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RANKS FOR BAIRE MULTIFUNCTIONS

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Abstract. Various ordinal ranks for Baire-1 real-valued functions, which have been used in the literature, are adapted to provide ranks for Baire-1 multifunctions. A new rank is also introduced which, roughly speaking, gives an estimate of how far a Baire-1 multifunction is from being upper semicontinuous.

1. Introduction. The purpose of this paper is to show that certain ordinal ranks which have been defined in the study of Baire-1, real-valued functions (see for instance [8] and the references therein) can be successfully adapted to give ranks for Baire-1 multifunctions $F : X \to P_k(Y)$ (where X is a Polish space and Y a separable metrizable space).

In Section 3 we present the separation rank by modifying Bourgain's definition of the α rank. In Section 4 an equivalent characterization of locally compact, Baire-1 multifunctions is given in terms of distance functions. This description permits us to associate (indirectly), to each such multifunction, any of the existing ranks for real-valued Baire-1 functions, via a supremum formula. Note that without additional assumptions, the so called convergence rank γ cannot be directly associated to a Baire-1 multifunction, since the description of a real-valued Baire-1 function as the limit of a sequence of continuous functions is not in general valid for multifunctions (see however [5]).

In Section 5 we define the rank δ (when Y is a compact metrizable space) which is not symmetric with respect to both kinds of semicontinuity. The rank δ has the property that the bigger δ, the less F looks like an upper semicontinuous multifunction (in particular $\delta(F) = 1$ if and only if F is upper semicontinuous). In the last section we discuss some possible applications of the tools developed in this paper in nonsmooth analysis.

Finally, we should point out that the oscillation rank, although not presented in this paper, may well be defined for Baire-1 multifunctions since the oscillation function is defined for any function which takes values in a metrizable space.

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2. Preliminaries. Let X and Y be sets. For any multifunction (set-valued map) $F:X\to 2^Y\setminus\{\emptyset\}$ and any set $A\subseteq Y$ one defines the *weak inverse image* and *strong inverse image* of A under F by $F^-(A)=\{x\in X: F(x)\cap A\neq\emptyset\}$ and $F^+(A)=\{x\in X: F(x)\subseteq A\}$ respectively. It is easy to see that $X\setminus F^-(A)=F^+(Y\setminus A)$ for any $A\subseteq Y$. In addition, if $\{A_i\}_{i\in I}\subseteq Y$, we have

(i) $F^-\Big(\bigcup_{i\in I}A_i\Big)=\bigcup_{i\in I}F^-(A_i),\quad F^-\Big(\bigcap_{i\in I}A_i\Big)\subseteq\bigcap_{i\in I}F^-(A_i)$;

(ii) $F^+\Big(\bigcup_{i\in I}A_i\Big)\supseteq\bigcup_{i\in I}F^+(A_i),\quad F^+\Big(\bigcap_{i\in I}A_i\Big)=\bigcap_{i\in I}F^+(A_i)$.

The *graph* of a multifunction, denoted by $\mathrm{Gr}\,F$, is defined to be the set $\mathrm{Gr}\,F=\{(x,y)\subseteq X\times Y: y\in F(x)\}$. For a not necessarily everywhere defined multifunction $F:X\to 2^Y$, the *domain* of F, denoted by $\mathrm{dom}(F)$, is defined to be the set $\mathrm{dom}(F)=\{x\in X: F(x)\neq\emptyset\}$.

Let X and Y be Hausdorff topological spaces. A multifunction $F:X\to 2^Y\setminus\{\emptyset\}$ is said to be *lower semicontinuous* if $F^-(U)$ is an open subset of X for every $U\subseteq Y$ open; it is *upper semicontinuous* if $F^+(U)$ is an open subset of X for every $U\subseteq Y$ open; and it is *continuous* if it is both upper and lower semicontinuous. Of course we can have local versions of the above notions. So, for instance, F is said to be *lower semicontinuous at* $x\in X$ if for every $U\subseteq Y$ with $F(x)\cap U\neq\emptyset$, there exists $V\in N(x)$ such that $F(z)\cap U\neq\emptyset$ for every $z\in V$ (i.e. $V\subseteq F^-(U)$). As usual, $N(x)$ denotes the filter of neighborhoods of x. For more information about continuity concepts for multifunctions one can consult [6].

We widely use the definitions and notations from descriptive set theory. So for a metrizable space X, we denote by $\boldsymbol{\Sigma}^0_1(X)$ the open subsets of X, by $\boldsymbol{\Pi}^0_1(X)$ the closed subsets, by $\boldsymbol{\Sigma}^0_2(X)$ the F_σ, by $\boldsymbol{\Pi}^0_2(X)$ the G_δ, etc. For more information we refer to [7]. Recall that if X and Y are metrizable spaces, then a function $f:X\to Y$ is said to be a *Baire*-1 *function* if $f^{-1}(U)\in\boldsymbol{\Sigma}^0_2(X)$ for every $U\subseteq Y$ open. Now let X be a Polish space. To any real-valued Baire-1 function $f:X\to\mathbb{R}$, one associates three different ordinal ranks: (i) the separation rank α (introduced by Bourgain, see [3]), (ii) the oscillation rank β, and (iii) the convergence rank γ. For their definitions and properties, we refer to [8].

For any topological space X, we denote by $P_k(X)$ the collection of all non-empty, compact subsets of X. If X is separable metrizable, then $P_k(X)$ equipped with the Vietoris hyperspace topology, denoted by $(P_k(X),\tau_V)$, becomes a separable metrizable space (see [7, p. 25]). A compatible metric is the Hausdorff metric $d_{\mathrm H}:P_k(X)\times P_k(X)\to\mathbb{R}$, defined by

$$d_{\mathrm H}(A,B)=\inf\{\varepsilon>0: A\subseteq B_\varepsilon \text{ and } B\subseteq A_\varepsilon\}$$

where as usual, for any set $A\subseteq X$, A_ε denotes the ε-ball around A.

Now let X and Y be metrizable spaces.

