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THE NATURAL LINEAR OPERATORS $T^* \rightsquigarrow TT^{(r)}$

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Abstract. For natural numbers $n \geq 3$ and r a complete description of all natural bilinear operators $T^* \times_{\mathcal{M}f_n} T^{(0,0)} \rightsquigarrow T^{(0,0)}T^{(r)}$ is presented. Next for natural numbers r and $n \geq 3$ a full classification of all natural linear operators $T^*_{|\mathcal{M}f_n} \rightsquigarrow TT^{(r)}$ is obtained.

Introduction. Let n and r be natural numbers. Given an n-dimensional manifold M we have the r-tangent vector bundle $T^{(r)}M = (J^r(M,\mathbb{R})_0)^*$ over M. Every embedding $\varphi : M \to N$ of n-manifolds induces a vector bundle map $T^{(r)}\varphi : T^{(r)}M \to T^{(r)}N$ covering φ such that $\langle T^{(r)}\varphi(\omega), j^r_{\varphi(x)}\gamma\rangle = \langle \omega, j^r_x(\gamma \circ \varphi)\rangle$ for $\omega \in T^{(r)}_x M$, $j^r_{\varphi(x)}\gamma \in J^r_{\varphi(x)}(N,\mathbb{R})_0$, $x \in M$. The correspondence $T^{(r)} : \mathcal{M}f_n \to \mathcal{FM}$ is a bundle functor from the category $\mathcal{M}f_n$ of n-manifolds and embeddings into the category $\mathcal{FM}$ of fibered manifolds and fibered maps [3].

In [4], we studied the problem of how a 1-form $\omega \in \Omega^1(M)$ on an n-manifold M can induce a 1-form $A(\omega) \in \Omega^1(T^{(r)}M)$ on $T^{(r)}M$. This problem was reflected in the concept of natural linear operators $T^*_{|\mathcal{M}f_n} \rightsquigarrow T^*T^{(r)}$ in the sense of Kolář, Michor and Slovák [3]. We presented a complete description of such operators.

In the present note we start with the problem of how a 1-form $\omega \in \Omega^1(M)$ and a map $f : M \to \mathbb{R}$ on an n-manifold M can induce a map $B(\omega, f) : T^{(r)}M \to \mathbb{R}$. This problem concerns natural bilinear operators $B : T^* \times_{\mathcal{M}f_n} T^{(0,0)} \rightsquigarrow T^{(0,0)}T^{(r)}$. We prove that the vector space of all such operators is 0-dimensional if $n \geq 3$ and $r \geq 3$, 3-dimensional if $n \geq 3$ and $r = 2$, and 2-dimensional if $n \geq 3$ and $r = 1$. We construct explicit bases of the vector spaces in question.

Next, using this classification we investigate how a 1-form ω on an n-manifold M can induce a vector field $C(\omega)$ on $T^{(r)}M$. This problem relates to natural linear operators $C : T^*_{|\mathcal{M}f_n} \rightsquigarrow TT^{(r)}$. We deduce that the vector space of all such operators is 0-dimensional if $n \geq 3$ and $r \geq 3$, 2-dimensional

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if $n \geq 3$ and $r = 2$, and 1-dimensional if $n \geq 3$ and $r = 1$, and we construct the corresponding bases.

Natural operators lifting functions, vector fields and 1-forms to some natural bundles were used practically in all papers in which problem of prolongations of geometric structures was studied (e.g. [6]). That is why such natural operators have been classified by many authors (see e.g. [1]–[5]).

The usual coordinates on $\mathbb{R}^n$ are denoted by $x^1, \ldots, x^n$. On $T^{(r)}\mathbb{R}^n$ we have the induced coordinates (x^i, X^α),

$$x^i(\Theta) = x_0^i, \qquad X^\alpha(\Theta) = \langle \Theta, j^r_{x_0}((x - x_0)^\alpha)\rangle, \tag{1}$$

$i = 1, \ldots, n$, $\alpha \in (\mathbb{N} \cup \{0\})^n$, $1 \leq |\alpha| \leq r$, $\Theta \in T^{(r)}_{x_0}\mathbb{R}^n$, $x_0 = (x_0^1, \ldots, x_0^n) \in \mathbb{R}^n$.

All manifolds are assumed to be finite-dimensional and smooth, i.e. of class $\mathcal{C}^\infty$. Maps between manifolds are assumed to be smooth.

1. The natural bilinear operators $T^* \times_{\mathcal{M}f_n} T^{(0,0)} \rightsquigarrow T^{(0,0)}T^{(r)}$. For $r = 1$ we have the following examples of natural bilinear operators $T^* \times_{\mathcal{M}f_n} T^{(0,0)} \rightsquigarrow T^{(0,0)}T^{(1)}$.

Example 1. For a 1-form $\omega \in \Omega^1(M)$ and a map $f : M \to \mathbb{R}$ on an n-manifold M we define $B^{(1)}(\omega, f) : T^{(1)}M \to \mathbb{R}$ by

$$B^{(1)}(\omega, f)_\eta = \langle \omega_{x_0}, \eta\rangle f(x_0), \qquad \eta \in T^{(1)}_{x_0}M \cong T_{x_0}M,\ x_0 \in M.$$

Then $B^{(1)} : T^* \times_{\mathcal{M}f_n} T^{(0,0)} \rightsquigarrow T^{(0,0)}T^{(1)}$ is a natural bilinear operator.

