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STRUCTURE OF GEODESICS IN THE CAYLEY GRAPH OF INFINITE COXETER GROUPS

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Abstract. Let (W,S) be a Coxeter system such that no two generators in S commute. Assume that the Cayley graph of (W,S) does not contain adjacent hexagons. Then for any two vertices x and y in the Cayley graph of W and any number $k \le d = \operatorname{dist}(x,y)$ there are at most two vertices z such that $\operatorname{dist}(x,z) = k$ and $\operatorname{dist}(z,y) = d-k$. Allowing adjacent hexagons, but assuming that no three hexagons can be adjacent to each other, we show that the number of such intermediate vertices at a given distance from x and y is at most 3. This means that the group W is hyperbolic in a sense stronger than that of Gromov.

1. Introduction. A *Coxeter system* is a pair (W,S), where W is a group and S is a set of generators, and the only relations are of the form

$$s^2 = 1, \qquad (ss')^{\mathrm{m}(s,s')} = 1, \qquad s \ne s'.$$

If no relation occurs for s and s' we set $\mathrm{m}(s,s') = \infty$.

The system of generators determines the Cayley graph Γ of the group W. If $\mathrm{m}(s,s') = \infty$ for any $s,s' \in S$ the graph Γ is a homogeneous tree of degree $\operatorname{card}(S)$. In this case any two vertices in Γ are connected by a unique geodesic in the graph. In general, when we allow $\mathrm{m}(s,s') < \infty$, we often have many geodesic lines connecting two vertices in the graph.

The aim of this work is to show that under the conditions:

(a) $\mathrm{m}(s,s') \ge 3$ for any $s,s' \in S$,

(b) there are no $a,b,c \in S$ such that $\mathrm{m}(a,b) = \mathrm{m}(a,c) = 3$,

for any two vertices x and y at distance d from each other and for any number $1 \le k \le d$, there are at most two vertices z at distance k from x and at distance $d-k$ of y. Replacing (b) by

(b′) there are no $a,b,c \in S$ such that $\mathrm{m}(a,b) = \mathrm{m}(a,c) = \mathrm{m}(b,c) = 3$,

we show that there at most three such intermediate vertices z in Γ. We will

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also show that if, in fact, there are at least two such vertices, then their distances are bounded by the maximum of those $\mathrm{m}(s, s')$ which are finite.

Assumptions (a), (b) and (b$'$) have a natural geometric interpretation as follows. The Cayley graph of (W, S) does not contain

(a) squares,
(b) adjacent hexagons,
(b$'$) three hexagons adjacent to each other.

The fact that geodesic lines connecting two vertices are not far apart means that the group (W, S) is hyperbolic in the sense of Gromov. Roughly a metric space is hyperbolic if geodesic triangles are "thin". Papasoglu showed [7, Theorem 1.4] that a graph is hyperbolic if geodesic biangles are "thin". In our case the biangles are not only "thin", but there is a geodesic biangle connecting two vertices, containing all other such biangles, under assumptions (a) and (b).

Not all hyperbolic Coxeter groups have this stronger property. Indeed, according to Moussong [6], a Coxeter group is hyperbolic exactly when it does not contain an abelian subgroup isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$. Basing on this one can easily construct a hyperbolic Coxeter group which admits many geodesic lines connecting two vertices (it suffices that this group contains $\mathbb{Z}_2 \oplus \ldots \oplus \mathbb{Z}_2$ as a subgroup).

The absence of low values of $\mathrm{m}(s, s')$ enables a good description of elements in the group. This is achieved in Propositions 1 and 2.

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2. Analysis of elements in Coxeter groups

Definition 1. For $a, b \in S$ and $a \neq b$ we define

$$\mathrm{N}(a,b) = \begin{cases} (ab)^k & \text{if } \mathrm{m}(a,b) = 2k+1, \\ b(ab)^{k-1} & \text{if } \mathrm{m}(a,b) = 2k. \end{cases}$$

We have the following.

