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EMBEDDING PROPER HOMOTOPY TYPES

BY

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Abstract. We show that the proper homotopy type of any properly c-connected locally finite n-dimensional CW-complex is represented by a closed polyhedron in $\mathbb{R}^{2n-c}$ (Theorem I). The case $n-c \geq 3$ is a special case of a general proper homotopy embedding theorem (Theorem II). For $n-c \leq 2$ we need some basic properties of "proper" algebraic topology which are summarized in Appendices A and B. The results of this paper are the proper analogues of classical results by Stallings [17] and Wall [20] for finite CW-complexes; see also Dranišnikov and Repovš [7].

Introduction. The classical Kuratowski theorem characterizing planar finite graphs ([13]) turns out to be trivial when one considers homotopy types since any finite connected graph is homotopically equivalent to a finite pointed union of circles. Actually this observation is just the 1-dimensional case of a theorem due to Stallings [17] (also proved by Dranišnikov and Repovš [7]) stating that the homotopy type of any c-connected n-dimensional finite CW-complex can be represented by a subpolyhedron in $\mathbb{R}^{2n-c}$.

When one directs interest to locally finite graphs one finds the analogue of Kuratowski's theorem; that is, the characterization due to Halin and Thomassen ([19]) of those locally finite graphs which admit proper planar embeddings. Recall that a continuous map $f : X \to Y$ is *proper* if $f^{-1}(K)$ is compact for any compact subset $K \subset Y$. The Halin–Thomassen theorem is trivial for proper homotopy types. This fact is an immediate consequence of the classification of proper homotopy types of locally finite graphs by circles attached to trees in [2]; see also Proposition 3.1. We will show below that this remark extends to the following proper analogue of Stallings's theorem. Recall that a locally finite CW-complex X is said to be *properly c-connected* ($c \geq 1$) if the proper homotopy class of X can be represented by a CW-complex Y whose c-skeleton $Y^c = T$ is reduced to an end-faithful tree. See Appendix A for details.

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THEOREM I. *The proper homotopy type of any n-dimensional locally finite CW-complex X can be represented by a closed polyhedron $P \subset \mathbb{R}^{2n}$. Moreover if X is properly c-connected then P can be chosen in $\mathbb{R}^{2n-c}$.*

The main goal of this paper is the following general proper homotopy embedding theorem from which we easily derive Theorem I for $n - c \geq 3$.

THEOREM II. *Let M be an m-dimensional open pl-manifold and let P be a k-dimensional locally compact polyhedron with $k \leq m - 3$. Let $f : P \to M$ be a properly $(2k - m + 1)$-connected pl-map such that f induces a homeomorphism between the spaces of Freudenthal ends of P and $f(P)$. Then there exists a closed subpolyhedron $\widetilde{P} \subset M$ and a proper homotopy equivalence $h : P \to \widetilde{P}$ such that h is properly homotopic to f inside M.*

We refer to Appendix A for the definition of a properly c-connected map.

For $n - c \leq 2$ Theorem I will follow from the explicit description of these highly connected proper homotopy types by use of basic facts of "proper" algebraic topology (Propositions 3.1 and 3.5) and a purely combinatorial proof of the special case $n = 2$, $c = 0$.

The results of this paper are the proper analogues of classical results by Stallings [17] and Wall [20] for finite CW-complexes; see also Dranišnikov and Repovš [7]. In fact, the arguments in [17] are used here as guidelines for the proofs. We can do that since the crucial points in [17] are general position in pl-manifolds as well as some features of the ordinary homotopy category which lie in the cofibration part of homotopy theory. In the proper category we shall use the (proper) general position theorem of [12] as well as the structure of cofibration category in the sense of Baues of the proper category; see [3].

1. Preliminaries. Recall that a continuous map $f : X \to Y$ is said to be *proper* if $f^{-1}(K)$ is compact for each compact subset $K \subset Y$. Proper homotopies, proper homotopy equivalences, etc. are then defined in the obvious way. Let "$\simeq_{\mathrm{p}}$" denote the proper homotopy relation. We will work within the category $\mathcal{P}$ of locally compact σ-compact Hausdorff spaces and proper maps. In other words, any space X in $\mathcal{P}$ admits a countable basis of neighbourhoods at infinity. Recall that $A \subset X$ is said to be a *neighbourhood at infinity* if the closure $\overline{X - A}$ is compact. In this paper we will deal specially with the class of finite-dimensional locally finite CW-complexes. For these, it is always possible to choose a countable basis of neighbourhoods at infinity consisting of subcomplexes; see [9]. Recall that the *space of ends* of a space X in $\mathcal{P}$ is the inverse limit $\mathcal{F}(X) = \varprojlim \pi_0(U_j)$ where $\{U_j\}_{j\geq 1}$ is a countable basis of neighbourhoods at infinity. Here $\pi_0(-)$ denotes the discrete set of connected components. In case X is connected and locally path connected the space $\mathcal{F}(X)$ if homeomorphic to a closed subset of the Cantor set. We shall use later the following

LEMMA 1.1. *Let M be an n-dimensional open pl-manifold and let $X \subset M$ be a closed non-compact subpolyhedron. Then there exists an increasing sequence $M_i \subset \operatorname{int} M_{i+1}$ $(i \geq 1)$ of compact n-submanifolds with $M = \bigcup_{i=1}^{\infty} M_i$ such that all connected components of $X - \operatorname{int} M_i$ are non-compact. Moreover, for each $i \geq 1$ there is a bijection*

$$\pi_0(X - \operatorname{int} M_i) \cong \pi_0(X \cap (M_{i+1} - \operatorname{int} M_i)). \tag{1}$$

Proof. Let $K_i \subset \operatorname{int} K_{i+1}$ be an increasing sequence of compact connected subpolyhedra in M with $M = \bigcup_{i=1}^{\infty} K_i$. Let $\{W_i\}$ be the sequence of submanifolds obtained by setting $W_i = N_i \cup N_i'$ where N_i is a regular neighbourhood of $W_{i-1} \cup K_i$ $(W_0 = \emptyset)$ and N_i' is the union of regular neighbourhoods in $M - \operatorname{int} N_i$ of all compact components of $X - \operatorname{int} N_i$. For the sequence $\{W_i\}$ one readily shows that all components of $X - \operatorname{int} W_i$ are non-compact.

Next we replace this sequence by a new sequence $\{M_i\}$ such that in addition we have a bijection (1) for all $i \geq 1$. For this we start with $M_1 = W_1$ and proceed inductively as follows. Assume that for a sequence $\emptyset = M_0 \subset M_1 \subset \ldots \subset M_k$ all components in $\pi_0(X - \operatorname{int} M_i)$ are non-compact and (1) holds for $0 \leq i \leq k$. In order to obtain M_{k+1} we consider W_{i_k} with $M_k \subset \operatorname{int} W_{i_k}$. For the components $C \in \pi_0(X - \operatorname{int} M_K)$ for which the intersection $C' = C \cap (W_{i_k} - \operatorname{int} M_k)$ is not connected we take a compact connected polyhedron $Z_C \subset C$ containing C'. Then we define M_{k+1} to be the union $M_{k+1} = M_k \cup (\bigcup\{N_C\}) \cup M_k'$ where N_C is a regular neighbourhood of Z_C in $M - \operatorname{int} M_k$ and M_k' is the union of regular neighbourhoods in $M - \operatorname{int}(M_k \cup (\bigcup\{N_C\}))$ of all connected components of $X - \operatorname{int}(M_k \cup (\bigcup\{N_C\}))$.