Example 2. For $\omega \in \Omega^1(M)$ and $f : M \to \mathbb{R}$ we define $B^{(1)}(\omega, f) : T^{(1)}M \to \mathbb{R}$ by

$$B^{[1]}(\omega, f)_\eta = \langle \omega_{x_0}, \eta\rangle \langle d_{x_0}f, \eta\rangle, \qquad \eta \in T^{(1)}_{x_0}M,\ x_0 \in M.$$

Then $B^{[1]} : T^* \times_{\mathcal{M}f_n} T^{(0,0)} \rightsquigarrow T^{(0,0)}T^{(1)}$ is a natural bilinear operator.

Now, let $r = 2$. In [4], we proved that there exists a linear (first order) natural operator $D : T^*_{\mathcal{M}f_n} \rightsquigarrow T^{2*}$ such that $D(fdg) = j^2_{x_0}((f + f(x_0))(g - g(x_0)))$ for $f, g : M \to \mathbb{R}$, $x_0 \in M$, $M \in \mathrm{obj}(\mathcal{M}f_n)$. Using D we now present three examples of natural bilinear operators $T^* \times_{\mathcal{M}f_n} T^{(0,0)} \rightsquigarrow T^{(0,0)}T^{(2)}$.

Example 3. For $\omega \in \Omega^1(M)$ and $f : M \to \mathbb{R}$ we define $B^{(2)}(\omega, f) : T^{(2)}M \to \mathbb{R}$ by

$$B^{(2)}(\omega, f)_\eta = \langle D(\omega)_{x_0}, \eta\rangle f(x_0), \qquad \eta \in T^{(2)}_{x_0}M = (T^{2*}_{x_0}M)^*,\ x_0 \in M.$$

Then $B^{(2)} : T^* \times_{\mathcal{M}f_n} T^{(0,0)} \rightsquigarrow T^{(0,0)}T^{(2)}$ is a natural bilinear operator.

Example 4. For $\omega \in \Omega^1(M)$ and $f : M \to \mathbb{R}$ we define $B^{[2]}(\omega, f) : T^{(2)}M \to \mathbb{R}$ by

$$B^{[2]}(\omega, f)_\eta = \langle D(\omega)_{x_0}, \eta\rangle \langle j^2_{x_0}(f - f(x_0)), \eta\rangle, \qquad \eta \in T^{(2)}_{x_0}M,\ x_0 \in M.$$

Then $B^{[2]} : T^* \times_{\mathcal{M}f_n} T^{(0,0)} \rightsquigarrow T^{(0,0)}T^{(2)}$ is a natural bilinear operator.

EXAMPLE 5. For $\omega \in \Omega^1(M)$ and $f : M \to \mathbb{R}$ we define $B^{\langle 2\rangle}(\omega, f) : T^{(2)}M \to \mathbb{R}$ by

$$B^{\langle 2\rangle}(\omega, f)_\eta = \langle D((f - f(x_0))\omega)_{x_0}, \eta\rangle, \quad \eta \in T^{(2)}_{x_0}M,\ x_0 \in M.$$

Then $B^{\langle 2\rangle} : T^* \times_{\mathcal{M}f_n} T^{(0,0)} \rightsquigarrow T^{(0,0)}T^{(2)}$ is a natural bilinear operator.

In the induced coordinates on $T^{(r)}\mathbb{R}^n$ we have

$$B^{(2)}(x^2dx^1, x^3) = X^{(1,1,0,\ldots,0)}x^3 + 2x^2x^3X^{(1,0,\ldots,0)},$$
$$B^{[2]}(x^2dx^1, x^3) = X^{(1,1,\ldots,0)}X^{(0,0,1,0,\ldots,0)} + 2x^2X^{(1,0,\ldots,0)}X^{(0,0,1,0,\ldots,0)},$$
$$B^{\langle 2\rangle}(x^2dx^1, x^3) = x^2X^{(1,0,1,0,\ldots,0)},$$
$$B^{(1)}(x^2dx^1, x^3) = x^2x^3X^{(1,0,\ldots,0)},$$
$$B^{[1]}(x^2dx^1, x^3) = x^2X^{(1,0,\ldots,0)}X^{(0,0,1,0,\ldots,0)}.$$

Hence, the operators $B^{(1)}$ and $B^{[1]}$ are linearly independent, and so are $B^{(2)}$, $B^{[2]}$ and $B^{\langle 2\rangle}$.

The first main result of this note is the following theorem.

THEOREM 1. *Let $n \geq 3$ be a natural number.*

(i) *Every natural bilinear operator $T^* \times_{\mathcal{M}f_n} T^{(0,0)} \rightsquigarrow T^{(0,0)}T^{(1)}$ is a linear combination of $B^{(1)}$ and $B^{[1]}$ with real coefficients.*

(ii) *Every natural bilinear operator $T^* \times_{\mathcal{M}f_n} T^{(0,0)} \rightsquigarrow T^{(0,0)}T^{(2)}$ is a linear combination of $B^{(2)}$, $B^{[2]}$ and $B^{\langle 2\rangle}$ with real coefficients.*

(iii) *If $r \geq 3$ then every natural bilinear operator $T^* \times_{\mathcal{M}f_n} T^{(0,0)} \rightsquigarrow T^{(0,0)}T^{(r)}$ is zero.*

The proof of Theorem 1 will occupy Sections 3–6.