Lemma 1.

$$\mathrm{N}(a,b)a = a\mathrm{N}(a,b) \quad \textit{if } \mathrm{m}(a,b) = 2k, \tag{1}$$

$$\mathrm{N}(a,b)a = b\mathrm{N}(a,b) \quad \textit{if } \mathrm{m}(a,b) = 2k+1, \tag{2}$$

$$\mathrm{N}(a,b)a = \mathrm{N}(b,a)b. \tag{3}$$

Obviously the elements $\mathrm{N}(a,b)a$ are reduced, while the elements $\mathrm{N}(a,b)$ have unique reduced expansions. Let

$$\begin{aligned} &\mathrm{c}(a,b)=a, \quad \mathrm{d}(a,b)=b \quad \text{if } \mathrm{m}(a,b)=2k,\\ &\mathrm{c}(a,b)=b, \quad \mathrm{d}(a,b)=a \quad \text{if } \mathrm{m}(a,b)=2k+1. \end{aligned}$$

We have

$$\mathrm{N}(a,b)a=\mathrm{c}(a,b)\mathrm{N}(a,b), \tag{4}$$

$$\mathrm{N}(a,b)b=\mathrm{d}(a,b)\mathrm{N}(b,a). \tag{5}$$

The choice of the set S of generators determines the length function $\ell(w)$ defined for $w \in W$ as the minimum of the numbers n such that

$$w=w_1\dots w_n, \quad w_i \in S.$$

A product $v_1\dots v_n$, where $v_i \in W$, will be called *reduced* if

$$\ell(v_1\dots v_n)=\ell(v_1)+\dots+\ell(v_n).$$

An expression $w_1\dots w_n$ will be called a *reduced expansion* if $w_i \in S$ and $\ell(w_1\dots w_n)=n$.

By a *subexpression* of the product $w_1\dots w_n$, where $w_i \in S$, we mean any product of the form $w_k w_{k+1}\dots w_l$, where $1 \le k \le l \le n$.

The next proposition is well known. More general results, with proofs restricted to finite Coxeter groups, can be found in [2, 4]. They have been extended to the infinite case by V. Deodhar. We provide a new combinatorial proof.

Proposition 1 (V. Deodhar). *Let*

$$w_1\dots w_n s=s' w_1\dots w_n \tag{6}$$

be reduced expansions. Then $w_1\dots w_n$ *has a reduced expansion of the form*

$$w_1\dots w_n=\mathrm{N}(a_m,b_m)\dots\mathrm{N}(a_1,b_1), \tag{7}$$

where $\mathrm{c}(a_i,b_i)=a_{i+1}$, $\mathrm{d}(a_i,b_i)\neq b_{i+1}$ *and* $a_1=s$, $b_1=w_n$, $\mathrm{c}(a_m,b_m)=s'$.

In particular, if the reduced expansion for $w_1\dots w_n$ *is unique and* $\mathrm{m}(a_1,b_1)>2$, *then* $w_{n-1}=a_1=s$.

Proof. We proceed by induction on n. The statement is obviously true for $n=1$. By assumption $w_1\dots w_n s$ has a reduced expansion which ends in w_n. Hence by the Exchange Condition (see [5, Ch. 5.8]) there is k such that

$$w_1\dots\widehat{w}_k\dots w_n s w_n=w_1\dots w_n s,$$

where the hat denotes omission. This implies

$$(w_{k+1}\dots w_n s)w_n=w_k(w_{k+1}\dots w_n s). \tag{8}$$

It suffices to show that $w_{k+1}\dots w_n s$ is of the form

$$w_{k+1}\dots w_n s=v\mathrm{N}(w_n,s), \tag{9}$$

where the product $v\mathrm{N}(w_n, s)$ is reduced. Indeed, by (3) and (8) we then get

$$w_k w_{k+1} \ldots w_n = w_k(w_{k+1} \ldots w_n s)s = v\mathrm{N}(w_n, s)w_n s = v\mathrm{N}(s, w_n).$$

Therefore

$$w_1 \ldots w_k \ldots w_n = w'\mathrm{N}(s, w_n) \tag{10}$$

and the product $w'\mathrm{N}(s, w_n)$ is reduced. Combining (2), (6) and Lemma 1 implies

$$(w_1 \ldots w_n)s = w'\mathrm{N}(s, w_n)s = w'\mathrm{c}(s, w_n)\mathrm{N}(s, w_n) = s'w'\mathrm{N}(s, w_n).$$

Thus

$$w'\mathrm{c}(s, w_n) = s'w', \qquad \ell(w') < \ell(w).$$

By the induction hypothesis, w' has a reduced expansion of the form

$$w' = \mathrm{N}(a_N, b_N) \ldots \mathrm{N}(a_2, b_2),$$

where $a_2 = \mathrm{c}(s, w_n)$. Setting $a_1 = s$, $b_1 = w_n$ and using (10) gives the conclusion.