In order to set up the homotopy theory of $\mathcal{P}$ one observes that this category is not closed under push-outs but it contains enough push-outs to allow the basic homotopical constructions. In fact, the category $\mathcal{P}$ is a cofibration category in the sense of Baues ([4] and [3]). Moreover in $\mathcal{P}$ the role of the base point is played by the half-line $\mathbb{R}_+ = [0, \infty)$ since $[X, \mathbb{R}_+]_{\mathrm{p}} = \{*\}$ is a one-point set [8]. Here $[\ ,\ ]_{\mathrm{p}}$ denotes the set of homotopy classes in $\mathcal{P}$. However we will need trees in order to deal with spaces with many ends. Indeed, for any connected finite-dimensional locally finite CW-complex X there exists a (maximal) tree $T \subset X$ such that $\mathcal{F}(T) = \mathcal{F}(X^1) = \mathcal{F}(X)$; see [11]. The tree T is called an *end-faithful tree*. The following proposition states a crucial property of end-faithful trees.

Proposition 1.2. *Let X be a finite-dimensional locally finite CW-complex and let $T \subset X$ be an end-faithful tree. Then the set $[X,T]_{\mathrm{p}}^{T} = \{*\}$ of proper homotopy classes under T is a one-point set.*

Proof. We argue by induction on $n = \dim(X - T)$. For $n = 0$, $X = T$ since T is a maximal tree and the result is obvious. Assume the result holds for X^{n-1}. In order to extend any proper map $r : X^{n-1} \to T$ to the n-skeleton X^n we proceed as follows. Given an n-cell $e \subset X^n - T$ let $T_e \subset T$ be the smallest subtree containing $rf_e(S_e^{n-1})$ where f_e is the attaching map of e. As $\mathcal{F}(T) = \mathcal{F}(X)$, the family $\{T_e\}$ is readily checked to be locally finite. Moreover, as each T_e is contractible we can extend $rf_e : S_e^{n-1} \to T_e$ to a map $r_e : B_e^n \to T_e$ which induces a map $\widetilde{r}_e : X^{n-1} \cup e \to T$. Then the union $\widetilde{r} = \bigcup \widetilde{r}_e$ yields a proper extension of r. Similar arguments show that all such extensions are properly homotopic.

Proposition 1.2 allows us to define the *proper cone* $C_T X$ and the *proper suspension* $\Sigma_T X$ of a connected finite-dimensional CW-complex X as the push-outs

$$\begin{tikzcd} X \times \{0\} \arrow[r] \arrow[d, "r"'] \arrow[dr, phantom, "\text{p.o.}"] & X \times I \arrow[d] \\ T \arrow[r] & M_r = C_T X \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} X \times \{0,1\} \arrow[r] \arrow[d, "r \cup r"'] \arrow[dr, phantom, "\text{p.o.}"] & X \times I \arrow[d] \\ T \cup T \arrow[r] & \Sigma_T X \end{tikzcd}$$

Here $T \subset X$ is an end-faithful tree, the map r is given in Proposition 1.2, and M_f denotes the cylinder of a map f. By the gluing lemma B.1, the proper homotopy types of $C_T X$ and $\Sigma_T X$ do not depend on the map r; in fact $C_T X \simeq_{\mathrm{p}} T$. Moreover, since any two trees with the same space of Freudenthal ends are properly homotopy equivalent ([2]) the proper homotopy types of $C_T X$ and $\Sigma_T X$ only depend on the proper homotopy type of X. Moreover if r above is cellular then both $C_T X$ and $\Sigma_T X$ turn out to be finite-dimensional locally finite CW-complexes.

The class of finite-dimensional locally finite CW-complexes (or more generally strongly locally finite CW-complexes) is a good class of complexes associated to a theory of coactions as defined in [5] and hence the fundamental homotopical properties of ordinary CW-complexes (Blakers–Massey, Hurewicz, or Whitehead theorems) also hold for suitable proper analogues of homotopy and homology groups. See [5]. In particular, we define proper c-connectedness in Appendix A in terms of the proper analogues Π_*^T of the homotopy groups. Moreover in Appendix B we include explicit proofs since these results are not easily found in the literature except for homology and homotopy towers; see [8]. For this we have chosen the usual language for "proper" algebra based on the category of trees of groups instead of the new approach based on theories and models of theories as done in [5].

2. A proper homotopy embedding theorem. In this section we prove Theorem II from which we easily derive Theorem I for the stable case $n - c \geq 3$. Theorem II is the proper analogue of the embedding theorem of [17]; see also [20]. For the proof we will need Addenda 2.2–2.4 below to the following well known general position theorem:

THEOREM 2.1 ([12, 4.8]). *Let Q be a closed subpolyhedron of the p-dimensional polyhedron P. Assume* $\dim(P - Q) = p_0$. *Let $f : P \to M$ be a pl-map into an m-dimensional pl-manifold M with boundary. Moreover assume that $f|_Q$ is a pl-embedding and $f(P - Q) \subset \operatorname{int} M$. Then*, *given $\varepsilon > 0$ and any closed subpolyhedron $R \subset M$ there exists a non-degenerate pl-map $g : P \to M$ which is ε-homotopic to f relative to Q and satisfies the following properties*:

(i) $\dim S_{g|P-Q} \leq 2p_0 - m$.
(ii) $\dim(g(P - Q) \cap R) \leq p + \dim R - m$.
(iii) $g(P - Q) \subset \operatorname{int} M$.

Here $S_g = \mathrm{Cl}\{x;\ g^{-1}(g(x)) \neq x\}$ *denotes the singular set of g.*

ADDENDUM 2.2. If we apply Theorem 2.1 to a proper map f we obtain a proper map g; see p. 96 in [12]. Namely, let n be large enough to embed M as a closed subpolyhedron in $\mathbb{R}^n$. Then given a compact subset $K \subset M$ let $r > 0$ be such that $K \subset B_r$ is contained in the closed ball B_r of radius r centred at $0 \in \mathbb{R}^n$. As g is ε-homotopic to f we have $g^{-1}(K) \subset f^{-1}(B_{r+2\varepsilon})$ and the result follows.

ADDENDUM 2.3 (cf. [15, 1.6.5(c)]). By taking $R = f(Q) = g(Q)$ we derive from (i) and (ii) that $\dim S_g \leq p + p_0 - m$. Here we use the equality $S_g = S_{g|P-Q} \cup g^{-1}(g(P - Q) \cap g(Q))$.

ADDENDUM 2.4. If f is proper and the induced map $f_* : \mathcal{F}(P) \to \mathcal{F}(f(P))$ is a homeomorphism then the same holds for $g_* : \mathcal{F}(P) \to \mathcal{F}(g(P))$. For this we simply apply Addendum 2.2 to $f : P \to N$ where N is a regular neighbourhood of $f(P)$ in M. Then we get a proper homotopy $H : f \simeq_{\mathrm{p}} g : P \to N$ and hence the composite $f_* = g_* : \mathcal{F}(P) \to \mathcal{F}(g(P)) \to \mathcal{F}(N)$ is a homeomorphism. Therefore $g_* : \mathcal{F}(P) \to \mathcal{F}(g(P))$ is a continuous bijection, and hence a homeomorphism.

Besides the general position theorem, another ingredient in the proof of Theorem II is the following

THEOREM 2.5. *Let M be an open pl-manifold and let $f : P \to M$ be a properly $(s + 1)$-connected pl-map whose singular set S_f has dimension $\leq s$. Assume in addition that f induces a homeomorphism $f_* : \mathcal{F}(P) \to \mathcal{F}(f(P))$. Then there exists a polyhedron Q with $f(P) \subset Q$ such that the composite $f : P \to f(P) \subset Q$ is a proper homotopy equivalence and has*

$\dim(Q - f(P)) \leq s + 2$. *Moreover there exists a proper pl-map* $g : Q \to M$ *such that* $g|_{f(P)}$ *is properly homotopic to the inclusion* $f(P) \subset M$.