2. The natural linear operators $T^* \rightsquigarrow TT^{(r)}$. Every natural linear operator $C : T^*_{|\mathcal{M}f_n} \rightsquigarrow TT^{(r)}$ of vertical type induces a natural bilinear operator $B^{(C)} : T^* \times_{\mathcal{M}f_n} T^{(0,0)} \rightsquigarrow T^{(0,0)}T^{(r)}$ defined by

$$B^{(C)}(\omega, f)_\eta = \langle j^r_{x_0}(f - f(x_0)), \mathrm{pr}_2 \circ C(\omega)_\eta\rangle$$

for $\omega \in \Omega^1(M)$, $f : M \to \mathbb{R}$, $\eta \in T^{(r)}_{x_0}M$, $x_0 \in M$, $M \in \mathrm{obj}(\mathcal{M}f_n)$, where $\mathrm{pr}_2 : VT^{(r)}M \cong T^{(r)}M \times_M T^{(r)}M \to T^{(r)}M$ is the projection on the second factor. Of course, $B^{(C)}(\cdot, 1) = 0$.

Conversely, any natural bilinear operator $B : T^* \times_{\mathcal{M}f_n} T^{(0,0)} \rightsquigarrow T^{(0,0)}T^{(r)}$ such that $B(\cdot, 1) = 0$ induces a natural linear operator $C^{(B)} : T^*_{|\mathcal{M}f_n} \rightsquigarrow TT^{(r)}$ of vertical type such that $\langle \mathrm{pr}_2 \circ C^{(B)}_\eta, j^r_{x_0}(f - f(x_0))\rangle = B(\omega, f)_\eta$ for $\omega \in \Omega^1(M)$, $f : M \to \mathbb{R}$, $\eta \in T^{(r)}_{x_0}M$, $x_0 \in M$, $M \in \mathrm{obj}(\mathcal{M}f_n)$.

LEMMA 1. *The above correspondences are mutually inverse.*

Proof. Clear. ∎

So we have the following examples of natural linear operators $T^*_{|\mathcal{M}f_n} \rightsquigarrow TT^{(r)}$ of vertical type.

EXAMPLE 6. $C^{(1)} = C^{(B^{[1]})} : T^*_{|\mathcal{M}f_n} \rightsquigarrow TT^{(1)}$ is a natural linear operator of vertical type. Here $B^{[1]}$ is defined in Example 2.

EXAMPLE 7. $C^{[2]} = C^{(B^{[2]})} : T^*_{|\mathcal{M}f_n} \rightsquigarrow TT^{(2)}$ is a natural linear operator of vertical type. Here $B^{[2]}$ is defined in Example 4.

EXAMPLE 8. $C^{\langle 2\rangle} = C^{(B^{\langle 2\rangle})} : T^*_{|\mathcal{M}f_n} \rightsquigarrow TT^{(2)}$ is a natural linear operator of vertical type. Here $B^{\langle 2\rangle}$ is defined in Example 5.

The second main result is

THEOREM 2. *Let $n \geq 3$ and r be natural numbers. Then every natural linear operator $T^*_{|\mathcal{M}f_n} \rightsquigarrow TT^{(r)}$ is of vertical type. Namely*:

(i) *Every natural linear operator $T^*_{|\mathcal{M}f_n} \rightsquigarrow TT^{(1)}$ is proportional to $C^{(1)}$.*

(ii) *Every natural linear operator $T^*_{|\mathcal{M}f_n} \rightsquigarrow TT^{(2)}$ is a linear combination of $C^{[2]}$ and $C^{\langle 2\rangle}$ with real coefficients.*

(iii) *If $r \geq 3$ then every natural linear operator $T^*_{|\mathcal{M}f_n} \rightsquigarrow TT^{(r)}$ is zero.*

The proof of Theorem 2 will occupy Section 7.

3. A reducibility lemma. We begin the proof of Theorem 1 with the following lemma.

LEMMA 2 (Reducibility Lemma). *Let $n \geq 3$. Let $B : T^* \times_{\mathcal{M}f_n} T^{(0,0)} \rightsquigarrow T^{(0,0)}T^{(r)}$ be a natural bilinear operator such that $B(x^2dx^1, x^3) = 0$. Then $B = 0$.*

Proof. Let $f, g : \mathbb{R}^n \to \mathbb{R}$ and $y \in \mathbb{R}^n$. There is a dense subset of (τ_1, τ_2) in $\mathbb{R}^2$ such that $\varphi_{\tau_1,\tau_2} = (x^1, f + \tau_1 x^2, g + \tau_2 x^3, x^4, \ldots, x^n) : \mathbb{R}^n \to \mathbb{R}^n$ is a local diffeomorphism near y. Using the naturality of B with respect to φ_{τ_1,τ_2} and the assumption we get

$$B((f + \tau_1 x^2)dx^1, g + \tau_2 x^3)_\eta = 0 \tag{2}$$

for every $\eta \in T^{(r)}_y\mathbb{R}^n$. The left hand side of (2) is a polynomial in τ_1, τ_2 for fixed η. Considering the constant term we derive that $B(fdx^1, g)_\eta = 0$.