DEFINITION 1. A multifunction $F : X \to 2^Y \setminus \{\emptyset\}$ is said to be a *Baire*-1 *multifunction* if $F^+(U) \in \mathbf{\Sigma}^0_2(X)$ and $F^-(U) \in \mathbf{\Sigma}^0_2(X)$ for every $U \subseteq Y$ open.

The class of Baire-1 multifunctions was introduced by Kuratowski. We have the following basic fact concerning Baire-1 multifunctions. For the proof we refer to [9].

PROPOSITION 2. *Let X and Y be metrizable spaces, with Y separable, and $F : X \to P_k(Y)$ a multifunction. Then F is a Baire-1 multifunction if and only if F is a Baire-1 function viewed as a single-valued function from X into $(P_k(Y), \tau_V)$.*

The following lemma will be useful in what follows. It is a straightforward consequence of the definition.

LEMMA 3. *A multifunction $F : X \to 2^Y \setminus \{\emptyset\}$ is a Baire-1 multifunction if and only if $F^-(C) \in \mathbf{\Pi}^0_2(X)$ and $F^+(C) \in \mathbf{\Pi}^0_2(X)$ for every $C \subseteq Y$ closed.*

It follows directly from the Kuratowski–Ryll-Nardzewski selection theorem (see [10]) that each Baire-1, compact-valued multifunction admits a Baire-1 selection. If Y is Polish, then this is also true for closed-valued multifunctions.

3. The separation rank. Throughout this section X will be a Polish space and Y a separable metrizable space. Fix a countable dense subset D of Y and put

$$\mathcal{B} = \{B(y, r) : y \in D,\ r \in \mathbb{Q}_+\}.$$

Clearly $\mathcal{B}$ is a base for Y. Now let

$$\breve{\mathcal{B}} = \{U^{\mathrm{c}} : U \in \mathcal{B}\} \quad \text{and} \quad \overline{\mathcal{B}} = \{\overline{U} : U \in \mathcal{B}\}.$$

Recall that for any given class $\mathcal{F}$ of subsets of Y, $\mathcal{F}_s$ denotes the collection of all finite unions of members of $\mathcal{F}$, and $\mathcal{F}_d$ the collection of all finite intersections of members of $\mathcal{F}$. One easily checks that $\breve{\mathcal{B}}_d = \{U^{\mathrm{c}} : U \in \mathcal{B}_s\}$ and $\overline{\mathcal{B}}_s = \{\overline{U} : U \in \mathcal{B}_s\}$. The following lemmas are obvious.

LEMMA 4. *$\breve{\mathcal{B}}_d$ and $\overline{\mathcal{B}}_s$ are countable families of closed sets.*

LEMMA 5. *If $U \in \mathcal{B}_s$, then there exist $\{U_n\}_{n\ge1} \subseteq \mathcal{B}_s$ and $\{C_n\}_{n\ge1} \subseteq \overline{\mathcal{B}}_s$, with $C_{n+1} \supseteq U_n \supseteq C_n$ for every $n \ge 1$, such that $U = \bigcup_n U_n = \bigcup_n C_n$.*

Let A and B be disjoint subsets of X and $P \subseteq X$ closed. One associates with them a derivation on closed sets by

$$P'_{A,B} = \overline{P \cap A} \cap \overline{P \cap B}$$

and then by transfinite induction $P^{(0)}_{A,B} = P$, $P^{(\xi+1)}_{A,B} = (P^{(\xi)}_{A,B})'_{A,B}$ and $P^{(\lambda)}_{A,B} = \bigcap_{\xi<\lambda} P^{(\xi)}_{A,B}$ for limit λ. Then we set

$$\alpha(P,A,B) = \begin{cases} \text{least } \xi \text{ with } P^{(\xi)}_{A,B} = \emptyset & \text{if such a } \xi \text{ exists,} \\ \omega_1 & \text{otherwise,} \end{cases}$$

and let $\alpha(A,B) = \alpha(X,A,B)$. It is well known that $\alpha(A,B) < \omega_1$ if and only if one can separate A from B by a set which is a transfinite difference of closed sets.

Now let $F : X \to P_k(Y)$ be a multifunction. Let $K \in \breve{\mathcal{B}}_d$ and $C \in \overline{\mathcal{B}}_s$ with $K \cap C = \emptyset$. Put

$$\alpha(F,K,C) = \sup\{\alpha(F^+(K), F^-(C)), \alpha(F^+(C), F^-(K))\}.$$

Note that $\alpha(F,K,C)$ is symmetric with respect to both inverse images. Finally define the *separation rank* by

$$\alpha(F) = \sup\{\alpha(F,K,C) : K \in \breve{\mathcal{B}}_d,\ C \in \overline{\mathcal{B}}_s \text{ with } K \cap C = \emptyset\}.$$

LEMMA 6. *Let $F : X \to P_k(Y)$ be a multifunction. If $\{C_n\}_{n\geq 1}$ is a sequence of closed subsets of Y with $C_n \downarrow C$, then $F^-(\bigcap_n C_n) = \bigcap_n F^-(C_n)$. Also, if $\{U_n\}_{n\geq 1}$ is a sequence of open subsets of Y with $U_n \uparrow U$, then $F^+(\bigcup_n U_n) = \bigcup_n F^+(U_n)$.*

Proof. Clearly $F^-(\bigcap_n C_n) \subseteq \bigcap_n F^-(C_n)$. So let $x \in \bigcap_n F^-(C_n)$. Then $L_n := F(x) \cap C_n \neq \emptyset$ for every $n \geq 1$. The sequence $\{L_n\}_{n\geq 1}$ is a decreasing sequence of closed subsets of $F(x)$, so it has the finite intersection property. Since $F(x)$ is compact, we conclude that it has non-empty intersection. But observe that $F(x) \cap \bigcap_n C_n = \bigcap_n L_n \neq \emptyset$, which implies that $x \in F^-(\bigcap_n C_n)$ and we are done. By taking complements we get the other half of the statement. ∎

THEOREM 7. *Let $F : X \to P_k(Y)$ be a multifunction. Then F is a Baire-1 multifunction if and only if $\alpha(F) < \omega_1$.*

Proof. [$\Rightarrow$] If F is a Baire-1 multifunction, then from Lemma 3, for any two sets $K \in \breve{\mathcal{B}}_d$ and $C \in \overline{\mathcal{B}}_s$ with $K \cap C = \emptyset$, $F^+(K)$ and $F^-(C)$ are disjoint G_δ sets, since K and C are closed. So they can be separated by a $\mathbf{\Delta}^0_2(X)$ set. The same holds for $F^+(C)$ and $F^-(K)$. So we conclude that $\alpha(F,K,L) < \omega_1$. Finally $\alpha(F)$ is also less than ω_1, since both $\breve{\mathcal{B}}_d$ and $\overline{\mathcal{B}}_s$ are countable.