The main arguments in the proof of Theorem 2.5 are provided by the cofibration category structure of the proper category. The proof will also use the following

LEMMA 2.6. *Let* $f : X \to Y$ *be a proper cellular map between finite-dimensional locally finite CW-complexes such that the induced map* $f_* : \mathcal{F}(X) \to \mathcal{F}(Y)$ *is a homeomorphism. Then for each* 0*-cell* $v \in Y$ *there exists a finite tree* $T_v \subset X$ *containing all the* 0*-cells in* $f^{-1}(v)$ *and such that the family* $\{T_v\}_{v \in Y}$ *is locally finite in* X.

Proof. The result will easily follow if we show that for any compact set $K \subset X$ there exists a compact set $L \subset Y$ such that for any 0-cell $v \in Y - L$ the set $f^{-1}(v)$ is contained in a unique connected component of $X - K$. Otherwise, there is a compact set $K_0 \subset X$ and an increasing sequence $L_1 \subset L_2 \subset \ldots$ of compact sets of Y such that there are 0-cells $v_n \in Y - L_n$ and 0-cells $x_n, x'_n \in f^{-1}(v_n)$ which lie in different components of $X - K_0$. Moreover, we can assume without loss of generality that the sequence $\{v_n\}$ converges to an end $\varepsilon \in \mathcal{F}(Y)$ in the Freudenthal compactification $\widehat{Y}$ of Y. Let $\eta, \eta' \in \mathcal{F}(X)$ be two ends obtained as cluster points in $\widehat{X}$ of the sequences $\{x_n\}$ and $\{x'_n\}$ respectively. Then one reaches the contradiction $\eta \neq \eta'$ and $f_*(\eta) = f_*(\eta')$.

For the proof of Theorem 2.5 we follow the same arguments as in the proof of the compact case in [17]. We start by choosing triangulations K and J of P and M respectively for which f is a simplicial map and S_f is triangulated by a subcomplex of K; see 3.6 in [12]. Then we observe that the natural CW-complex structure of the mapping cylinder M_f of f contains $f(K)$ as a subcomplex. Let $f' : S_f \to f(S_f)$ be the restriction of f, and $M_{f'}$ the corresponding mapping cylinder.

LEMMA 2.7. *The obvious map obtained by the union of the inclusion* $j : (M_{f'}, S_f) \subset (M_f, P)$ *and the projection* $I \times S_f \to S_f \subset P$ *extends to a proper homotopy* $h : I \times M_{f'} \to M_f$ *relative to* $S_f \subset M_{f'}$ *with* $h_0 = j$ *and* $h_1(M_{f'}) \subset P$.

Proof. We proceed inductively as follows. For each 0-cell $v \in M_{f'} - S_f \times \{0\}$ we have $v \in f(S_f)$. Then we pick any 0-cell $w_v \in S_f$ with $w_v \in f^{-1}(v)$ and we set $h^0_1(v) = w_v$. Moreover we define $h^0 : I \times M^0_{f'} \to M_f$ on the 0-skeleton of $M^0_{f'}$ by sending the segment $I \times \{v\}$ onto the 1-cell in M_f running between v and w_v. For the extension of h^0 to $h^1 : I \times M^1_{f'} \to M_f$ we use Lemma 2.6. Namely, given any 1-cell $e = e(v, w)$ with vertices v and $w \in f^{-1}(v)$ we define $h^1_1|e(v, w)$ to be a path in the tree $T_v \subset P$

given by Lemma 2.6 between w and $w_v = h_1^0(v)$. Moreover, given any 1-cell $e' = f(\sigma) \subset f(P)$ with vertices $v = f(w)$ and $v' = f(w')$ we define $h_1^1|e'$ to be the path in P defined by the juxtaposition of $h_1^1|e(v, w_v)$, the 1-cell σ and $h_1^1|e(v', w_{v'})$.

The extension of $\phi_1 = h_1^1 \cup h_0 \cup h^0 : \{0,1\} \times M_{f'}^1 \cup I \times M_{f'}^0 \to M_f$ to $h^1 : I \times M_{f'}^1 \to M_f$ can now be carried out since the map f, and hence the pair (M_f, P), is properly $(s+1)$-connected. Indeed, given an end-faithful tree $T \subset P$ we form a 1-spherical object S_α^1 under T with a circle S_e^1 for each 1-cell $e \subset M_{f'} - S_f$. This circle is attached at a 0-cell $w_{v(e)}$ where $v(e)$ is a vertex of e in $f(S_f)$. This way the map ϕ_1 above defines a proper map $g : S_\alpha^1 \to M_f$. Since (M_f, P) is properly $(s+1)$-connected there exists a proper homotopy $H : g \simeq_{\mathrm{p}} g'$ with $g'(S_\alpha^1) \subset P$. Then we replace the map ϕ_1 by a new proper map $\widetilde{\phi}_1 = \widetilde{h}_1^1 \cup h_0 \cup h^0$ where $\widetilde{h}_1^1$ is obtained by the juxtaposition of the path $h_1^1|e$ with the restriction $g'|S_e^1$ for all 1-cells $e \subset M_{f'} - S_f$. Moreover, by use of the homotopy H it is readily checked that $\widetilde{\phi}_1$ now extends to a proper map $h^1 : I \times M_{f'}^1 \to M_f$ with $h_1^1(M_{f'}) \subset P$.

Once we have gone through the critical case in dimension 1, the result will follow inductively as usual. Assume we have already constructed an extension $h^k : I \times M_{f'}^k \to M_f$ with $h_1^k(M_f^k) \subset P$. Then $h^k \cup h_0 : I \times M_{f'}^k \cup \{0\} \times M_{f'}^{k+1} \to M_f$ can be regarded as a proper map $\phi_k : (B_\alpha^{k+1}, S_\alpha^k) \to (M_f, P)$ for a suitably chosen spherical object S_α^k. Therefore Proposition A.2 yields an extension $h^{k+1} : I \times M_{f'}^{k+1} \to M_f$ of $h^k \cup h_0$ with the required properties.

We are now ready for the proof of Theorem 2.5.

Proof of Theorem 2.5. Let C_1 be the mapping cylinder of the map $h_1 : M_{f'} \to P \subseteq M_f$ given by Lemma 2.7. Moreover, let $\psi : C_1 \to M_f$ denote the proper map induced by the homotopy $h : I \times M_{f'} \to M_f$ in Lemma 2.7 and the inclusion $P \subset M_f$. In addition, let C_2 be the mapping cylinder relative to S_f; that is, C_2 is obtained by the left push-out diagram below.

$$\begin{tikzcd} I \times S_f \arrow[r] \arrow[d, "\pi"', "\simeq_{\mathrm{p}}"] \arrow[dr, phantom, "\text{p.o.}"] & C_1 \arrow[d, "\alpha"', "\simeq_{\mathrm{p}}"] \\ S_f \arrow[r] & C_2 \end{tikzcd} \qquad \begin{tikzcd} M_{f'} \arrow[r] \arrow[d, "r'"', "\simeq_{\mathrm{p}}"] \arrow[dr, phantom, "\text{p.o.}"] & C_2 \arrow[d, "\beta"', "\simeq_{\mathrm{p}}"] \\ f(S_f) \arrow[r] & C_3 \end{tikzcd}$$

By the push-out property, the map ψ and the inclusion $S_f \subset M_f$ induce a proper map $\widetilde{h} : C_2 \to M_f$. Note that C_2 is endowed with a natural CW-complex structure for which $\widetilde{h}$ is a cellular map. Finally we obtain a new CW-complex via the right push-out diagram above in which $r' : M_{f'} \to f(S_f)$ is the restriction of the canonical retraction $r : M_f \to M$. Moreover the map $\Psi : C_3 \to M$ obtained from the previous diagram for the composite $r\widetilde{h}$ and the inclusion $f(S_f) \subset M$ is a proper map for which one readily checks $\Psi\beta\alpha j = f$ where j denotes the inclusion $P \subset C_1$ and α and β are defined in the diagrams above. Moreover Ψ induces a cellular homeomorphism $\Psi : \beta\alpha j(P) \cong f(P)$ with respect to the triangulation K. Finally

$$\dim(C_3 - f(P)) = \dim(C_2 - \alpha j(P)) = \dim(C_1 - j(P)) \leq \dim M_{f'} + 1 = s + 2.$$

We finish by using Proposition B.2 to replace C_3 by a polyhedron Q of the same dimension and by choosing $g : Q \to M$ as the composite $g : Q \simeq_{\mathrm{p}} C_3 \xrightarrow{\Psi} M$.