So, $B(fdx^1, g) = 0$ for every $f, g : \mathbb{R}^n \to \mathbb{R}$. Now, by the invariance of B under coordinate permutations we get $B(fdx^i, g) = 0$. Then the bilinearity and naturality of B imply that $B = 0$. ∎

4. A form of $B(x^2dx^1, x^3)$. By the reducibility lemma every natural bilinear operator $B : T^* \times_{\mathcal{M}f_n} T^{(0,0)} \rightsquigarrow T^{(0,0)}T^{(r)}$ is uniquely determined by $B(x^2dx^1, x^3)$.

LEMMA 3. *Let* $B : T^* \times_{\mathcal{M}f_n} T^{(0,0)} \rightsquigarrow T^{(0,0)}T^{(r)}$ *be a natural bilinear operator with* $n \geq 3$.

(i) *If* $r = 1$ *then*

$$B(x^2dx^1, x^3) = \mu_1 X^{e_1}x^2x^3 + \mu_2 X^{e_1}X^{e_2}x^3 + \mu_3 X^{e_1}X^{e_3}x^2 + \mu_4 X^{e_1}X^{e_2}X^{e_3} \tag{3}$$

for some $\mu_1, \ldots, \mu_4 \in \mathbb{R}$, *where* e_i *is the ith standard unit vector in* $\mathbb{R}^n$.

(ii) *If* $r = 2$ *then*

$$\begin{aligned} B(x^2dx^1, x^3) = {} & \mu_1 X^{e_1}x^2x^3 + \mu_2 X^{e_1}X^{e_2}x^3 + \mu_3 X^{e_1}X^{e_3}x^2 \\ & + \mu_4 X^{e_1}X^{e_2}X^{e_3} + \mu_5 X^{(1,0,1,0,\ldots,0)}x^2 \\ & + \mu_6 X^{(1,1,0,\ldots,0)}x^3 + \mu_7 X^{(1,0,1,0,\ldots,0)}X^{e_2} \\ & + \mu_8 X^{(1,1,0,\ldots,0)}X^{e_3} + \mu_9 X^{(0,1,1,0,\ldots,0)}X^{e_1} \end{aligned} \tag{4}$$

for some $\mu_1, \ldots, \mu_9 \in \mathbb{R}$.

(iii) *If* $r \geq 3$ *then*

$$\begin{aligned} B(x^2dx^1, x^3) = {} & \mu_1 X^{e_1}x^2x^3 + \mu_2 X^{e_1}X^{e_2}x^3 \\ & + \mu_3 X^{e_1}X^{e_3}x^2 + \mu_4 X^{e_1}X^{e_2}X^{e_3} \\ & + \mu_5 X^{(1,0,1,0,\ldots,0)}x^2 + \mu_6 X^{(1,1,0,\ldots,0)}x^3 \\ & + \mu_7 X^{(1,0,1,0,\ldots,0)}X^{e_2} + \mu_8 X^{(1,1,0,\ldots,0)}X^{e_3} \\ & + \mu_9 X^{(0,1,1,0,\ldots,0)}X^{e_1} + \mu_{10} X^{(1,1,1,0,\ldots,0)} \end{aligned} \tag{5}$$

for some $\mu_1, \ldots, \mu_{10} \in \mathbb{R}$.

Proof. We can write $B(x^2dx^1, x^3) = f(x^i, X^\alpha)$, where f is of class $\mathcal{C}^\infty$. By the invariance of B with respect to translations of the first coordinate we deduce that f is independent of x^1. By the invariance of B with respect to the homotheties $(t^ix^i) : \mathbb{R}^n \to \mathbb{R}^n$ for $(t^i) \in \mathbb{R}^n_+$ we get the homogeneity condition

$$t^1t^2t^3 f(x^i, X^\alpha) = f(t^ix^i, t^\alpha X^\alpha).$$

This type of homogeneity implies that f is a linear combination of monomials in x^i and X^α having the homogeneity type $t^1t^2t^3$. These facts yield the result. ∎

5. Transformation rules. We have a local diffeomorphism

$$G = \left(x^1 - \frac{1}{2}(x^1)^2, \frac{x^2}{1 - x^1}, x^3, \ldots, x^n\right) \tag{6}$$

defined in some neighbourhood U of $0 \in \mathbb{R}^n$. This diffeomorphism preserves the germ of (x^2dx^1, x^3) at 0. Therefore we will study the invariance of $B(x^2dx^1, x^3)$ with respect to G. We need some transformation rules.

Lemma 3. *Let $n \geq 3$ and G and U be as above.*

(i) *If $r = 1$ then*

$$X^{e_1} \circ T^{(1)}G = (1 - x^1)X^{e_1}, \tag{7}$$

$$X^{e_2} \circ T^{(1)}G = \frac{x^2}{(1-x^1)^2}\, X^{e_1} + \frac{1}{1-x^1}\, X^{e_2}, \tag{8}$$

$$X^{e_3} \circ T^{(1)}G = X^{e_3} \tag{9}$$

over U.