Thus we have to show that $w_{k+1} \ldots w_n s$ is of the form (9). This follows from (8) by the induction hypothesis if $k > 1$ because $\ell(w_{k+1} \ldots w_n s) = n - k + 1 < n$. Therefore it remains to consider the case $k = 1$, i.e.

$$(w_2 \ldots w_n s)w_n = w_1(w_2 \ldots w_n s), \tag{11}$$

and show that $w_2 \ldots w_n s$ is of the form (7), which obviously implies (9) for $k = 1$. Observe that (11) has the same form as (6). Therefore we can apply the first part of the proof to (11) and conclude, by arriving at (11) again, that it remains to consider the case

$$(w_3 \ldots w_n s w_n)s = w_2(w_3 \ldots w_n s w_n). \tag{12}$$

and show that $w_3 \ldots w_n s w_n$ is of the form (7). Repetition of this argument again and again reduces our considerations to the case

$$(w_n s \ldots w_n s)w_n = s(w_n s \ldots w_n s)$$

or

$$(w_n s \ldots w_n s w_n)s = s(w_n s \ldots w_n s w_n),$$

according as $\mathrm{m}(w_n, s)$ is odd or even, and it suffices to show that $w_n s \ldots w_n s$ or $w_n s \ldots w_n s w_n$ is of the form (7). The latter is obviously true. ■

PROPOSITION 2. *Let* $\mathrm{m}(a, b) > 2$ *for any* $a, b \in S$, $a \neq b$. *Any product of the form*

$$\mathrm{N}(a_m, b_m) \ldots \mathrm{N}(a_1, b_1), \tag{13}$$

where $\mathrm{c}(a_i, b_i) = a_{i+1}$ *and* $d(a_i, b_i) \neq b_{i+1}$, *is reduced and has a unique reduced expansion.*

Proof. Obviously a single factor $\mathrm{N}(a_i, b_i)$ has a unique reduced expansion. Observe also that the assumption $\mathrm{m}(a, b) > 2$ implies that no two different generators in S commute.

Let $w_1 \dots w_n$ be a subexpression of (13), where $w_i \in S$. It is clear that $w_i \neq w_{i+1}$ for $1 \leq i \leq n$. We will show, by induction on n, that this subexpression is reduced and has a unique expansion. Let $n = 2$. Then $w_1 w_2$ is obviously reduced and has a unique expansion because by assumption $w_1 w_2 \neq w_2 w_1$.

Assume the statement is true for n. Let $w_1 \dots w_{n+1}$ be a subexpression of (13). Assume it is not reduced. Hence by the Deletion Condition (see [5, Ch. 5.8]) there are $i < j$ such that

$$w_1 \dots \widehat{w}_i \dots \widehat{w}_j \dots w_{n+1} = w_1 \dots w_{n+1}. \tag{14}$$

We may assume that $j - i \geq 2$, since $w_i \neq w_{i+1}$ by (13). Hence

$$w_{i+1} \dots w_{j-1} w_j = w_i w_{i+1} \dots w_{j-1}. \tag{15}$$

Since $\ell(w_{i+1} \dots w_{j-1} w_j) \leq j - i \leq n$, we may apply the induction hypothesis to conclude that $w_{i+1} \dots w_{j-1} w_j$ is reduced and has a unique expansion. Therefore by (15) we obtain $w_i = w_{i+1}$, which leads to a contradiction. Hence $w_1 \dots w_{n+1}$ is reduced.

Assume now that $w_1 \dots w_{n+1}$ has another reduced expansion, i.e.

$$v_1 \dots v_{n+1} = w_1 \dots w_{n+1}. \tag{16}$$

By the Exchange Condition there exists k such that

$$w_1 \dots \widehat{w}_k \dots w_{n+1} v_{n+1} = w_1 \dots w_{n+1}.$$

Thus

$$w_{k+1} \dots w_{n+1} v_{n+1} = w_k \dots w_{n+1}.$$

If $k > 1$, then by the induction hypothesis the right side has a unique reduced expansion. Hence $w_k = w_{k+1}$, which gives a contradiction. Therefore it suffices to consider the case $k = 1$. Then