Proof of Theorem II. By Addendum 2.4 we can assume that f is a non-degenerate pl-map with $\dim S_f \leq s$ for $s = 2k - m$. Next we apply Theorem 2.5 to f getting a diagram

$$\begin{tikzcd} P \arrow[r, "h_1", "\simeq_{\mathrm{p}}"'] \arrow[d, "f"'] & Q_1 \arrow[d, "g"] \\ f(P) \arrow[r, "i"'] \arrow[ur, "j"'] & M \end{tikzcd}$$

which is commutative up to proper homotopy and where i and j are the obvious inclusions and h_1 is a proper homotopy equivalence. Moreover, we have $\dim(Q_1 - f(P)) \leq s + 2 \leq k - 1$ since $k \leq m - 3$. Note that $\dim f(P) = \dim P$ since f is a non-degenerate map. Hence $\dim Q_1 = \max\{s + 2, k\} = k$. By applying Addendum 2.3 we can replace g by a non-degenerate proper pl-map g_1 with $\dim S_{g_1} \leq k + s + 2 - m \leq s - 1$. Here we use $k \leq m - 3$. At this point we face two possible cases.

1. $S_{g_1} = \emptyset$ and hence g_1 is a closed pl-embedding. Then we choose $\widetilde{P} = g_1(P_1)$ and $h : P \to \widetilde{P}$ to be the composite $h = g_1 h_1$.

2. $S_{g_1} \neq \emptyset$. Then we proceed as follows. Since f is properly $(s+1)$-connected and h_1 is a proper homotopy equivalence we derive from $g_1 h_1 \simeq_{\mathrm{p}} f$ that g_1 is also a properly $(s+1)$-connected map. As $\dim S_{g_1} \leq s - 1$ we can argue as above with g_1 in place of f to obtain a new diagram

$$\begin{tikzcd} P \simeq_{\mathrm{p}} Q_1 \arrow[r, "h_2", "\simeq_{\mathrm{p}}"'] \arrow[d, "g_1"'] & Q_2 \arrow[d, "g_2"] \\ g_1(Q_1) \arrow[r, "i"'] \arrow[ur, "j"'] & M \end{tikzcd}$$

which is commutative up to proper homotopy where $\dim(Q_2 - g_1(Q_1)) \leq s + 1$, and $\dim S_{g_2} \leq s - 2$. By iterating this procedure we reach Case 1 at least for g_{s+1} and the proof is complete.

3. Embedding proper homotopy types into Euclidean spaces. In this section we give a proof of Theorem I. For $n-c \geq 3$ this is an immediate consequence of Theorem II:

Proof of Theorem I for $n-c \geq 3$. Let $T \subset P$ be any end-faithful tree. Let $i : T \subset \mathbb{R}^{2n-c}$ be any proper embedding and let $r : P \to T$ be any retraction as in Proposition 1.2. Then r induces a homeomorphism $r_* : \mathcal{F}(X) \cong \mathcal{F}(T)$ and the result follows immediately by applying Theorem II to the composite ir since $\mathbb{R}^{2n-c}$ is properly $(2n-c-2)$-connected and $n-c \geq 3$.

The case $n-c \leq 2$ is proved below by use of arguments from the folklore of "proper" algebraic topology together with an elementary proof of the embeddability up to proper homotopy of any 2-dimensional locally finite CW-complex in $\mathbb{R}^4$. Actually for $n-c \leq 1$ Theorem I is an immediate consequence of the following classification of highly connected proper homotopy types for which we use the existence of kernels in the category of "free" trees of abelian groups.

Proposition 3.1. *Let X be a properly $(n-1)$-connected locally finite CW-complex of dimension $n \geq 1$. Then X has the proper homotopy type of a spherical object.*

We refer to Appendix A for the definition of a spherical object. In the proof of Proposition 3.1 we use the category $\mathbf{Tree}_T(\mathbf{Ab})$ of trees of abelian groups and the notion of homology tree of X. Also some results from Appendix B are used.

Proof of Proposition 3.1. Let $T \subset X$ be an end-faithful tree. If $n = 1$ the gluing lemma B.1 applied to the attaching maps of the 1-cells of $X - T$ yields the result; compare the proof of Proposition 1.2. For $n \geq 2$ we apply Proposition B.3 to find a proper homotopy equivalence $f : X \to Y$ with $\dim Y = n+1$ and $Y^{n-1} = T$. Moreover by Proposition 1.2 we can assume that $Y^n = S^n_\alpha$ is a spherical object. The proper cellular approximation theorem [9] shows that i_* in the following diagram in the category $\mathbf{Tree}_T(\mathbf{Ab})$ is an epimorphism:

$$\begin{tikzcd} & H_n^T(S^n_\alpha) \arrow[d, "i_*"] \\ Z_n^T(X) = H_n^T(X) \arrow[ur, "\varphi"] \arrow[r, "\cong", "f_*"'] & H_n^T(Y) \end{tikzcd}$$

In this diagram the tree $Z_n^T(X)$ of n-cycles is a free and hence projective tree by Proposition A.1. Hence there exists a lifting φ as in the diagram. Moreover we can easily identify, via the homology tree, the free tree $Z_n^T(X)$

with a certain spherical object S^n_β in such a way that we get a bijection

$$[S^n_\beta, S^n_\alpha] \cong \mathbf{Tree}_T(\mathbf{Ab})(Z^T_n(X), H^T_n(S^n_\alpha))$$

and so the lifting φ is realizable by a proper map $g : S^n_\beta \to S^n_\alpha$. Therefore, the composite $\widetilde{g} = ig$ induces an isomorphism

$$\widetilde{g}_* : H^T_n(S^n_\beta) \to H^T_n(Y)$$

and hence $\widetilde{g}$ is a proper homotopy equivalence by Proposition B.8.

For $n - c = 2$ we first give a purely combinatorial proof for the case $n = 2$, $c = 0$ in the following

PROPOSITION 3.2. *The proper homotopy type of any* 2*-dimensional locally finite CW-complex* X *can be represented by a closed polyhedron in* $\mathbb{R}^4$.

Proof. By Proposition B.2 we can assume that X is a polyhedron. Let K be a triangulation of X, and let $T \subseteq K$ be an end-faithful tree in K. By Proposition 3.1, K^1 is properly homotopy equivalent to a 1-dimensional spherical object S^1_α under T. More explicitly, S^1_α is obtained from K^1 as follows. We order the vertices of T. Then for each ordered edge $e = (v_e, w_e)$ of K^1 we get one 1-sphere $S^1_e \subset S^1_\alpha$ by sliding the vertex w_e to v_e along the reduced edge path from w_e to v_e in T. By using Lemma B.1 we replace K by a 2-dimensional CW-complex W^2 with $W^1 = S^1_\alpha$ and for which the attaching maps of the 2-cells have degree 0, ± 1 on each (oriented) 1-sphere S^1_e.