(ii) *If $r = 2$ then*

$$X^{e_1} \circ T^{(2)}G = (1 - x^1)X^{e_1} - \frac{1}{2}\, X^{(2,0,\dots,0)}, \tag{10}$$

$$X^{e_2} \circ T^{(2)}G = \frac{x^2}{(1-x^1)^2}\, X^{e_1} + \frac{1}{1-x^1}\, X^{e_2} + \frac{x^2}{(1-x^1)^3}\, X^{(2,0,\dots,0)} + \frac{1}{(1-x^1)^2}\, X^{(1,1,0,\dots,0)}, \tag{11}$$

$$X^{e_3} \circ T^{(2)}G = X^{e_3}, \tag{12}$$

$$X^{(1,1,0,\dots,0)} \circ T^{(2)}G = \frac{x^2}{1-x^1} X^{(2,0,\dots,0)} + X^{(1,1,0,\dots,0)}, \tag{13}$$

$$X^{(1,0,1,0,\dots,0)} \circ T^{(2)}G = (1-x^1)X^{(1,0,1,0,\dots,0)}, \tag{14}$$

$$X^{(0,1,1,0,\dots,0)} \circ T^{(2)}G = \frac{x^2}{1-x^1}\, X^{(1,0,1,0,\dots,0)} + \frac{1}{1-x^1}\, X^{(0,1,1,0,\dots,0)} \tag{15}$$

over U.

(iii) *If $r = 3$ then*

$$X^{e_1} \circ T^{(3)}G = (1 - x^1)X^{e_1} - \frac{1}{2}\, X^{(2,0,\dots,0)}, \tag{16}$$

$$X^{e_3} \circ T^{(3)}G = X^{e_3}, \tag{17}$$

$$X^{(1,1,0,\dots,0)} \circ T^{(3)}G = \frac{x^2}{1-x^1}\, X^{(2,0,\dots,0)} + X^{(1,1,0,\dots,0)} + \frac{1}{2}\frac{x^2}{(1-x^2)^2}\, X^{(3,0,\dots,0)} + \frac{1}{2}\frac{1}{1-x^1}\, X^{(2,1,0,\dots,0)}, \tag{18}$$

$$X^{(1,0,1,0,\dots,0)} \circ T^{(3)}G = (1-x^1)X^{(1,0,1,0,\dots,0)} - \frac{1}{2}\, X^{(2,0,1,0,\dots,0)}, \tag{19}$$

$$X^{(1,1,1,0,\dots,0)} \circ T^{(3)}G = X^{(1,1,1,0,\dots,0)} + \frac{x^2}{1-x^1}\, X^{(2,1,0,\dots,0)} \tag{20}$$

over U.

Proof. This is an extension of Lemma 4.1 in [4].

For example, we prove (7). Let $x_0 = (x_0^1, \dots, x_0^n) \in U$ and $\Theta \in T^{(1)}_{x_0}\mathbb{R}^n$. Comparing the respective jet coordinates (by computing derivatives) we obtain $j^1_{x_0}(G^1 - G^1(x_0)) = (1 - x_0^1) j^1_{x_0}(x^1 - x_0^1)$. Therefore

$$\begin{aligned} X^{e_1} \circ T^{(1)}G(\Theta) &= \langle T^{(1)}G(\Theta), j^1_{G(x_0)}(x^1 - G^1(x_0))\rangle \\ &= \langle \Theta, j^1_{x_0}(G^1 - G^1(x_0))\rangle \\ &= (1 - x_0^1)\langle \Theta, j^1_{x_0}(x^1 - x_0^1)\rangle \\ &= (1 - x_0^1)X^{e_1}(\Theta) = ((1 - x^1)X^{e_1})(\Theta). \end{aligned}$$

The proofs of the other formulas are similar. ∎

6. Proof of Theorem 1. Consider a natural bilinear operator $B : T^* \times_{\mathcal{M}f_n} T^{(0,0)} \rightsquigarrow T^{(0,0)}T^{(r)}$, $n \geq 3$. We know that it is sufficient to study $B(x^2dx^1, x^3)$. The form of $B(x^2dx^1, x^3)$ is given in Lemma 3.

(i) The local diffeomorphism G preserves (x^2dx^1, x^3) on U. Thus it preserves $B(x^2dx^1, x^3)$ over U. Hence it preserves the right hand side of (3). The transformation rules (7)–(9) yield

$$\begin{aligned} &\mu_1 X^{e_1}x^2x^3 + \mu_2 X^{e_1}X^{e_2}x^3 + \mu_3 X^{e_1}X^{e_3}x^2 + \mu_4 X^{e_1}X^{e_2}X^{e_3} \\ &\quad = \mu_1 X^{e_1}x^2x^3 + \mu_2(1 - x^1)\, X^{e_1}\left(\frac{x^2}{(1 - x^1)^2}\, X^{e_1} + \frac{1}{1 - x^1}\, X^{e_2}\right)x^3 \\ &\qquad + \mu_3(1 - x^1)X^{e_1}\, X^{e_3}\frac{x^2}{1 - x^1} \\ &\qquad + \mu_4(1 - x^1)X^{e_1}\left(\frac{x^2}{(1 - x^1)^2}\, X^{e_1} + \frac{1}{1 - x^1}\, X^{e_2}\right)X^{e_3} \end{aligned}$$

over U. Thus

$$\mu_2(X^{e_1})^2\,\frac{x^2}{1 - x^1}\, x^3 + \mu_4(X^{e_1})^2\,\frac{x^2}{1 - x^1}\, X^{e_3} = 0.$$

So, $\mu_2 = \mu_4 = 0$ in (3). Consequently, the vector space of all bilinear natural operators $T^* \times_{\mathcal{M}f_n} T^{(0,0)} \rightsquigarrow T^{(0,0)}T^{(1)}$ is at most 2-dimensional. On the other hand, the operators $B^{(1)}$ and $B^{[1]}$ are linearly independent. This proves (i).