[$\Leftarrow$] Assume that $\alpha(F) < \omega_1$. Then, from the definition of α, given $K \in \breve{\mathcal{B}}_d$ and $C \in \overline{\mathcal{B}}_s$ with $K \cap C = \emptyset$, we can find $A, B \in \mathbf{\Delta}^0_2(X)$ such that $F^-(C) \subseteq A$, $F^+(K) \subseteq A^{\mathrm{c}}$ and $F^+(C) \subseteq B$, $F^-(K) \subseteq B^{\mathrm{c}}$. We need to prove that $F^-(U) \in \mathbf{\Sigma}^0_2(X)$ and $F^+(U) \in \mathbf{\Sigma}^0_2(X)$ for any $U \subseteq Y$ open. This will be done in a number of claims.

Claim 1. *If $U \in \mathcal{B}_s$, then $F^-(U) \in \mathbf{\Sigma}^0_2(X)$.*

Proof. Pick $\{U_n\}_{n\geq 1} \subseteq \mathcal{B}_s$ and $\{C_n\}_{n\geq 1} \subseteq \overline{\mathcal{B}}_s$, with $C_{n+1} \supseteq U_n \supseteq C_n$, such that $U = \bigcup_n U_n = \bigcup_n C_n$ (Lemma 5). For every $n \geq 1$ put $K_n = U_n^{\mathrm{c}}$. Note that $K_n \in \breve{\mathcal{B}}_d$, $C_n \in \overline{\mathcal{B}}_s$ and $K_n \cap C_n = \emptyset$. So there exists $A_n \in \mathbf{\Delta}^0_2(X)$ such that $F^-(C_n) \subseteq A_n$ and $F^+(K_n) \subseteq A_n^{\mathrm{c}}$. Finally let $C = U^{\mathrm{c}}$ and observe that

$$F^-(U) = F^-\Big(\bigcup_{n\geq 1} C_n\Big) = \bigcup_{n\geq 1} F^-(C_n) \subseteq \bigcup_{n\geq 1} A_n,$$

$$F^+(C) = F^+\Big(\bigcap_{n\geq 1} K_n\Big) = \bigcap_{n\geq 1} F^+(K_n) \subseteq \bigcap_{n\geq 1} A_n^{\mathrm{c}}.$$

Since $F^-(U) = F^+(C)^{\mathrm{c}}$, we conclude that $F^-(U) = \bigcup_n A_n \in \mathbf{\Sigma}^0_2(X)$, as desired.

Claim 2. *If $U \subseteq Y$ is open, then $F^-(U) \in \mathbf{\Sigma}^0_2(X)$.*

Proof. Since the weak inverse image behaves well with respect to unions, this follows easily from Claim 1 and the fact that $\mathcal{B}_s$ is a base for Y.

Claim 3. *If $U \in \mathcal{B}_s$, then $F^+(U) \in \mathbf{\Sigma}^0_2(X)$.*

Proof. Let U_n, C_n and K_n be as in the proof of Claim 1. Again, since $K_n \in \breve{\mathcal{B}}_d$ and $C_n \in \overline{\mathcal{B}}_s$ with $K_n \cap C_n = \emptyset$, pick $B_n \in \mathbf{\Delta}^0_2(X)$ such that $F^+(C_n) \subseteq B_n$ and $F^-(K_n) \subseteq B_n^{\mathrm{c}}$ for every $n \geq 1$. Note that

$$F^+(U) = F^+\Big(\bigcup_{n\geq 1} C_n\Big) = F^+\Big(\bigcup_{n\geq 1} U_n\Big).$$

As the sets U_n are open and $U_n \uparrow U$, from Lemma 6 we get $F^+(U) = \bigcup_n F^+(U_n)$. But observe that $F^+(C_n) \subseteq F^+(U_n) \subseteq F^+(C_{n+1})$ for every $n \geq 1$. So we finally get

$$F^+(U) = \bigcup_{n\geq 1} F^+(C_n) \subseteq \bigcup_{n\geq 1} B_n.$$

On the other hand, letting $C = U^{\mathrm{c}}$, we have $K_n \downarrow C$, with K_n closed for every $n \geq 1$. Invoking Lemma 6, we get

$$F^-(C) = F^-\Big(\bigcap_{n\geq 1} K_n\Big) = \bigcap_{n\geq 1} F^-(K_n) \subseteq \bigcap_{n\geq 1} B_n^{\mathrm{c}}.$$

So we conclude that $F^+(U) = \bigcup_n B_n \in \mathbf{\Sigma}^0_2(X)$ and the claim is proved.

Claim 4. *If $U \subseteq Y$ is open, then $F^+(U) \in \mathbf{\Sigma}^0_2(X)$.*

Proof. Since $\mathcal{B}$ is a base for Y, pick $\{U_m\}_{m\geq 1} \subseteq \mathcal{B}$ such that $U = \bigcup_m U_m$. Let $\{V_i\}_{i\geq 1}$ be the (countable) collection of all finite unions of U_m's. Note that $U = \bigcup_m U_m = \bigcup_i V_i$. We claim that

$$(1) \qquad F^+(U) = F^+\Big(\bigcup_{m\geq 1} U_m\Big) = F^+\Big(\bigcup_{i\geq 1} V_i\Big) = \bigcup_{i\geq 1} F^+(V_i).$$

It is clear that $F^+(U) \supseteq \bigcup_i F^+(V_i)$. So let $x \in F^+(U)$, which implies that $F(x) \subseteq \bigcup_m U_m$. Since $F(x)$ is compact, there exists a finite subcover, say $\{U_m\}_{m=1}^k$. From the definition of V_i's, there exists $i \geq 1$ such that $V_i = \bigcup_{m=1}^k U_m$, which implies that $F(x) \subseteq V_i$. So $F^+(U) \subseteq \bigcup_i F^+(V_i)$ and (1) is proved. Since each V_i is a finite union of members of $\mathcal{B}$, we conclude that $V_i \in \mathcal{B}_s$ for every $i \geq 1$. So, from (1) and Claim 3, we see that $F^+(U) \in \mathbf{\Sigma}^0_2(X)$ and this completes the proof. ■

The next result is in the spirit of Proposition 2.2 in [8].