Any proper embedding $S^1_\alpha \subset \mathbb{R}^2 \times \{(0,0)\} \subset \mathbb{R}^4$ induces a cyclic ordering of the 1-spheres on the boundary of each 2-cell $d \subset W^2$ as one goes around the base vertex $v_d \in d$ following the clockwise orientation of $\mathbb{R}^2$. Then we choose the proper embedding above in such a way that the cyclic ordering induced by $\mathbb{R}^2$ coincides with the cyclic ordering defined by the ordering of the vertices in K^1. For this we use the fact that no pair of 1-spheres is shared by the 2-cells of W^2.

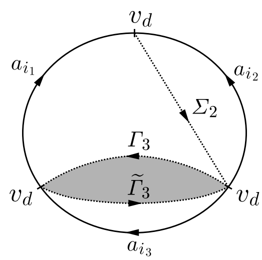
This way the attaching map $f_d : \partial B^2 \to S^1_\alpha$ of each 2-cell d can be described as a sequence $a^{\varepsilon(1)}_{i_1} \dots a^{\varepsilon(r)}_{i_r}$ with $r \leq 3$ and $\varepsilon(j) = \pm 1$ and where each a_{i_j} represents a 1-sphere in $\partial d \subset \mathbb{R}^2$. The next lemma is immediate:

LEMMA 3.3. *Assume that for the* 2*-cell* $d \subset W^2$ *all* $\varepsilon(j)$ *are the same for* $j = 1, 2, 3$ *and that the ordering* $a_{i_1} a_{i_2} a_{i_3}$ *represents the cyclic ordering around the base vertex* v_d. *Then for a given* 3*-dimensional vector space* H *containing* $\mathbb{R}^2 \times \{(0,0)\}$ *there exists a pl-map* $F_d : B^2 \to H$ *with* $F_d | \partial B^2 = f_d$ *such that* $F_d | \operatorname{int} B^2$ *is an embedding into the upper half-space* H^+ *in case* $\varepsilon(j) = 1$. *Moreover, if* $\varepsilon(j) = -1$ *for all* j *then the embedding is chosen to map into the lower half-space* H^-.

Assume now that the attaching map f_d of the 2-cell d does not satisfy the condition of Lemma 3.3. We describe below a process after which the original CW-complex W^2 is replaced by a new one $\widetilde{W}^2$ of the same proper homotopy type as W^2 and such that the attaching maps of $\widetilde{W}^2$ satisfy the conditions of Lemma 3.3. It is then left to choose a countable family $\{H_e\}$ of 3-dimensional vector spaces in $\mathbb{R}^4$, one for each 2-cell $e \subset \widetilde{W}^2$, so that $H_e \cap H_{e'} = \mathbb{R}^2 \times \{(0,0)\}$ for $e \neq e'$ and apply Lemma 3.3. Then Proposition 3.2 will follow since the union of all embeddings obtained from Lemma 3.3 defines a proper embedding of $\widetilde{W}$ as a closed subpolyhedron of $\mathbb{R}^4$. In order to replace W^2 by $\widetilde{W}^2$ we proceed as follows. The process is sketched in the picture below.

(1) Assume $\varepsilon(2) = -1$. Then we enlarge the spherical object S^1_α via an embedded 1-sphere Σ_2 wedged at T at the same vertex as $S^1_{e_{i_2}}$, and we replace the original cell d by two new 2-cells with attaching maps $a_{i_1}^{\varepsilon(1)}\Sigma_2 a_{i_3}^{\varepsilon(3)}$ and $\Sigma_2^{-1}a_{i_2}^{-1}$ respectively.

(2) After (1) we can now assume that the attaching map f_d is described by a sequence $a_{i_1}a_{i_2}a_{i_3}$ with $\varepsilon(j) = 1$ for $j = 1, 2, 3$. If i_3 precedes i_2 in the cyclic ordering on ∂d, then we enlarge the spherical object S^1_α by adding two new 1-spheres Γ_3 and $\widetilde{\Gamma}_3$ wedged at the same vertex as $S^1_{e_{i_3}}$ and $S^1_{e_{i_2}}$ respectively, and we replace the original 2-cell d by three new 2-cells with attaching maps $a_{i_1}a_{i_2}\widetilde{\Gamma}_3$, $\widetilde{\Gamma}_3^{-1}\Gamma_3^{-1}$ and $\Gamma_3 a_{i_3}$ respectively.



We now exhibit a contractible 2-dimensional CW-complex X whose proper homotopy type cannot be represented in $\mathbb{R}^3$.

Example 3.4. Let $\{S_k\}_{k\geq 1}$ be a countable family of circles. We consider the space $X = \bigcup_{n=1}^{\infty} Z_n$ where $Z_1 \subset Z_2 \subset \ldots$ are defined inductively as follows. Let $Z_1 = D$ be a disk with boundary $\partial D = S_1$. Assume Z_n is constructed; then Z_{n+1} is defined as the obvious adjunction space $Z_{n+1} = Z_n \cup M_{f_n}$ where $f_n : S_{n+1} \to S_n$ is a map of degree 2. Clearly, X is a contractible 2-dimensional CW-complex whose proper homotopy type cannot be represented by a closed polyhedron in $\mathbb{R}^3$. Indeed, if $Y \subset \mathbb{R}^3$ is a polyhedron proper homotopy equivalent to X, then a regular neighbourhood M of Y in $\mathbb{R}^3$ would be an orientable 3-manifold proper homotopy equivalent to Y and hence $H^2_c(Y) \cong H^2_c(M) \cong H_1(M, \partial M)$ by Lefschetz duality ([14, 11.3]), which is a contradiction, since $H^2_c(Y) \cong H^2_c(X)$ is the group of dyadic numbers (i.e., the rational numbers in reduced form p/q with q a power of 2) and $H_1(M, \partial M) \cong \widetilde{H}_0(\partial M)$ is free abelian.

Once we have Proposition 3.2 at hand, Theorem I for coconnectedness $n - c = 2$ is an immediate consequence of the following two propositions:

PROPOSITION 3.5. *Let X be a properly $(n-2)$-connected n-dimensional locally finite CW-complex. Then X has the proper homotopy type of the $(n-2)$th suspension $\Sigma_T^{n-2} Y$ of a 2-dimensional locally finite CW-complex Y.*

Proof. By Proposition B.3 the proper homotopy type of X can be represented by an n-dimensional locally finite CW-complex Y such that $Y^{n-2} = T$ is a tree. In particular the $(n-1)$-skeleton Y^{n-1} has the proper homotopy type of a spherical object $S_\alpha^{n-1} \simeq_{\mathrm{p}} Y^{n-1}$. Here we use Proposition 3.1. Moreover, by using based attaching maps for the n-cells of Y if necessary (see Lemma B.1) the CW-complex Y can be described as the mapping cone $Y = C_f$ of a proper map $f : S_\beta^{n-1} \to S_\alpha^{n-1}$ between $(n-1)$-dimensional spherical objects. By Corollary B.5 the map f is properly homotopic to the iterated suspension of a proper map $\widetilde{f} : S_\beta^1 \to S_\alpha^1$. Hence $Y \simeq_{\mathrm{p}} \Sigma_T^{n-2} Z$ where $Z = C_{\widetilde{f}}$ is the 2-dimensional CW-complex defined by the cone of $\widetilde{f}$.