(ii) We have the natural inclusion $i : T^{(1)} \to T^{(2)}$ which is dual to the jet projection $J^2(\cdot, \mathbb{R})_0 \to J^1(\cdot, \mathbb{R})_0$. So, by pull-back we have the bilinear natural operator $i^*B : T^* \times_{\mathcal{M}f_n} T^{(0,0)} \rightsquigarrow T^{(0,0)}T^{(1)}$. Hence $\mu_2 = \mu_4 = 0$ in (4) by (i).

In this case the local diffeomorphism G preserves the right hand side of (4) with $\mu_2 = \mu_4 = 0$, and the transformation rules (10)–(15) yield

$$
\begin{aligned}
&\mu_1 X^{e_1} x^2 x^3 + \mu_3 X^{e_1} X^{e_3} x^2 + \mu_5 X^{(1,0,1,0,\dots,0)} x^2 + \mu_6 X^{(1,1,0,\dots,0)} x^3 \\
&\quad + \mu_7 X^{(1,0,1,0,\dots,0)} X^{e_2} + \mu_8 X^{(1,1,0,\dots,0)} X^{e_3} + \mu_9 X^{(0,1,1,0,\dots,0)} X^{e_1} \\
&\ = \mu_1 \Big((1-x^1) X^{e_1} - \frac{1}{2} X^{(2,0,\dots,0)} \Big) \frac{x^2}{1-x^1}\, x^3 \\
&\quad + \mu_3 \Big((1-x^1) X^{e_1} - \frac{1}{2}\, X^{(2,0,\dots,0)} \Big) X^{e_3}\, \frac{x^2}{1-x^1} \\
&\quad + \mu_5 (1-x^1) X^{(1,0,1,0,\dots,0)} \frac{x^2}{1-x^1} \\
&\quad + \mu_6 \Big(\frac{x^2}{1-x^1}\, X^{(2,0,\dots,0)} + X^{(1,1,0,\dots,0)} \Big) x^3 \\
&\quad + \mu_7 (1-x^1) X^{(1,0,1,0,\dots,0)} \Big(\frac{x^2}{(1-x^1)^2}\, X^{e_1} + \frac{1}{1-x^1}\, X^{e_2} \\
&\quad + \frac{x^2}{(1-x^1)^3}\, X^{(2,0,\dots,0)} + \frac{1}{(1-x^1)^2}\, X^{(1,1,0,\dots,0)} \Big) \\
&\quad + \mu_8 \Big(\frac{x^2}{1-x^1}\, X^{(2,0,\dots,0)} + X^{(1,1,0,\dots,0)} \Big) X^{e_3} \\
&\quad + \mu_9 \Big(\frac{x^2}{1-x^1}\, X^{(1,0,1,0,\dots,0)} + \frac{1}{1-x^1}\, X^{(0,1,1,0,\dots,0)} \Big) \\
&\quad \times \Big((1-x^1) X^{e_1} - \frac{1}{2}\, X^{(2,0,\dots,0)} \Big).
\end{aligned}
$$

Analysing the coefficients of $X^{(0,1,1,0,\dots,0)} X^{(2,0,\dots,0)}$ in the above equality we get $0 = \mu_9(-\frac{1}{2}) \frac{1}{1-x^1}$. So, $\mu_9 = 0$. Then considering the coefficients of $X^{(1,0,1,0,\dots,0)} X^{(2,0,\dots,0)}$ we obtain $0 = \mu_7 \frac{x^2}{(1-x^1)^2}$. So, $\mu_7 = 0$. Comparing the coefficients of $X^{(2,0,\dots,0)} X^{e_3}$ we deduce $0 = \mu_8 \frac{x^2}{1-x^1} + \mu_3\big(-\frac{1}{2}\big) \frac{x^2}{1-x^1}$. Consequently, $\mu_3 = 2\mu_8$. Finally looking at $X^{(2,0,\dots,0)}$ we have $0 = \mu_6 \frac{x^2 x^3}{1-x^1} + \mu_1\big(-\frac{1}{2}\big) \frac{x^2 x^3}{1-x^1}$, i.e. $\mu_1 = 2\mu_6$.

Therefore the vector space of all natural bilinear operators $T^* \times_{\mathcal{M}f_n} T^{(0,0)} \rightsquigarrow T^{(0,0)} T^{(2)}$ is at most 3-dimensional. On the other hand, the operators $B^{(2)}$, $B^{[2]}$ and $B^{\langle 2 \rangle}$ are linearly independent, which proves (ii).