Proposition 8. *Let $F : X \to P_k(Y)$ be a multifunction.*

(i) *F is continuous if and only if $\alpha(F) = 1$.*
(ii) *If F is upper or lower semicontinuous, then $\alpha(F) \leq 2$.*

Proof. (i) The "only if" part is trivial. For the "if" part, observe that if $\alpha(F) = 1$, then for any $K \in \breve{\mathcal{B}}_d$ and $C \in \overline{\mathcal{B}}_s$ with $K \cap C = \emptyset$, we have

$$\overline{F^+(K)} \cap \overline{F^-(C)} = \emptyset \quad \text{and} \quad \overline{F^+(C)} \cap \overline{F^-(K)} = \emptyset.$$

So the sets A_n and B_n in the proof of Theorem 7 can be chosen to be open. Following the proof of Theorem 7 we conclude that for any $U \subseteq Y$ open, $F^+(U)$ and $F^-(U)$ are open subsets of Y respectively, which implies that F is continuous.

(ii) If F is, say, lower semicontinuous, then $F^+(K)$ is closed for any $K \subseteq Y$ closed. So given $K \in \breve{\mathcal{B}}_d$ and $C \in \overline{\mathcal{B}}_s$ disjoint, letting $A = F^+(K)$ and $B = F^-(C)$, we have

$$P'_{A,B} = \overline{P \cap A} \cap \overline{P \cap B} \subseteq A \cap \overline{P \cap B} \subseteq A$$

and so $P''_{A,B} = (P'_{A,B})'_{A,B} \subseteq \overline{A \cap B} = \emptyset$, for any $P \subseteq X$ closed. The same is true if $A = F^+(C)$ and $B = F^-(K)$. So we conclude that $\alpha(F) \leq 2$. Similarly we deal with the case of F upper semicontinuous. ■

An immediate corollary is the following well-known result.

Corollary 9. *If $F : X \to P_k(Y)$ is an upper or lower semicontinuous multifunction, then F is a Baire-1 multifunction.*

Remark 1. For every set $A \subseteq X$, define the *indicator multifunction* $F_A : X \to P_k(\mathbb{R})$ by

$$F_A(x) = \begin{cases} [0,1] & \text{if } x \in A, \\ \{0\} & \text{otherwise.} \end{cases}$$

Note that for $C \subseteq X$ closed, F_C is upper semicontinuous, while for $U \subseteq X$ open, F_U is lower semicontinuous. Also observe that for every $A \subseteq X$, F_A is a Baire-1 multifunction if and only if $A \in \mathbf{\Delta}^0_2(X)$. Moreover it is easy to see that $\alpha(F_A) = \alpha(A, A^c)$.

4. Distance functions. In this section we will give an equivalent characterization of Baire-1 multifunctions, via distance functions. As before let X be a Polish space and Y a separable metrizable space. Fix a countable dense subset D of Y and a compatible metric d for Y, with respect to which the d-diameter of Y is less than one. Given $F : X \to 2^Y \setminus \{\emptyset\}$ and $y \in Y$, we define the *distance function* $d_y : X \to \mathbb{R}_+$ by

$$d_y(x) = d(y, F(x)) = \inf\{d(y, z) : z \in F(x)\}.$$

Observe that if F has compact values then the above infimum is attained. Recall the following definitions.

Definition 10. Let $F : X \to 2^Y$ be a not necessarily everywhere defined multifunction.

(i) F is said to be *locally compact* if for every $x \in X$, there exists a $U \in N(x)$ such that $\overline{F(U)}$ is compact.

(ii) F is said to be *closed at* $x \in X$ if given $\{(x_n, y_n)\}_{n\ge1} \subseteq \operatorname{Gr} F$ such that $x_n \to x$ and $y_n \to y$, we have $(x, y) \in \operatorname{Gr} F$.

The following proposition provides a very useful criterion of upper semicontinuity. For the proof we refer to [6].

Proposition 11. *Let $F : X \to P_k(Y)$ be a locally compact multifunction. Then F is upper semicontinuous at $x \in X$ if and only if F is closed at x.*

We will also need the following lemma.

Lemma 12. *Let C be a completely metrizable space, Y a Polish space and $F : C \to 2^Y \setminus \{\emptyset\}$ a multifunction such that $F^-(U) \in \mathbf{\Sigma}^0_2(C)$ for every $U \subseteq Y$ open. Then F is lower semicontinuous on a dense G_δ subset of C.*

Proof. Let $\{U_n\}_{n\ge1}$ be a countable base for Y. Put $C_n = F^-(U_n) \in \mathbf{\Sigma}^0_2(C)$ for every $n \ge 1$. Let $D = \bigcup_{n\ge1} C_n \setminus \operatorname{int}(C_n)$. It is easy to see that $D \in \mathbf{\Sigma}^0_2(C)$ is of first category. To see that F is lower semicontinuous on D^c, let $x \in D^c$ and $U \subseteq Y$ be open such that $F(x) \cap U \ne \emptyset$. Pick $U_n \subseteq U$ basic open with $F(x) \cap U_n \ne \emptyset$. Since $x \in C_n$ and $x \notin D$, we get $x \in \operatorname{int}(C_n)$. Then $F(z) \cap U \supseteq F(z) \cap U_n \ne \emptyset$ for every $z \in \operatorname{int}(C_n)$, which implies that F is lower semicontinuous at $x \in D^c$ and completes the proof. ∎

Using Proposition 11 and Lemma 12, we will give an equivalent characterization of locally compact, Baire-1 multifunctions.