PROPOSITION 3.6. *Let $X \subset \mathbb{R}^n$ be a closed subpolyhedron. Then the proper homotopy type of the proper suspension $\Sigma_T X$ is represented by a closed subpolyhedron in $\mathbb{R}^{n+1}$.*

Proof. Let $\{M_i\}_{i \geq 1}$ be an increasing sequence of compact n-submanifolds as in Lemma 1.1. By using collars of the boundaries ∂M_i we can write $\mathbb{R}^n = M_1 \cup \partial M_1 \times [0,1] \cup (M_2 - \operatorname{int} M_1) \cup \partial M_2 \times [0,1] \cup \ldots$ and hence we obtain a new polyhedron $X' \subset \mathbb{R}^n$ of the form

$$X' = X_1 \cup X_1' \times [0,1] \cup X_2 \cup X_2' \times [0,1] \cup \ldots$$

where $X_i = X \cap (M_i - \operatorname{int} M_{i-1})$ and $X_i' = X_i \cap \partial M_i$ for $i \geq 1$ with $M_0 = \emptyset$. Moreover, X' has the same proper homotopy type as X since both X and X' are proper deformation retracts of the cylinder $X \times [0,1]$. Finally we form the tree T with vertices of level $i \geq 0$ the elements of the set $\pi_0(X - \operatorname{int} M_i) \cong \pi_0(X \cap (M_{i+1} - \operatorname{int} M_i))$ where two vertices $C \in \pi_0(X - \operatorname{int} M_i)$, $D \in \pi_0(X - \operatorname{int} M_{i+1})$ are joined by an edge if $D \subset C$. There is an obvious proper map $r : X' \to T$ which carries each of the points in $C \in \pi_0(X \cap (M_{i+1} - \operatorname{int} M_i))$ to the vertex in T associated to C. One readily checks that the induced map $r_* : \mathcal{F}(X') \to \mathcal{F}(T)$ is a homeomorphism. Therefore for an end-faithful tree $T' \subset X'$ the restriction $r : T' \to T$ is a proper homotopy equivalence; see [2]. Hence the gluing lemma B.1 shows that the proper suspension $\Sigma_{T'} X'$ has the same proper homotopy type as the polyhedron $S(X')$ obtained by the push-out diagram

$$\begin{tikzcd} X' \times \{0,1\} \arrow[r] \arrow[d, "r \cup r"'] \arrow[dr, phantom, "\text{p.o.}"] & X' \times I \arrow[d] \\ T \arrow[r] & S(X') \end{tikzcd}$$

The proof is now complete since the embedding $X' \subset \mathbb{R}^n$ can be readily extended to an embedding $S(X') \subset \mathbb{R}^{n+1}$.

Appendix A: The category of trees of groups. Here we collect from [10] the basic facts concerning the category of trees of groups, and we use them to state the proper analogues of homology and homotopy groups. Let T be any locally finite tree. By choosing a base vertex $v_0 \in T$ the set of vertices of T is stratified as follows. If v is a vertex of T, let $|v|$ denote the number of edges in the unique arc $\gamma_v \subset T$ connecting v to v_0. The number $|v|$ is called the *height* of v. The set of vertices of T can be partially ordered by setting $v \leq w$ if v lies in the arc γ_w. Any tree T can be considered as a category whose objects are the vertices of T and the morphism set $T(x, y)$ consists of a single morphism whenever $x \geq y$. Given a category $\mathbf{C}$, a T-*tree* in $\mathbf{C}$ is just a functor $\underline{F} : T \to \mathbf{C}$ where the induced morphisms $\underline{F}(x \geq y)$ are called the *bonding morphisms* of $\underline{F}$. For the special case $T = \mathbb{R}_+$ the T-trees are termed *towers* in $\mathbf{C}$. In order to define the category with objects T-trees in a category $\mathbf{C}$ with coproducts we define morphisms $\underline{f} = [(\{f_n\}_{n\geq 0}, \alpha)] : \underline{F} \to \underline{G}$ by equivalence classes of pairs $(\{f_n\}_{n\geq 0}, \alpha)$ defined as follows. The map (called a *shift map*) $\alpha : \mathbb{N} \cup \{0\} \to \mathbb{N} \cup \{0\}$ is an increasing map with $\alpha(0) = 0$ and $n \leq \alpha(n)$ for all $n \geq 0$. In addition, for each $n \geq 0$, f_n is a morphism in $\mathbf{C}$

$$f_n : \bigvee_{|x|=\alpha(n)} \underline{F}(x) \to \bigvee_{|y|=n} \underline{G}(y)$$

which is decomposed into morphisms

$$f_n : \bigvee \{\underline{F}(x);\, x \geq y,\, |x| = \alpha(n)\} \to \underline{G}(y).$$

Moreover two pairs $(\{f_n\}_{n>0}, \alpha)$ and $(\{g_n\}_{n>0}, \beta)$ are said to be equivalent if there exists a shift map $\gamma : \mathbb{N} \cup \{0\} \to \mathbb{N} \cup \{0\}$ with $\alpha, \beta \leq \gamma$ and for all $n \geq 0$ and $x \in T$ with $|x| = n$ we have $f_n h_{zy} = g_n h_{zw}$ for $x \leq y \leq z$, $x \leq w \leq z$, $|y| = \alpha(n)$, $|w| = \beta(n)$, and $|z| = \gamma(n)$. Here h_{zy} and h_{zw} denote the corresponding bonding morphisms. Let $\mathbf{Tree}_T(\mathbf{C})$ denote the category of T-trees in the category $\mathbf{C}$.

By using the notion of a tree in the category $\mathbf{Ab}$ of abelian groups one defines the homology tree of a strongly locally finite CW-pair (X, A) with an end-faithful tree $T \subset X$ as follows. Recall that a CW-complex X is said to be *strongly locally finite* if it can be decomposed as a locally finite union of finite subcomplexes. Therefore it is easy to find a countable basis $\{U_j\}_{j\geq 0}$ of neighbourhoods at infinity consisting of subcomplexes with $U_0 = X$. Then the nth *homology tree* of (X, A) is the T-tree $H_n^T(X, A) \in \mathbf{Tree}_T(\mathbf{Ab})$ given by

$$H_n^T(X, A)(v) = H_n(C_v, C_v \cap A)$$

where $C_v \subset U_{j(v)}$ is the component of v with $j(v) = \max\{j;\, w \in U_j \text{ for all } w \geq v\}$. The bonding homomorphisms are induced by the corresponding inclusions.

The category $\mathbf{Tree}_T(\mathbf{Ab})$ is an abelian category and it is readily checked that $H_n^T(X, A)$ can also be obtained as the homology of the chain complex in $\mathbf{Tree}_T(\mathbf{Ab})$

$$\ldots \to \underline{C}_n^T(X, A) \xrightarrow{\underline{\partial}} \underline{C}_{n-1}^T(X, A) \to \ldots$$

where $\underline{C}_n^T(X, A)$ is the tree of free abelian groups obtained from the cellular n-chains of the components C_v, and $\underline{\partial}$ is defined by the cellular boundary operator.

A crucial feature of the T-tree $\underline{C}_n^T(X, A)$ is the following "freeness property". A T-tree $\underline{B} \in \mathbf{Tree}_T(\mathbf{Set})$ of sets is said to be a *T-basis* if the following conditions hold:

(a) the bonding maps are inclusions;
(b) if $|v| = |w|$ and $v \neq w$ then $\underline{B}(v) \cap \underline{B}(w) = \emptyset$;
(c) for all $n \geq 0$, $\underline{B}(v_0) - \bigcup\{\underline{B}(v);\, |v| = n\}$ is a finite set;
(d) $\bigcap_{i=0}^{\infty}\{\bigcup \underline{B}(v);\, |v| = i\} = \emptyset$.

A T-tree in $\mathbf{Ab}$, $\underline{F} : T \to \mathbf{Ab}$, is said to be a *free tree* with T-basis $\underline{B}$ if $\underline{F}(v)$ is the free abelian group generated by $\underline{B}(v)$ for all $v \in T$ and the bonding homomorphisms of $\underline{F}$ are the obvious basis inclusion homomorphisms. The tree $\underline{C}_n^T(X)$ of cellular chains is readily checked to be a free tree. Moreover, it is a simple exercise to show that any free tree is a projective object in $\mathbf{Tree}_T(\mathbf{Ab})$.