$$w_2 \dots w_{n+1} v_{n+1} = w_1 \dots w_{n+1}. \tag{17}$$

In the same way, using the fact that

$$w_{n+1} w_n \dots w_1 = v_{n+1} v_n \dots v_1,$$

we can show that $w_1 \dots w_{n+1}$ has a unique reduced expansion unless

$$v_1 w_1 \dots w_n = w_1 \dots w_{n+1}. \tag{18}$$

So we are done unless both (17) and (18) hold. Observe that the left sides of (17) or (18) can be put in (16) instead of $v_1 \dots v_n$. In particular, we can substitute the left side of (17) for (16) and apply (18). This shows that $w_1 \dots w_{n+1}$ has a unique reduced expansion unless

$$w_2 (w_1 \dots w_n) = w_1 \dots w_{n+1}. \tag{19}$$

Similarly substituting the left side of (18) for (16) and applying (17) implies that $w_1 w_2 \ldots w_{n+1}$ has a unique expansion unless

$$(w_2 \ldots w_{n+1}) w_n = w_1 \ldots w_{n+1}. \tag{20}$$

Combining (19) and (20) gives

$$w_2(w_1 \ldots w_n) = (w_2 \ldots w_{n+1}) w_n.$$

Hence

$$w_1 \ldots w_{n-1} = w_3 \ldots w_{n+1}.$$

We may now use the induction hypothesis to get

$$w_1 = w_3 = w_5 = \ldots, \qquad w_2 = w_4 = w_6 = \ldots$$

Therefore

$$w_1 \ldots w_{n+1} = \begin{cases} (w_1 w_2)^k & \text{if } n = 2k-1, \\ (w_1 w_2)^k w_1 & \text{if } n = 2k. \end{cases}$$

In both cases the expansions are unique since $k < \mathrm{m}(w_1, w_2)$. The latter follows from the fact that $w_1 \ldots w_{n+1}$ is a subexpression of (13). ■

The following lemma will be used frequently.

LEMMA 2. *Assume the product $w = w'\mathrm{N}(s, s')$ is reduced. Then ws is not reduced if and only if $w'\mathrm{c}(s, s')$ is not reduced.*

Proof. By (4) we have

$$ws = w'\mathrm{N}(s, s')s = w'\mathrm{c}(s, s')\mathrm{N}(s, s').$$

Hence if $w'\mathrm{c}(s, s')$ is not reduced neither is ws.

Assume now that ws is not reduced. Let $w' = w_1 \ldots w_k$. By the Deletion Condition and the fact that $\mathrm{N}(s, s')s$ is reduced, we have

$$w_1 \ldots w_l \ldots w_k \mathrm{N}(s, s')s = w_1 \ldots \widehat{w}_l \ldots w_k \mathrm{N}(s, s')$$

for some $l \le k$. This implies

$$w_1 \ldots w_l \ldots w_k \mathrm{c}(s, s') = w_1 \ldots \widehat{w}_l \ldots w_k.$$

Thus $w'\mathrm{c}(s, s')$ is not reduced. ■

DEFINITION 2. For $w \in G$ let $\mathrm{C}(w) = \{s \in S \mid \ell(ws) < \ell(w)\}$.

PROPOSITION 3. *Assume $\mathrm{m}(s, s') \ge 3$ for any $s \ne s' \in S$. Let $w = w_1 \ldots w_n$ be a reduced expansion of $w \in G$. Then either $\mathrm{C}(w) = \{w_n\}$ or $\mathrm{C}(w) = \{w_{n-1}, w_n\}$.*

Proof. Assume $\ell(ws) < \ell(w)$ for some $s \in S$ and $s \ne w_n$. Then there exists k such that

$$w_1 \ldots \widehat{w}_k \ldots w_n s = w_1 \ldots w_n.$$

Thus

$$(w_{k+1} \ldots w_n)s = w_k(w_{k+1} \ldots w_n).$$

By Propositions 1 and 2 we know that $w_{k+1}\dots w_n$ has a unique reduced expansion of the form

$$w_{k+1}\dots w_n = \mathrm{N}(a_m,b_m)\dots\mathrm{N}(a_1,b_1),$$

where $a_1 = s$. Since $\ell(\mathrm{N}(a_1,b_1)) \geq 2$ we get $w_{n-1} = a_1 = s$. Thus $\mathrm{C}(w) = \{w_{n-1}, w_n\}$. ∎

REMARK. Professor R. B. Howlett observed that Proposition 3 can be obtained otherwise as follows. Let $I = \{s \in S \mid \ell(ws) < \ell(w)\}$. By [5, Prop. 1.10(c)] the element w has a unique representation of the form $w = w^I w_I$, where w_I belongs to W_I, the group generated by I, and $\ell(w^I s) > \ell(w^I)$ for any $s \in I$. Thus $\ell(w_I s) < \ell(w_I)$ for any $s \in I$. Hence the element w_I has a maximal length in W_I, i.e. the group W_I is finite. Since no two generators in I commute, there can be two elements in I at most. Now the conclusion of Proposition 3 follows easily.