THEOREM 13. *Let $F : X \to P_k(Y)$ be a locally compact multifunction. Then the following are equivalent.*

(i) *F is a Baire-1 multifunction.*
(ii) *For every $y \in Y$, d_y is a Baire-1 function.*
(iii) *For every $y \in D$, d_y is a Baire-1 function.*

Proof. (i)$\Rightarrow$(ii). Assume that F is a Baire-1 multifunction. Let $y \in Y$ and $r > 0$. Then observe that

$$d_y^{-1}((-\infty, r)) = \{x \in X : F(x) \cap B(y,r) \neq \emptyset\} = F^-(B(y,r)) \in \mathbf{\Sigma}_2^0(X),$$
$$d_y^{-1}((-\infty, r]) = \{x \in X : F(x) \cap \overline{B(y,r)} \neq \emptyset\} = F^-(\overline{B(y,r)}) \in \mathbf{\Pi}_2^0(X),$$

where the last equality follows from the fact that F is compact-valued. So, given $b > a$, we have

$$\begin{aligned} d_y^{-1}((a,b)) &= d_y^{-1}((-\infty, b)) \cap d_y^{-1}((a,\infty)) \\ &= d_y^{-1}((-\infty, b)) \cap d_y^{-1}((-\infty, a])^{\mathrm{c}} \in \mathbf{\Sigma}_2^0(X), \end{aligned}$$

which implies that d_y is a Baire-1 function.

(ii)$\Rightarrow$(iii). Obvious.

(iii)$\Rightarrow$(i). In light of Proposition 2, it suffices to show that F is a Baire-1 function from X into $(P_k(Y), \tau_V)$. From the well-known characterization of Baire-1 functions (see for instance [7, p. 193]), it is enough to show that $F|_C$ has a point of continuity for every $C \subseteq X$ non-empty closed. So let $C \subseteq X$ be one. Note that $F|_C^-(U) = \{x \in C : F(x) \cap U \neq \emptyset\} = F^-(U) \cap C$. As before, for any $y \in D$ and $r > 0$, we have

$$F^-(B(y,r)) = d_y^{-1}((-\infty, r)) \in \mathbf{\Sigma}_2^0(X).$$

Since the family $\{B(y,r)\}_{y \in D, r \in \mathbb{Q}_+}$ is a base for Y we easily conclude that $F^-(U) \in \mathbf{\Sigma}_2^0(X)$ for every $U \subseteq Y$ open. So $F|_C^-(U) \in \mathbf{\Sigma}_2^0(C)$ for every $U \subseteq Y$ open (note that $C \in \mathbf{\Sigma}_2^0(X)$ and so $\mathbf{\Sigma}_2^0(C) = \mathbf{\Sigma}_2^0(X)|C$). From Lemma 12, we see that $F|_C$ is lower semicontinuous on a dense G_δ subset of C, say B. On the other hand, for every $y \in D$, the function $d_y|_C$ is Baire-1, so it is continuous on a dense G_δ subset of C, say B_y. Put $A = B \cap \bigcap_{y \in D} B_y$, which is dense G_δ.

We will show that $F|_C$ is continuous on A and this will finish the proof. Clearly it is enough to show that $F|_C$ is upper semicontinuous on A. Since $F|_C$ is locally compact, from Proposition 11, it suffices to show that $F|_C$ is closed on A. So let $\{(x_n, z_n)\}_{n \geq 1} \subseteq C \times Y$, with $z_n \in F(x_n)$, be such that $x_n \to x \in A$ and $z_n \to z \in Y$. We need to prove that $z \in F(x)$. Pick $\{y_n\}_{n \geq 1} \subseteq D$ such that $d(y_n, z_n) \leq 1/n$. Observe that $y_n \to z$. We have

$$d(z, F(x)) \leq d(z, y_n) + d(y_n, F(x)) = d(z, y_n) + d_{y_n}(x).$$

From the definition of A, d_{y_n} is continuous at $x \in A$, for every $n \geq 1$. Since

$x_n \to x$, for every $n \geq 1$ pick $m = m(n) > n$ such that

$$d_{y_n}(x) \leq d_{y_n}(x_m) + 1/n = d(y_n, F(x_m)) + 1/n \leq d(y_n, z_m) + 1/n.$$

Combining the above inequalities we get

$$\begin{aligned} d(z, F(x)) &\leq d(z, y_n) + d(y_n, z_m) + 1/n \\ &\leq d(z, y_n) + d(y_n, z_n) + d(z_n, z_m) + 1/n \\ &\leq d(z, y_n) + d(z_n, z_m) + 2/n. \end{aligned}$$

Letting $n \to \infty$, since $m > n$, we get $d(z, F(x)) = 0$ ($\{z_n\}_{n\geq 1}$ is Cauchy). Since F has non-empty, compact values we conclude that $z \in F(x)$. So F is closed at $x \in A$ and the proof is complete. ∎

Remark 2. We should point out that the implication (i)⇒(ii) is valid without the assumption of local compactness.

Besides its independent interest, Theorem 13 is useful because it permits us to use standard ranks for Baire-1 real-valued functions to define ranks for Baire-1 locally compact multifunctions. Specifically, for any multifunction $F : X \to P_k(Y)$, we define the ranks $\overline{\alpha}$, $\overline{\beta}$ and $\overline{\gamma}$ by

$$\begin{aligned} &\overline{\alpha}(F) = \sup\{\alpha(d_y) : y \in D\}, \quad \overline{\beta}(F) = \sup\{\beta(d_y) : y \in D\}, \\ &\overline{\gamma}(F) = \sup\{\gamma(d_y) : y \in D\}. \end{aligned}$$

The following proposition is a straightforward consequence of Theorem 13.

Proposition 14. *Let $F : X \to P_k(Y)$ be a locally compact multifunction. Then F is a Baire-1 multifunction if and only if $\overline{\alpha}(F) < \omega_1$ (and similarly for $\overline{\beta}$ and $\overline{\gamma}$).*

From the fact that $\alpha(f) \leq \beta(f) \leq \gamma(f)$ for any Baire-1 function $f : X \to \mathbb{R}$, we immediately see that $\overline{\alpha}(F) \leq \overline{\beta}(F) \leq \overline{\gamma}(F)$. In the next proposition we give the relationship between $\overline{\alpha}$ and the separation rank defined in the previous section.