Observe that a T-basis $\underline{B} : T \to \mathbf{Set}$ is completely determined by a proper map $\alpha : \underline{B}(v_0) \to T^0$ from the discrete set $\underline{B}(v_0)$ to the 0-skeleton T^0 of T. Namely $\alpha(b) = v$ if $b \in \underline{B}(v)$ and hence $\underline{B}(v) = \{\alpha^{-1}(w);\, w \geq v\}$. A free tree of basis $\underline{B}$ will also be denoted by $\underline{L}_\alpha$. Moreover by use of the nth homology tree ($n \geq 2$) we can identify the free tree $\underline{L}_\alpha$ with the space S_α^n obtained by attaching $\#\alpha^{-1}(v)$ n-spheres at each vertex $v \in T$. The space S_α^n is termed an *n-spherical object under T*. The n-ball B^n yields the corresponding object B_α^n.

We make use of the following result; see [1] for a proof.

Proposition A.1. *The full subcategory of* $\mathbf{Tree}_T(\mathbf{Ab})$ *consisting of the free T-trees admits kernels. In particular, for a finite-dimensional locally* *finite CW-complex $X = (X, T)$ the T-tree $Z_n^T(X)$ of cellular cycles is a free T-tree.*

Next we define the proper analogue of the homotopy groups as follows. Given an ordered tree T, a *T-based pair* is a pair (X, A) in $\mathcal{P}$ together with a proper map $t : T \to A$. Given a T-based pair (X, A, t) we define the T-tree of homotopy groups of (X, A, t) as the object $\Pi_n^T(X, A, t) \in \mathbf{Tree}_T(\mathbf{Gr})$ defined as follows. Let $\{U_j\}_{j \geq 0}$ with $U_0 = X$ be a countable basis of neighbourhoods at infinity in X. Then for each vertex $v \in T$ we set

$$\Pi_n^T(X, A, t)(v) = \pi_n(C_v, C_v \cap A, t(v))$$

where $C_v \subset U_{j(v)}$ is the component of $t(v)$ and $j(v) = \max\{j;\ t(w) \in U_j$ for all $w \geq v\}$. The bonding homomorphisms are composites of inclusion induced homomorphisms and base point change isomorphisms along the tree T. If $T \subseteq A$ is the inclusion of an end-faithful tree for the CW-complex X we write $\Pi_n^T(X, A)$ and moreover if $A = T$ is the identity we write $\Pi_n^T(X)$. It can be checked that the isomorphism type of $\Pi_n^T(X, A, t)$ does not depend on the basis $\{U_j\}$ but it does depend on the base map t; see p. 13 in [16]. Moreover the classical exact homotopy sequence yields the corresponding exact sequence of homotopy trees in $\mathbf{Tree}_T(\mathbf{Gr})$. By using the identification of the spherical object S_α^n under T with the free T-tree $\underline{L}_\alpha$ via the isomorphism

$$\Pi_n^T(B_\alpha^n, S_\alpha^{n-1}) \cong \Pi_{n-1}^T(S_\alpha^{n-1}) \cong \underline{L}_\alpha$$

we obtain

Proposition A.2. *Let $T \subset A \subset X$ be as above. Then there exists a one-one correspondence*

$$[(B_\alpha^n, S_\alpha^{n-1}), (X, A)]^T \cong \mathbf{Tree}_T(\mathbf{Gr})(\underline{L}_\alpha, \Pi_n^T(X, A, t)).$$

The correspondence carries the proper homotopy class $[f]$ to the induced morphism $f_* : \Pi_n^T(B_\alpha^n, S_\alpha^{n-1}) \to \Pi_n^T(X, A, t)$.

For any integer $c \geq 1$ a proper map $f : X \to Y$ is said to be *properly c-connected* if for any base tree $t : T \to X$ the induced morphism $f_* : \Pi_k^T(X, t) \to \Pi_k^T(Y, ft)$ is an isomorphism for $1 \leq k \leq c - 1$ and an epimorphism for $k = c$.

A pair (X, A) is said to be properly c-connected if the inclusion $A \subset X$ is a properly c-connected map. In particular if $A = T$ is an end-faithful tree of X we simply say that X is properly c-connected. It is readily checked that a proper map $f : X \to Y$ is properly c-connected if and only if the pair (M_f, X) is properly c-connected. Here M_f is the mapping cylinder of f.

Note that we set $c \geq 1$ in order to require no relation between the number of Freudenthal ends of X and Y. Otherwise, for $c = 0$ it is a well known fact that a proper map f for which $f_* : \Pi_0^T(X, t) \to \Pi_0^T(Y, ft)$ is onto for any base tree $t : T \to X$ induces an onto map between the corresponding spaces of Freudenthal ends. Therefore in this paper the case $c = 0$ simply refers to the general case of arbitrary (proper maps between) finite-dimensional locally finite connected CW-complexes.

Appendix B: Some basic results of proper homotopy theory. This appendix contains the proper analogues of classical results in homotopy theory (Hurewicz and Whitehead theorems among them) which are needed in §3. These results are essentially consequences of the fact that the proper homotopy category is a cofibration category in the sense of Baues ([3]) and so the theoretical machinery in [4] or [5] can be applied. With this Appendix we intend to help the reader by providing explicit proofs. We include the proofs since these results are not easily found in the literature except for homology and homotopy towers; see [8]. For this we have chosen the usual language for "proper" algebra based on the category of trees of groups instead of the new approach based on theories and models of theories as done in [5]. We start with the following gluing lemma available in any cofibration category [4].

Lemma B.1. *Consider the commutative diagram in* $\mathcal{P}$

$$\begin{tikzcd} Y_0 \arrow[d, "\alpha"'] & Y \arrow[l] \arrow[r] \arrow[d, "\gamma"] & Y_1 \arrow[d, "\beta"'] \\ X_0 & X \arrow[l] \arrow[r] & X_1 \end{tikzcd}$$

where the rows are push-out diagrams. Assume in addition that α, β, γ *are proper homotopy equivalences. Then the natural map* $\alpha \cup \beta : Y_0 \cup_Y Y_1 \to X_0 \cup_X X_1$ *between the corresponding push-outs is a proper homotopy equivalence. As a consequence, the proper homotopy type of* $X_0 \cup_X X_1$ *only depends on the proper homotopy classes of the maps involved in its definition.*

As in classical homotopy theory the gluing lemma is used in the proof of the following

Proposition B.2. *Any* n*-dimensional locally finite CW-complex has the same proper homotopy type as a locally compact polyhedron of the same dimension.*

Proof. For $n = 0$ the result is immediate. Assume now that X^{n-1} is a polyhedron. Then by gathering together all attaching maps of the n-cells of X one gets a proper map $f = \bigsqcup_{k\geq 1} f_k : \bigsqcup_{k\geq 1} S_k^{n-1} \to X^{n-1}$. Recall that the number of cells of X is countable. Then one uses the (proper) simplicial approximation theorem to obtain a simplicial global attaching map $g = \bigsqcup g_k$ and a proper homotopy equivalence $X^n = \bigcup_{k=1}^{\infty} C_{f_k} \simeq_{\mathrm{p}}$

$\bigcup_{k=1}^{\infty} C_{g_k}$. Here we use the gluing lemma. Finally, we apply ([21, VII.47]) to triangulate each mapping cone C_{g_k} by a simplicial complex Y_k^n containing X^{n-1} as a subcomplex. Then the polyhedron $Y^n = \bigcup Y_k^n$ has the same proper homotopy type as X^n.