COROLLARY 1. *Let* $\mathrm{m}(a,b) > 2$ *for any* $a,b \in S$, $a \neq b$. *Any product of the form*

$$\mathrm{N}(a_m,b_m)\dots\mathrm{N}(a_1,b_1)a_1, \tag{21}$$

where $\mathrm{c}(a_i,b_i) = a_{i+1}$ *and* $d(a_i,b_i) \neq b_{i+1}$, *is reduced. All reduced expansions of this product are given by the formula*

$$\mathrm{N}(a_m,b_m)\dots\mathrm{N}(a_{k+1},b_{k+1})a_{k+1}\mathrm{N}(a_k,b_k)\dots\mathrm{N}(a_1,b_1), \tag{22}$$

where $0 \leq k \leq m-1$ *and* $a_{m+1} = \mathrm{c}(a_m,b_m)$.

Proof. We proceed by induction on m. It is clear that $\mathrm{N}(a_1,b_1)a_1$ is a reduced product. Assume that any product of the form (21) is reduced for $m \leq M$. Assume that the product

$$\mathrm{N}(a_{M+1},b_{M+1})\dots\mathrm{N}(a_1,b_1)a_1$$

is not reduced. By Lemma 2 the product

$$\mathrm{N}(a_{M+1},b_{M+1})\dots\mathrm{N}(a_2,b_2)a_2$$

is not reduced. On the other hand, by the induction hypothesis, this product is reduced. Thus we have arrived at a contradiction.

In view of (4) the elements (22) represent different reduced expansions of (21). We have to show that any reduced representation of (21) is of the form (22). We use induction on m. Assume that

$$\mathrm{N}(a_m,b_m)\dots\mathrm{N}(a_1,b_1)a_1 = v_1\dots v_{n-1}v_n$$

is another reduced representation of (21). If $v_n = a_1$, then

$$\mathrm{N}(a_m,b_m)\dots\mathrm{N}(a_1,b_1) = v_1\dots v_{n-1}.$$

By Proposition 2 these words are identical. Hence $v_1\dots v_n$ is of the form (21). Assume that $v_n \neq a_1$. Then by Proposition 3 we get $v_n = b_1$. Moreover, since the product $v_1 \dots v_n a_1$ is not reduced, $v_1 \dots v_n$ must end in $\mathrm{N}(a_1, b_1)$. Hence we get

$$\mathrm{N}(a_m, b_m) \dots \mathrm{N}(a_2, b_2) a_2 \mathrm{N}(a_1, b_1) = v_1 \dots v_l \mathrm{N}(a_1, b_1).$$

This gives

$$\mathrm{N}(a_m, b_m) \dots \mathrm{N}(a_2, b_2) a_2 = v_1 \dots v_l.$$

Now we can apply the induction hypothesis. ∎

3. Hyperbolic Coxeter groups. The results in this section are true only for Coxeter groups such that $\mathrm{m}(s, s') \geq 3$ for any $s, s' \in S$, and there are no three generators a, b and c satisfying $\mathrm{m}(a, b) = \mathrm{m}(a, c) = 3$. By [6] this implies that the group W is hyperbolic in the sense of Gromov, because it does not contain $\mathbb{Z} \oplus \mathbb{Z}$ as a subgroup. The fact that the group W is hyperbolic will also follow from Theorem 1, by a result of Papasoglu [7, Theorem 1.4].

LEMMA 3. *Assume* $\mathrm{m}(s_i, s_j) \geq 3$ *for each* $i \neq j$, *and there are no* i, j *and* k *such that*

$$\mathrm{m}(s_i, s_j) = \mathrm{m}(s_i, s_k) = \mathrm{m}(s_j, s_k) = 3.$$

Let $w = w_1 \dots w_n$ *be a reduced expansion. Assume that* ws *and* ws' *are reduced for* $s \neq s' \in S$.