Proposition 15. *Let $F : X \to P_k(Y)$ be a locally compact, Baire-1 multifunction. Then $\overline{\alpha}(F) \leq \alpha(F)$.*

Proof. Let $y \in D$ and $r_1, r_2 \in \mathbb{Q}$ with $0 < r_1 < r_2$. Put $C = \overline{B(y, r_1)}$ and $K = B(y, r_2)^{\mathrm{c}} = \{z \in Y : d(y, z) \geq r_2\}$. Then, as in the proof of Theorem 13, we have

$$d_y^{-1}((-\infty, r_1]) = F^-(C), \quad d_y^{-1}([r_2, \infty)) = F^-(B(y, r_2))^{\mathrm{c}} = F^+(K).$$

Note that, under the notation of the previous section, $C \in \overline{\mathcal{B}}_s$, $K \in \breve{\mathcal{B}}_d$ and $K \cap C = \emptyset$. It follows immediately that $\alpha(F, K, C) \geq \alpha(d_y, r_1, r_2)$, which implies that $\alpha(F) \geq \overline{\alpha}(F)$, as desired. ∎

5. The rank δ. In this section we will define a rank for Baire-1 multifunctions which, roughly speaking, gives a quantitative estimate of how far a Baire-1 multifunction F is from being upper semicontinuous; the bigger the rank, the less F looks like an upper semicontinuous multifunction. Note that upper semicontinuous multifunctions occur naturally in analysis and are standard tools in a variety of problems.

Throughout this section X will be a Polish space and Y a compact metrizable space. In what follows, d will be a compatible metric for Y. We make the following definition.

DEFINITION 16. Let $F : X \to 2^Y$ be a not necessarily everywhere defined multifunction. Then the multifunction $\widehat{F} : X \to 2^Y$ defined by $\operatorname{Gr}\widehat{F} = \overline{\operatorname{Gr} F}$ is said to be the *closed hull* of F (not to be confused with $\overline{F}$).

In the following lemma we gather some elementary properties of the closed hull. The proof is left to the reader.

LEMMA 17. *Let $F : X \to 2^Y$ be a multifunction.*

(i) *For every $x \in X$, $\widehat{F}(x)$ is a closed subset of Y (possibly empty).*
(ii) $\operatorname{Gr}\widehat{F}$ *is closed in $X \times Y$.*
(iii) $F(x) = \widehat{F}(x)$ *if and only if F is closed at x.*
(iv) $\operatorname{dom}(\widehat{F}) = \overline{\operatorname{dom}(F)}$ *and so if* $\operatorname{dom}(F)$ *is closed they coincide.*

Now let $F : X \to P_k(Y)$ be a multifunction. For every $\varepsilon > 0$ and every $P \subseteq X$ closed, consider the derivative operation

$$P'_\varepsilon = \overline{\{x \in P : d_{\mathrm{H}}(F(x), \widehat{F}|_P(x)) \geq \varepsilon\}}$$

where $\widehat{F}|_P$ is the closed hull of the multifunction $F|_P : X \to 2^Y$ defined by

$$F|_P(x) = \begin{cases} F(x) & \text{if } x \in P, \\ \emptyset & \text{otherwise.} \end{cases}$$

Observe that as $F(x) \subseteq \widehat{F}|_P(x)$ for every $x \in P$, we have

$$d_{\mathrm{H}}(F(x), \widehat{F}|_P(x)) = \inf\{r > 0 : \widehat{F}|_P(x) \subseteq F(x)_r\}.$$

So for every $P \subseteq X$ closed and every $\varepsilon > 0$, P'_ε is the closure of the set of points where $\widehat{F}|_P(x) \not\subseteq F(x)_\varepsilon$ (i.e. the closure of the set of points where $F|_P$ is not ε-closed).

By iterating, define again $P_\varepsilon^{(\xi)}$ for $\xi < \omega_1$, and set

$$\delta(F, P, \varepsilon) = \begin{cases} \text{least } \xi \text{ with } P_\varepsilon^{(\xi)} = \emptyset & \text{if such a } \xi \text{ exists,} \\ \omega_1 & \text{otherwise,} \end{cases}$$

and let $\delta(F, \varepsilon) = \delta(F, X, \varepsilon)$. Finally put

$$\delta(F) = \sup_{n \geq 1} \delta(F, 1/n).$$

THEOREM 18. *Let $F : X \to P_k(Y)$ be a multifunction. Then F is a Baire-1 multifunction if and only if $\delta(F) < \omega_1$.*

Proof. [$\Rightarrow$] Assume that F is a Baire-1 multifunction. Fix $\varepsilon > 0$. We will show that for every $P \subseteq X$ non-empty closed, we have $P'_\varepsilon \subsetneq P$. Put $C = \{x \in P : d_{\mathrm{H}}(F(x), \widehat{F}|_P(x)) \geq \varepsilon\}$. Assume, towards a contradiction, that $P = P'_\varepsilon = \overline{C}$. Recall that the *oscillation function* of $F|_P$ at x, defined by

$$\operatorname{osc}(F|_P)(x) = \inf_{V \in N(x)} \sup_{x_1, x_2 \in V \cap P} d_{\mathrm{H}}(F(x_1), F(x_2)),$$

is an upper semicontinuous, real-valued function and that $F|_P$ is continuous at $x \in P$ if and only if $\operatorname{osc}(F|_P)(x) = 0$.

CLAIM. *If $x \in C$, then* $\operatorname{osc}(F|_P)(x) \geq \varepsilon/4$.