Next we deal with proper Hurewicz and Whitehead theorems involving the homology and homotopy trees defined in Appendix A. By using a well known argument due to J. H. C. Whitehead ([18, 6.13]) we prove

PROPOSITION B.3. *Let (X, A) be a pair of finite-dimensional locally finite CW-complexes with $\mathcal{F}(X) = \mathcal{F}(A)$. Assume that (X, A) is properly n-connected. Then (X, A) can be embedded in a CW-pair (Y, B) with the same properties as (X, A) and moreover X and A are proper deformation retracts of Y and B respectively and $Y^n \subset B$. Moreover* $\dim(Y - X) \leq n+2$.

Proof. Let $T \subseteq X$ be an end-faithful tree and suppose the proposition holds for $n-1$. Then we assume without loss of generality $X^{n-1} \subset A$. There is a canonical element $\xi \in \Pi_n^T(X, A)$ represented by a proper map $f : (B_\alpha^n, S_\alpha^{n-1}) \to (X, A)$ consisting of the union of all the characteristic maps of the locally finite family of n-cells $e \subset X - A$ (we base all cells e at T by using a locally finite family of paths γ_e if necessary). The vanishing of ξ yields a proper map $\widetilde{f} : B_\alpha^{n+1} \to X$ extending f onto the northern hemispheres of the $(n+1)$-balls in B_α^{n+1}. Moreover the map $\widetilde{f}$ carries the southern hemispheres into A. Now the Whitehead trick comes into play. Namely, we attach to X new $(n+1)$-cells ϱ_e by the restriction of $\widetilde{f}$ to the n-spheres in S_α^n. Moreover we attach new $(n+2)$-cells β_e along the union $\widetilde{f}(B_\alpha^{n+1}) \cup \{\varrho_e;\, e \subset X - A\}$. This way we obtain the CW-complexes $B = A \cup \{\varrho_e;\, e \subset X - A\}$ and $Y = B \cup \{\beta_e;\, e \subset X - A\}$. Finally we see that B and Y have the same proper homotopy type as A and X respectively by pushing the cells ϱ_e and β_e through their faces e and ϱ_e respectively.

Proposition B.3 and the classical result in ordinary homotopy theory yield the proper analogue of the Blakers–Massey theorem:

PROPOSITION B.4. *Let X be a finite-dimensional locally finite CW-complex and let $A, B \subset X$ be subcomplexes with $T \subset A \cap B$ a common end-faithful tree. Assume the pairs $(A, A \cap B)$ and $(B, A \cap B)$ are properly n-connected and m-connected respectively. Then*

$$i_* : \Pi_k^T(A, A \cap B) \to \Pi_k^T(X, B)$$

is an isomorphism for $1 \leq k \leq m+n-1$ and an epimorphism for $k = m+n$.

Proof. By using Proposition B.3 several times together with the gluing lemma we can embed the triple $(X; A, B)$ in a new triple $(X_0; A_0, B_0)$ such that $A_0^n \cup B_0^m \subset A_0 \cap B_0$ and moreover the inclusions $X \subset X_0$, $A \subset A_0$ and $B \subset B_0$ are proper deformation retracts. Then we can apply the classical

Blakers–Massey theorem to each of the neighbourhoods at infinity in the definition of $\Pi_k^T(-)$ and the result follows.

In order to state a proper suspension theorem we need the *reduced proper cone* $\widetilde{C}_T X$ and the *reduced proper suspension* $\widetilde{\Sigma}_T X$ which are defined by the push-out diagrams

$$\begin{tikzcd} T \times I \arrow[r, "k_1"'] \arrow[d, "\pi"', "\simeq_{\mathrm{p}}"] \arrow[dr, phantom, "\text{p.o.}"] & C_T X \arrow[d, "\widetilde{\pi}"', "\simeq_{\mathrm{p}}"] \\ T \arrow[r] & \widetilde{C}_T X \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} T \times I \arrow[r, "k_2"'] \arrow[d, "\pi"', "\simeq_{\mathrm{p}}"] \arrow[dr, phantom, "\text{p.o.}"] & \Sigma_T X \arrow[d, "\widetilde{\pi}"', "\simeq_{\mathrm{p}}"] \\ T \arrow[r] & \widetilde{\Sigma}_T X \end{tikzcd}$$

Here k_1 and k_2 denote the corresponding canonical embeddings and π is the natural projection. Moreover for any spherical object S_α^n under T one gets $\widetilde{\Sigma}_T S_\alpha^n = S_\alpha^{n+1}$ and the suspension operator

$$\widetilde{\Sigma}_T : \Pi_m^T(X) \to \Pi_{m+1}^T(\widetilde{\Sigma}_T X)$$

is a well defined morphism in $\mathbf{Tree}_T(\mathbf{Gr})$ for any finite-dimensional locally finite CW-complex X with end-faithful tree $T \subset X$. The standard arguments based on the exact homotopy sequence of the pair $(\widetilde{C}_T X, X)$ and Proposition B.3 yield

Corollary B.5. *Let $T \subset X$ be an end-faithful tree and let X be a properly c-connected finite-dimensional locally finite CW-complex. Then the natural suspension operator $\widetilde{\Sigma}_T$ is an isomorphism for $1 \leq m \leq 2c$ and an epimorphism for $m = 2c + 1$.*

Proposition B.3 can also be used to obtain the following proper analogue of the classical Hurewicz theorem:

Proposition B.6. *Let (X, A) be a pair with $T \subset A$ an end-faithful tree. Assume in addition that (X, A) is properly c-connected. Then the natural morphism*

$$h : \Pi_m^T(X, A) \to H_m^T(X, A)$$

induced by the Hurewicz homomorphism is an isomorphism for $m \leq c + 1$ and an epimorphism for $m = c + 2$.

Proof. According to Proposition B.3, (X, A) can be embedded in a CW-pair (Y, B) with $Y^c \subset Y$ and such that A and X are proper deformation retracts of B and Y respectively. Then we can apply the Hurewicz theorem in ordinary homotopy theory to each of the neighbourhoods at infinity in the definition of $\Pi_k^T(-)$ and $H_k^T(-)$ and the result follows.

We finish this overview of the proper homotopy theory by restating the proper Whitehead theorem of [6].

PROPOSITION B.7. *Let X and Y be two finite-dimensional locally finite CW-complexes with the same end-faithful tree T. Assume that $f : X \to Y$ is a proper map under T, and hence $f_* : \mathcal{F}(X) \to \mathcal{F}(Y)$ is a homeomorphism, such that $f_* : \Pi_n^T(X) \to \Pi_n^T(Y)$ is an isomorphism for all $n \geq 1$. Then f is a proper homotopy equivalence.*

Proof. Let $\varepsilon \in \mathcal{F}(T)$ be any Freudenthal end. By using the natural bijection in Proposition A.2 one finds that f induces a group isomorphism $f_* : [S^n_{\alpha_\varepsilon}, X]^{r_\varepsilon} \xrightarrow{\cong} [S^n_{\alpha_\varepsilon}, Y]^{r_\varepsilon}$ where $S^n_{\alpha_\varepsilon}$ is the spherical object with one n-sphere attached at each vertex in the canonical ray $r_\varepsilon : \mathbb{R} \to T$ from v_0 to ε in T. The group $[S^n_{\alpha_\varepsilon}, X]^{r_\varepsilon}$ coincides with the Brown groups as defined in [6] and the result follows from [6].

As usual, from Propositions B.7 and B.6 one derives the following homological version of the proper Whitehead theorem.

PROPOSITION B.8. *Let X and Y be properly 1-connected finite-dimensional locally finite CW-complexes with the same end-faithful tree T and let $f : X \to Y$ be a proper map under T such that $f_* : H_n^T(X) \to H_n^T(Y)$ is an isomorphism for all $n \geq 1$. Then f is a proper homotopy equivalence.*

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