v^{jt} \oplus X_v^{iu}. \end{cases}$$

Further according to condition (c) we conclude that X_v^i is not isomorphic to X_v^j. Consequently, $\{(p,q,r),(s,t,u)\}$ is not a SBpt. ∎

PROPOSITION 3.3. *If a pair of triples $\{(p,q,r),(s,t,u)\}$ with $p > q$, $s > t$ and with w-number $\mathcal{N}$ different from zero is a SBpt, then $\mathcal{N}$ divides $q-p+r$ and $t-s+u$.*

Proof. By the symmetry of the definition of SBpt, it suffices to prove that $\mathcal{N}$ divides $q-p+r$. Thus assume that $\mathcal{N}$ does not divide $q-p+r$. We now distinguish two cases.

CASE 1: $\mathcal{N} \geq 2$. Take $i = \mathcal{N}+r$, $j = p-q$ and $v = \mathcal{N}$. Then $i(q-p)+jr = \mathcal{N}(q-p)$, $iu + j(t-s) = \mathcal{N}(u-1)$ and $j - 1 = -\mathcal{N} - (q-p+r)$. Thus Lemma 3.2 implies that $\{(p,q,r),(s,t,u)\}$ is not a SBpt.

CASE 2: $\mathcal{N} \leq -2$. Take $i = -\mathcal{N} + r$, $j = p - q$ and $v = -\mathcal{N}$. Then $i(q-p)+jr = \mathcal{N}(p-q)$, $iu+j(t-s) = -\mathcal{N}(u+1)$ and $j-i = -\mathcal{N}+q-p+r$. So by Lemma 3.2 we see that $\{(p,q,r),(s,t,u)\}$ is not a SBpt. ∎

PROPOSITION 3.4. *If a pair of triples $\{(p,q,r),(s,t,u)\}$ with $p > q$, $s > t$ and $\mathcal{N} = 0$ is a SBpt, then $p = q + r$.*

Proof. Suppose that $p \neq q+r$. Take $i = r$, $j = p-q$ and $v \in \mathbb{N}$ such that v does not divide $p-q-r$. So $i(q-p)+jr = 0$ and since $\mathcal{N} = 0$, it follows that $iu + j(t-s) = 0$. Hence according to Lemma 3.2, $\{(p,q,r),(s,t,u)\}$ is not a SBpt. ∎

PROPOSITION 3.5. *If a pair of triples $\{(p,q,r),(s,t,u)\}$ with $p > q$, $s > t$ and $\mathcal{N} = 0$ is a SBpt, then $gcd(r,u) = 1$.*

Proof. By Proposition 3.4, $p = q + r$ and therefore $s = t + u$. Assume that $\gcd(r,u) \neq 1$. Take $i = 1$, $j = 2$ and $v \in \mathbb{N}$, $v \geq 2$ such that v divides r and u. By Lemma 3.2 we conclude that $\{(p,q,r),(s,t,u)\}$ is not a SBpt. ∎

Acknowledgements. The author is grateful to the referee for his valuable comments and suggestions.

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Received 15 January 2013;
revised 12 October 2013 (5850)