Proof. Let $x \in C$. Then there exists $y \in \widehat{F}|_P(x)$ such that $d(y, F(x)) \geq \varepsilon/2$. From the definition of the closed hull, there exist $\{(z_n, y_n)\}_{n\geq 1} \subseteq \operatorname{Gr} F|_P$ with $z_n \to x$ and $y_n \to y$. Let $n_0 \geq 1$ be such that $d(y, y_n) \leq \varepsilon/4$ for every $n \geq n_0$. Then note

$$d_{\mathrm{H}}(F(z_n), F(x)) \geq d(y_n, F(x)) \geq d(y, F(x)) - d(y, y_n) \geq \varepsilon/4$$

for all $n \geq n_0$. As $z_n \to x$ and $z_n \in P$, it follows that $\operatorname{osc}(F|_P)(x) \geq \varepsilon/4$ and the claim is proved.

Now if C were dense in P, then for any $x \in P$, pick $\{x_n\}_{n\geq 1} \subseteq C$ such that $x_n \to x$. From the upper semicontinuity of $\operatorname{osc}(F|_P)$, we get

$$\operatorname{osc}(F|_P)(x) \geq \limsup_{n\to\infty} \operatorname{osc}(F|_P)(x_n) \geq \varepsilon/4,$$

which implies that $F|_P$ is nowhere continuous on P, contradicting the fact that F is a Baire-1 multifunction. So $P'_\varepsilon \subsetneq P$, which implies that $P_\varepsilon^{(\xi)}$ must be stabilized at $\emptyset$. Thus $\delta(F, \varepsilon) < \omega_1$ for every $\varepsilon > 0$, and finally $\delta(F) < \omega_1$ too.

[$\Leftarrow$] Assume that F is not a Baire-1 multifunction. Then there exist $P \subseteq X$ non-empty closed and $\varepsilon > 0$ such that $\operatorname{osc}(F|_P)(x) \geq \varepsilon$ for every $x \in P$.

Let $K \subseteq X$ be closed with $K \supseteq P$. Put $C = \{x \in K : d_{\mathrm{H}}(F(x), \widehat{F}|_K(x)) \geq \varepsilon/4\}$. We claim that $\overline{C} = K'_{\varepsilon/4} \supseteq P$. Assume not. Put $U = P \cap (X \setminus \overline{C}) \neq \emptyset$. Then U is a relatively open subset of P and moreover

$$d_{\mathrm{H}}(F(x), \widehat{F}|_K(x)) < \varepsilon/4$$

for every $x \in U$. From the definition of the closed hull and the fact that $K \supseteq P$, we have $\widehat{F}|_K(x) \supseteq \widehat{F}|_P(x) \supseteq F(x)$ for every $x \in U$. It follows that

$$d_{\mathrm{H}}(F(x), \widehat{F}|_P(x)) < \varepsilon/4$$

for every $x \in U$. From Lemma 17, we know that $\widehat{F}|_P$ is closed (i.e. $\mathrm{Gr}\,\widehat{F}|_P$ is closed) and that $\mathrm{dom}(\widehat{F}|_P) = P$. So, from Proposition 11, $\widehat{F}|_P$ is upper semicontinuous on P. Corollary 9 shows that $\widehat{F}|_P$ is a Baire-1 multifunction. Since the points of continuity of $\widehat{F}|_P$ form a dense G_δ subset of P, it follows that the set $V = \{x \in P : \mathrm{osc}(\widehat{F}|_P)(x) < \varepsilon/4\}$ is relatively open and dense in P. Put $W = U \cap V \neq \emptyset$. Given $x_1, x_2 \in W$ we have

$$\begin{aligned} d_{\mathrm{H}}(F(x_1), F(x_2)) &\leq d_{\mathrm{H}}(F(x_1), \widehat{F}|_P(x_1)) + d_{\mathrm{H}}(\widehat{F}|_P(x_1), \widehat{F}|_P(x_2)) \\ &\quad + d_{\mathrm{H}}(F(x_2), \widehat{F}|_P(x_2)) \\ &\leq d_{\mathrm{H}}(\widehat{F}|_P(x_1), \widehat{F}|_P(x_2)) + \varepsilon/2. \end{aligned}$$

From the definition of the oscillation function and the fact that W is a relatively open subset of P, we see that for any $x \in W \subseteq P$,

$$\mathrm{osc}(F|_P)(x) \leq \mathrm{osc}(\widehat{F}|_P)(x) + \varepsilon/2 < 3\varepsilon/4 < \varepsilon,$$

which is a contradiction. So for any $K \subseteq X$ closed with $K \supseteq P$, we have $K'_{\varepsilon/4} \supseteq P$. Thus, from the induction hypothesis, we conclude that $X^{(\xi)}_{\varepsilon/4} \supseteq P$ for all $\xi < \omega_1$ and so $\delta(F) = \omega_1$ from the definition of the rank. ∎

In the following corollary we isolate a useful property of the derivative operation P'_ε defined above.

Corollary 19. *Let $F : X \to P_k(Y)$ be a Baire-1 multifunction. Then for every $P \subseteq X$ non-empty closed and every $\varepsilon > 0$, the set P'_ε is a nowhere dense subset of P.*

Proof. Assume on the contrary that P'_ε has non-empty interior in the relative topology of P (recall that P'_ε is closed). Put $U = \mathrm{int}(P'_\varepsilon) \neq \emptyset$. Working as in the proof of Theorem 18, we find that $\mathrm{osc}(F|_P)(x) \geq \varepsilon/4$ for every $x \in U$. This contradicts the fact that the points of continuity of $F|_P$ form a dense G_δ subset of P. ∎

The rank δ has the following interesting property.

Proposition 20. *Let $F : X \to P_k(Y)$ be a multifunction. Then F is upper semicontinuous if and only if $\delta(F) = 1$.*

Proof. If F is upper semicontinuous, then $F(x) = \widehat{F}(x)$ for every $x \in X$. So $\delta(F, \varepsilon) = 1$ for every $\varepsilon > 0$, which implies that $\delta(F) = 1$.