(iii) First, let $r = 3$. We have the natural inclusion $i : T^{(2)} \to T^{(3)}$ which is dual to the jet projection $J^3(\cdot, \mathbb{R})_0 \to J^2(\cdot, \mathbb{R})_0$. So, by pull-back we have the natural bilinear operator $i^* B : T^* \times_{\mathcal{M}f_n} T^{(0,0)} \rightsquigarrow T^{(0,0)} T^{(2)}$. Hence in (5) we have $\mu_2 = \mu_4 = \mu_9 = \mu_7 = 0$, $\mu_3 = 2\mu_8$ and $\mu_1 = 2\mu_6$ by (ii).

Now the local diffeomorphism G preserves the right hand side of (5) with the above conditions, and the transformation rules (16)–(20) yield

$$\begin{aligned}
&\mu_1 X^{e_1}x^2x^3 + \mu_3 X^{e_1}X^{e_3}x^2 + +\mu_5 X^{(1,0,1,0,\dots,0)}x^2 \\
&\qquad + \mu_6 X^{(1,1,0,\dots,0)}x^3 + \mu_8 X^{(1,1,0,\dots,0)}X^{e_3} + \mu_{10}X^{(1,1,1,0,\dots,0)} \\
&\quad = \mu_1\left((1-x^1)X^{e_1} - \frac{1}{2}\,X^{(2,0,\dots,0)}\right)\frac{x^2}{1-x^1}\,x^3 \\
&\qquad + \mu_3\left((1-x^1)X^{e_1} - \frac{1}{2}\,X^{(2,0,\dots,0)}\right)X^{e_3}\,\frac{x^2}{1-x^1} \\
&\qquad + \mu_5\left((1-x^1)X^{(1,0,1,0,\dots,0)} - \frac{1}{2}\,X^{(2,0,1,0,\dots,0)}\right)\frac{x^2}{1-x^1} \\
&\qquad + \mu_6\left(\frac{x^2}{1-x^1}\,X^{(2,0,\dots,0)} + X^{(1,1,0,\dots,0)}\right. \\
&\qquad \left. + \frac{1}{2}\,\frac{x^2}{(1-x^1)^2}\,X^{(3,0,\dots,0)} + \frac{1}{2}\,\frac{1}{1-x^1}\,X^{(2,1,0,\dots,0)}\right)x^3 \\
&\qquad + \mu_8\left(\frac{x^2}{1-x^1}\,X^{(2,0,\dots,0)} + X^{(1,1,0,\dots,0)}\right. \\
&\qquad \left. + \frac{1}{2}\,\frac{x^2}{(1-x^1)^2}\,X^{(3,0,\dots,0)} + \frac{1}{2}\,\frac{1}{1-x^1}\,X^{(2,1,0,\dots,0)}\right)X^{e_3} \\
&\qquad + \mu_{10}\left(X^{(1,1,1,0,\dots,0)} + \frac{x^2}{1-x^1}\,X^{(2,1,0,\dots,0)}\right).
\end{aligned}$$

Analysing the coefficients of $X^{(3,0,\dots,0)}$ in the above equality we get $0 = \mu_6 \frac{1}{2}\frac{x^2}{(1-x^1)^2}x^3$. So, $\mu_6 = 0$. Then considering the coefficients of $X^{(3,0,\dots,0)}X^{e_3}$ we obtain $0 = \mu_8 \frac{1}{2}\frac{x^2}{(1-x^1)^2}$. So, $\mu_8 = 0$. Comparing the coefficients of $X^{(2,1,0,\dots,0)}$ we deduce $0 = \mu_{10}\frac{x^2}{1-x^1}$. Consequently, $\mu_{10} = 0$. Finally, looking at $X^{(2,0,1,0\dots,0)}$ we have $0 = \mu_5(-\frac{1}{2})\frac{x^2}{1-x^1}$, i.e. $\mu_5 = 0$.

Therefore $\mu_1 = \ldots = \mu_{10} = 0$, i.e. $B = 0$.

Let now $r \geq 4$. We have the natural inclusion $i : T^{(3)} \to T^{(r)}$ which is dual to the jet projection $J^r(\cdot,\mathbb{R})_0 \to J^3(\cdot,\mathbb{R})_0$. So, by pull-back we have the bilinear natural operator $i^*B : T^* \times_{\mathcal{M}f_n} T^{(0,0)} \rightsquigarrow T^{(0,0)}T^{(3)}$. Hence in (6) we have $\mu_1 = \ldots = \mu_{10} = 0$ by the case $r = 3$. Thus $B = 0$. ■

7. Proof of Theorem 2. Let $C : T^*_{|\mathcal{M}f_n} \rightsquigarrow TT^{(r)}$ be a natural linear operator, $n \geq 3$. It induces a natural bilinear operator $B^{[C]} : T^* \times_{\mathcal{M}f_n}$

$T^{(0,0)} \rightsquigarrow T^{(0,0)}T^{(r)}$ by

$$B^{[C]}(\omega, f)_\eta = \langle T\pi(C(\omega)_\eta), d_x f\rangle, \tag{21}$$

$\omega \in \Omega(M)$, $f : M \to \mathbb{R}$, $\eta \in T^{(r)}_x M$, $x \in M$, $M \in \mathrm{obj}(\mathcal{M}f_n)$. Here $\pi : T^{(r)}M \to M$ is the bundle projection.

We first prove (iii). By Theorem 1, $B^{[C]} = 0$. Thus C is of vertical type. Then by Section 2, C induces the bilinear operator $B^{(C)}$ which is 0 in view of Theorem 1. Hence $C = 0$.