(i) *If* $\mathrm{m}(s, w_n) \geq 4$ *or* $\mathrm{m}(s, w_n) \geq 4$ *then either* $\mathrm{C}(ws) = \{s\}$ *or* $\mathrm{C}(ws') = \{s'\}$. *In other words the last letter of either* ws *or* ws' *is uniquely determined.*

(ii) *If neither* ws *nor* ws' *determines its last letter then* $\mathrm{m}(s, w_n) = \mathrm{m}(s', w_n) = 3$ *and* w *ends in* $\mathrm{N}(s, s')w_n$ *or in* $\mathrm{N}(s', s)w_n$.

Proof. Assume that there are $t, t' \in S$ such that $t \neq s$, $t' \neq s'$ and $\ell(wst) < \ell(ws)$, $\ell(ws't') < \ell(ws')$. Let $w_1 \dots w_n$ be any reduced expansion for w. By the Deletion Condition there are i and j such that

$$w_i w_{i+1} \dots w_n s = w_{i+1} \dots w_n st, \tag{23}$$

$$w_j w_{j+1} \dots w_n s' = w_{j+1} \dots w_n s't'. \tag{24}$$

By Propositions 1 and 2 the elements $w_{i+1} \dots w_n s$ and $w_{j+1} \dots w_n s'$ have unique reduced expansions which end in $\mathrm{N}(t, s)$ and $\mathrm{N}(t', s')$, respectively. By assumptions and by Definition 1 we get $\ell(\mathrm{N}(t, s)), \ell(\mathrm{N}(t', s')) \geq 2$. Hence the last letter of w_n is t and t', simultaneously. Thus $t = t'$.

We will break the proof into three cases.

(a) $\mathrm{m}(t, s) \geq 4$ and $\mathrm{m}(t, s') \geq 4$.

Then $\ell(\mathrm{N}(t, s)), \ell(\mathrm{N}(t, s')) \geq 3$. Thus $w_{n-1} = s$ and $w_{n-1} = s'$, which gives a contradiction.

(b) $\mathrm{m}(t, s) = 3$ and $\mathrm{m}(t, s') \geq 4$.

Then $\ell(\mathrm{N}(t,s)) = 2$ and $\ell(\mathrm{N}(t,s')) \geq 3$. This implies $w_{n-1} = s'$. Thus the expression $w_{i+1}\dots w_n s$ ends in $s'\mathrm{N}(t,s)$. By Corollary 1 it ends in $\mathrm{N}(s,s')\mathrm{N}(t,s)$. Hence $w_{n-2} = s$. Now we consider two subcases.

(b1) $\mathrm{m}(s,s') \geq 4$.

Thus $\ell(\mathrm{N}(s,s')) \geq 3$. Hence $w_{n-3} = s'$. This implies that w ends in $s'ss't$. Applying Propositions 1 and 2 to (24) yields $\ell(\mathrm{N}(t,s')) = 3$. Hence ws' ends in $s's\mathrm{N}(t,s')$, so it can be written as a reduced product $w's's\mathrm{N}(t,s')$, where $w' = w_1\dots w_{n-4}$. Since $ws't$ is not reduced, Lemma 2 shows that $w's'st$ is not reduced. By Propositions 1 and 2 the element $w's's$ must end in $\mathrm{N}(t,s)$. This implies $t = s'$, which gives a contradiction.

(b2) $\mathrm{m}(s,s') = 3$.

Thus $\ell(\mathrm{N}(s,s')) = 2$. We already know that ws ends in $\mathrm{N}(s,s')\mathrm{N}(t,s')$. Hence it can be written as a reduced product $w'\mathrm{N}(s,s')\mathrm{N}(t,s)$, where $w' = w_1\dots w_{n-3}$. The product wst is not reduced. Hence by Lemma 2 the product $w'\mathrm{N}(s,s')s$ is not reduced. Applying Lemma 2 again shows that $w's'$ is not reduced either. The product $w_{i+1}\dots w_{n-3}s'$ is reduced. Thus by Propositions 1, 2 and Corollary 1, either $w_{n-3} = s'$ or w' ends in $\mathrm{N}(s',w_{n-3})$. The first case has already been considered in (b1). Therefore we can assume that ws ends in $\mathrm{N}(s',w_{n-3})\mathrm{N}(s,s')\mathrm{N}(t,s)$. Thus ws' ends in $\mathrm{N}(s',w_{n-3})ss'ts'$, because $\mathrm{N}(s,s') = ss'$ and $\mathrm{N}(t,s) = ts$. The product $ws't$ is not reduced, hence by Propositions 1 and 2 we get $\ell(\mathrm{N}(t,s')) = 3$. This implies that ws' can be written in reduced form as