Conversely, if $\delta(F) = 1$, then $\widehat{F}(x) \subseteq F(x)_{1/n}$ for all $x \in X$ and $n \geq 1$. Since F has compact values, we deduce that $\widehat{F}(x) \subseteq \bigcap_{n \geq 1} F(x)_{1/n} = F(x)$. On the other hand, from the properties of the closed hull, we have $F(x) \subseteq \widehat{F}(x)$. Thus $F(x) = \widehat{F}(x)$ for every $x \in X$. It follows that F is closed at every $x \in X$ and so, from Proposition 11, we conclude that F is upper semicontinuous. ∎

Remark 3. We point out that δ is well defined if Y is a separable metrizable space and $F : X \to P_k(Y)$ a locally compact, Baire-1 multifunction. In this case, it is clear that both Theorem 18 and Proposition 20 are still valid.

One might expect to get a relatively small bound for $\delta(F)$ when F is a lower semicontinuous multifunction. This is not the case, as is shown in the following example (similar to the example given in Proposition 2.2 of [8]).

Example 1. Let $\{K_n\}_{n\geq 1}$ be a decreasing sequence of non-empty closed subsets of X such that K_1 is nowhere dense in X and K_{n+1} is nowhere dense in K_n for every $n \geq 1$. Put $K = \bigcap_n K_n$ and $K_0 = X$. Define the multifunction $F : X \to P_k(\mathbb{R})$ by

$$F(x) = \begin{cases} [0,1] & \text{if } x \in K_0 \setminus K_1, \\ [0,1/2^n] & \text{if } x \in K_n \setminus K_{n+1}, \\ \{0\} & \text{if } x \in K. \end{cases}$$

Note that F is well defined even if $K = \emptyset$. We claim that F is lower semicontinuous. Indeed, let $U \subseteq \mathbb{R}$ be open. Note that

$$F^{-}(U) = \bigcup_{n\geq 0} \{K_n^{\mathrm{c}} : [0,1/2^n] \cap U \neq \emptyset\}$$

if $U \cap \{0\} = \emptyset$ and $F^{-}(U) = X$ otherwise. As the sets K_n are closed, it follows that F is a lower semicontinuous multifunction. Now it is easy to see that if $m \geq 1$ and $0 < \varepsilon < 1/2^{m+1}$, then $X'_\varepsilon = K_1$ and $X^{(n)}_\varepsilon = (K_n)'_\varepsilon = K_{n+1}$ for every $1 \leq n \leq m$. As the sets K_n are non-empty, we see that $\delta(F,\varepsilon) \geq m$ for every $0 < \varepsilon < 1/2^{m+1}$. Using similar arguments, it is also easy to verify that if $m \geq 1$ and $\varepsilon \geq 1/2^{m+1}$, then $\delta(F,\varepsilon) \leq m+1$. It follows directly from the definition of the rank that $\delta(F) = \omega$.

The above example shows that the ranks α and δ are incomparable. Indeed, let F be an upper semicontinuous but not continuous multifunction. Then, from Propositions 8 and 20, we have $\alpha(F) = 2$ and $\delta(F) = 1$. On the other hand, if F is the multifunction of the above example, then $\alpha(F) = 2$ (as F is lower semicontinuous but not continuous) and $\delta(F) = \omega$. We could say that this expresses the fact that upper and lower semicontinuity of multifunctions is quite different from upper and lower semicontinuity of real-valued functions. This difference is a typical phenomenon and occurs in many aspects of multivalued analysis.

6. Comments. Clearly the ordinal ranks defined in this paper may be used to classify multifunctions according to their complexity. However we feel that these ranks, and especially the rank δ, may also be used to classify Lipschitz functions. Let us be more precise.

Let X be a separable Banach space and U a non-empty bounded open subset of X. For $r > 0$, denote by $\mathrm{Lip}(U, r)$ the set of all Lipschitz functions $f : U \to \mathbb{R}$ with Lipschitz constant less than or equal to r. For every $f \in \mathrm{Lip}(U, r)$, the Clarke subdifferential $\partial_c f$ of f is an upper semicontinuous multifunction from X into $X^*_{w^*}$ ($X^*_{w^*}$ stands for the topological dual of X equipped with the weak* topology). $\partial_c f$ is an extensively investigated multifunction (see [4]). It has gained its widespread utility due to its rich calculus and powerful analytical properties. However, Clarke's subdifferential is too large to reveal any structure. As has been shown in [2] by J. M. Borwein, W. B. Moors and X. F. Wang, for almost every $f \in \mathrm{Lip}(U, 1)$ (in a precise topological sense), $\partial_c f(x)$ is identical to the dual ball, for every $x \in U$.

So, instead of working with Clarke's subdifferential, one may use the Michel–Penot subdifferential $\partial_{mp} f$ of f, which has the following remarkable property: $\partial_{mp} f(x)$ coincides with the Gateaux derivative of f whenever the latter exists (for the proof as well as for the definition of the Michel–Penot subdifferential we refer to [11]). There is however one disadvantage. As a multifunction ∂_{mp} is not always upper semicontinuous.

Nevertheless, we may restrict ourselves to those functions f for which the Michel–Penot subdifferential is a Baire-1 multifunction (the study of subclasses of $\mathrm{Lip}(U, r)$ with good properties is an important part of non-smooth analysis; see [1] and the references therein). Specifically, for $r > 0$, let Y be the closed ball in X^* of radius r, endowed with the weak* topology. Then Y is a compact metrizable space and moreover for every $f \in \mathrm{Lip}(U, r)$, $\partial_{mp} f$ is a multifunction from U to $P_k(Y)$. So, all the tools developed in this paper can be applied. Thus, for every countable ordinal ξ, we may consider the classes

$$L_\xi(U, r) = \{f \in \mathrm{Lip}(U, r) : \delta(\partial_{mp} f) \le \xi\}.$$

From the properties of δ, for every $f \in L_\xi(U, r)$, one may replace the pathological $\partial_c f$ with the certainly well-behaved $\partial_{mp} f$ and still have good analytical properties. Furthermore, one has additional information on the set of points where $\partial_{mp} f$ is not ε-closed. As closedness and ε-closedness of the subdifferential are the crucial properties in many applications, we believe that these classes may be of interest.

Remark 4. It follows from the properties of the Michel–Penot subdifferential that the class $L_1(U, r)$ contains all continuous convex functions and all continuously Gateaux differentiable functions. Moreover, it can be shown that every $f \in L_1(U, r)$ is an essentially smooth Lipschitz function.

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