We now show (ii). Since $B^{[C]}$ is of order 1 with respect to f and $B^{[C]}(\cdot, 1) = 0$, it follows that $B^{[C]} = aB^{\langle 2\rangle}$ for some $a \in \mathbb{R}$ by Theorem 1. Then $B^{[C]}(dx^1, x^1)_{(j^2_0((x^1)^2))^*} = a$, where $(j^2_0 x^\alpha)^* \in T^{(2)}_0\mathbb{R}^n$ is the basis dual to $j^2_0 x^\alpha$. Therefore

$$C(dx^1, x^1)_{(j^2_0((x^1)^2))^*} = a\left(\frac{\partial}{\partial x^1}\right)^C_{(j^2_0((x^1)^2))^*} + \mathcal{V} + \sum_{i=2}^n a_i\left(\frac{\partial}{\partial x^1}\right)^C_{(j^2_0((x^1)^2))^*} \tag{22}$$

for some vertical vector $\mathcal{V} \in V_{(j^2_0((x^1)^2))^*}T^{(2)}\mathbb{R}^n$ and $a_i \in \mathbb{R}$, where $(\ )^C$ is the complete lifting of a vector field to $T^{(2)}$.

Consider the diffeomorphism $\varphi = (x^1, x^2 + (x^1)^3, x^3, \ldots, x^n) : \mathbb{R}^n \to \mathbb{R}^n$. It preserves x^1, dx^1 and $(j^2_0((x^1)^2))^*$ because $j^2_0\varphi = \mathrm{id}$. Thus it preserves the left hand side of (22). On the other hand it preserves $\partial/\partial x^i$ for $i = 2, \ldots, n$, $(j^2_0((x^1)^2))^*$ and $\mathcal{V}$ because $j^2_0\varphi = \mathrm{id}$, and sends $\partial/\partial x^1$ to $\partial/\partial x^1 + 3(x^1)^2\partial/\partial x^2$. Therefore

$$a\left(\frac{\partial}{\partial x^1}\right)^C_{(j^2_0((x^1)^2))^*} = a\left(\frac{\partial}{\partial x^1}\right)^C_{(j^2_0((x^1)^2))^*} + 3a\left(\left((x^1)^2\frac{\partial}{\partial x^2}\right)^C\right)_{(j^2_0((x^1)^2))^*}.$$

So, $a = 0$ because $(((x^1)^2\partial/\partial x^2)^C)_{(j^2_0((x^1)^2))^*} \neq 0$. Hence C is of vertical type.

Now, by Lemma 1, $C = C^{(B)}$ for some bilinear natural operator $B : T^* \times_{\mathcal{M}f_n} T^{(0,0)} \rightsquigarrow T^{(0,0)}T^{(2)}$ with $B(\cdot, 1) = 0$. Applying Theorem 1 ends the proof of (ii).

Finally, we prove (i). Since $B^{[C]}(\cdot, 1) = 0$, we have $B^{[C]} = aB^{[1]}$ for some $a \in \mathbb{R}$ by Theorem 1. Thus $B^{[C]}(dx^1, x^1)_{(j^2_0((x^1)^2))^*} = a$. Hence

$$C(dx^1, x^1)_{(j^1_0(x^1))^*} = a\left(\frac{\partial}{\partial x^1}\right)^C_{(j^1_0(x^1))^*} + \mathcal{V} + \sum_{i=2}^n a_i\left(\frac{\partial}{\partial x^1}\right)^C_{(j^1_0(x^1))^*} \tag{23}$$

for some vertical vector $\mathcal{V} \in V_{(j^1_0(x^1))^*}T^{(1)}\mathbb{R}^n$ and $a_i \in \mathbb{R}$.

Consider the diffeomorphism $\varphi = (x^1, x^2 + (x^1)^2, x^3, \ldots, x^n) : \mathbb{R}^n \to \mathbb{R}^n$. It preserves x^1, dx^1 and $(j^1_0(x^1))^*$ because $j^1_0\varphi = \mathrm{id}$. Thus it preserves the left hand side of (23). On the other hand it preserves $\partial/\partial x^i$ for $i = 2, \ldots, n$, $(j_0^1(x^1))^*$ and $\mathcal{V}$ because $j_0^1\varphi = \mathrm{id}$ and sends $\partial/\partial x^1$ into $\partial/\partial x^1 + 2x^1\partial/\partial x^2$. Therefore

$$a\left(\frac{\partial}{\partial x^1}\right)^C_{(j_0^1(x^1))^*} = a\left(\frac{\partial}{\partial x^1}\right)^C_{(j_0^2(x^1))^*} + 2a\left(\left(x^1\frac{\partial}{\partial x^2}\right)^C\right)_{(j_0^1(x^1))^*}.$$

So, $a = 0$ because $((x^1\partial/\partial x^2)^C)_{(j_0^1(x^1))^*} \neq 0$. Hence C is of vertical type.

Now, by Lemma 1, $C = C^{(B)}$ for some bilinear natural operator $B : T^* \times_{\mathcal{M}f_n} T^{(0,0)} \rightsquigarrow T^{(0,0)}T^{(1)}$ with $B(\cdot, 1) = 0$, and it remains to apply Theorem 1. ■

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