$$ws' = w''\mathrm{N}(s',w_{n-3})s\mathrm{N}(t,s'). \tag{25}$$

The product $ws't$ is not reduced. By Lemma 2 the product $w''\mathrm{N}(s',w_{n-3})st$ is not reduced either. In view of Propositions 1 and 2 this yields $w_{n-3} = t$. Summarizing, we have shown that the product $ws't = w''ts'tst$ is not reduced (note that $\mathrm{N}(s',t) = ts't$). By Lemma 2 the product $w''ts's$ is not reduced either. By Propositions 1 and 2 this implies $t = s$, which is a contradiction.

(c) $\mathrm{m}(t,s) = \mathrm{m}(t,s') = 3$.

We already know that $w_n = t = t'$. Assume that $w_{n-1} = s$. Hence $w_1\dots w_n s$ ends in sts. Thus ws' can be written as $ws' = w'sts'$. The word $ws't$ is not reduced, and neither is $w'ss'$, by Lemma 2. By Propositions 1 and 2 the element $w's$ must end in $\mathrm{N}(s',s)$. Hence w ends in $\mathrm{N}(s',s)t$. If $w_{n-1} = s'$ the reasoning is the same. Assume now that $w_{n-1} \neq s$ and $w_{n-1} \neq s'$. We know that ws is of the form $w''\mathrm{N}(t,s)$, where $w'' = w_1\dots w_{n-1}$. Moreover, by Lemma 2, the product $w''s$ is not reduced because wst is not. Hence w'' ends in $\mathrm{N}(s,w_{n-1})$. In the same way we show that w'' ends in $\mathrm{N}(s',w_{n-1})$. This implies $s = s'$, which gives a contradiction. ∎

Theorem 1. *Let W be a Coxeter group such that* $\mathrm{m}(s,s')\ge 3$ *for any* $s\ne s'\in S$.

(i) *Assume that there are no s, s' and s'' such that* $\mathrm{m}(s,s')=\mathrm{m}(s,s'')=3$. *Then for each $w\in W$ and each $k<\ell(w)$ there are at most two different decompositions $w=uv$ such that $\ell(u)=k$ and $\ell(v)=\ell(w)-k$.*

(ii) *Assume that there are no s, s' and s'' such that* $\mathrm{m}(s,s')=\mathrm{m}(s,s'')=\mathrm{m}(s',s'')=3$. *Then for each $w\in W$ and each $k<\ell(w)$ there are at most three different decompositions $w=uv$ such that $\ell(u)=k$ and $\ell(v)=\ell(w)-k$.*

Moreover if $w=uv=u'v'$ are decompositions such that

$$\ell(u)+\ell(v)=\ell(u')+\ell(v')=\ell(w)$$

and $\ell(u)=\ell(u')$, then $\ell(u^{-1}u')\le M$, where $M=\max\{\mathrm{m}(s,s')\mid s,s'\in S,\ \mathrm{m}(s,s')<\infty\}$.

Proof. (i) For a subset $X\subset W$ let

$$\mathrm{T}(X)=\{ws\mid w\in X,\ s\in \mathrm{C}(w)\}.$$

Let $\mathrm{T}^{j+1}=\mathrm{T}(\mathrm{T}^j)$. The statement will be proved if we show that $\mathrm{card}\{\mathrm{T}^j(w)\}\le 2$ for every $w\in W$ and $j<\ell(w)$. The proof is by induction on $\ell(w)$. Assume that $\mathrm{k}(w)=\ell(w)$, i.e. the last letter of w is unique. Hence $\mathrm{T}(w)$ contains a single element, say w', such that $\ell(w')=\ell(w)-1$. By the induction hypothesis we get the conclusion.

Assume now that there are $a\ne b\in S$ such that $\ell(wa)=\ell(wb)=\ell(w)-1$. By Proposition 3 any expansion for w ends in either a or b. Moreover by Propositions 1 and 2 any expansion for w which ends in a must end in $\mathrm{N}(b,a)$. Similarly any expansion which ends in b must end in $\mathrm{N}(a,b)$. Therefore any reduced expansion for w must be of the form either

$$w=w_{(1)}\mathrm{N}(a,b)\quad\text{or}\quad w=w_{(2)}\mathrm{N}(b,a). \tag{26}$$

Set $l=\mathrm{m}(a,b)-1$. Thus $\mathrm{card}\{\mathrm{T}^j(w)\}=2$ for $1\le j\le l$. Moreover

$$\mathrm{T}^l(w)=\{w_1,w_2\}. \tag{27}$$

Define $s=\mathrm{c}(a,b)$ and $s'=\mathrm{d}(a,b)$. We have (see (1))

$$wa=w_{(1)}\mathrm{N}(a,b)a=w_{(1)}s\mathrm{N}(a,b),$$
$$wa=w_{(2)}\mathrm{N}(b,a)a=w_{(2)}s'\mathrm{N}(a,b).$$

Therefore $w_{(1)}s=w_{(2)}s'$. Moreover by Lemma 2 and (26) the products $w_{(1)}s$ and $w_{(2)}s'$ are not reduced. Let $v=w_{(1)}s=w_{(2)}s'$. Thus the expressions $w_{(1)}=vs$, $w_{(2)}=vs'$ are reduced. Since $s\ne s'$, Lemma 3(i) shows that either

$$\mathrm{C}(w_{(1)})=\{s\}\quad\text{or}\quad \mathrm{C}(w_{(2)})=\{s'\}.$$

Thus either

$$\mathrm{T}(w_{(1)}) = \{v\} \quad \text{or} \quad \mathrm{T}(w_{(2)}) = \{v\}.$$

As $w_{(1)} = vs$ and $w_{(2)} = vs'$ we have $v \in \mathrm{T}(w_{(i)})$ for $i = 1, 2$. Thus either

$$\mathrm{T}(w_{(1)}, w_{(2)}) = \mathrm{T}(w_{(1)}) \quad \text{or} \quad \mathrm{T}(w_{(1)}, w_{(2)}) = \mathrm{T}(w_{(2)}).$$

Combining this with (27) and applying the induction hypothesis implies $\operatorname{card}\{\mathrm{T}^j(w)\} \le 2$ for every $j < \ell(w)$.

We now turn to (ii). Again we will use induction to show that $\operatorname{card}\{\mathrm{T}(w)\} \le 3$. In doing so we can follow the lines of the proof of (ii) until we arrive at the place where Lemma 3(i) is applied. Let $v = v_1 \dots v_n$ be a reduced expansion for v. If $\mathrm{m}(v_n, s) \ge 4$ or $\mathrm{m}(v_n, s') \ge 4$, then we can apply Lemma 3(i) and conclude as in the proof of (i). Thus it suffices to consider the case $\mathrm{m}(v_n, s) = 3$ and $\mathrm{m}(v_n, s') = 3$. We can also assume that $\mathrm{C}(vs) = \{v_n, s\}$ and $\mathrm{C}(vs') = \{v_n, s'\}$, because if either vs or vs' has a unique last letter then, again, we can conclude as in the proof of (i). By Lemma 3(ii), v ends in $\mathrm{N}(s, s')v_n$ or in $\mathrm{N}(s', s)v_n$. Assume the former, i.e. $v = v''\mathrm{N}(s, s')v_n$ is reduced for some $v'' \in W$. Then, as $\mathrm{N}(v_n, s) = v_n s$, we have

$$vsv_n = v''\mathrm{N}(s, s')\mathrm{N}(v_n, s)v_n.$$

Since $\ell(vsv_n) < \ell(vs)$, applying Lemma 2 we find that $v''\mathrm{c}(s, s')$ is not reduced. Hence v'' can be written as $v'' = v'\mathrm{c}(s, s')$, where $\ell(v') = \ell(v'') - 1$. Therefore we can represent v as a reduced product as follows:

$$v = v'\mathrm{c}(s, s')\mathrm{N}(s, s')v_n.$$

Now, since $w_{(1)} = vs$ and $w_{(2)} = vs'$, we can compute easily that

$$\begin{aligned} \mathrm{T}(w_{(1)}, w_{(2)}) &= \{v'\mathrm{c}(s, s')\mathrm{N}(s, s')v_n,\ v'\mathrm{N}(s, s')v_n s,\ v'\mathrm{N}(s', s)v_n s'\}, \\ \mathrm{T}^2(w_{(1)}, w_{(2)}) &= \{v'\mathrm{c}(s, s')\mathrm{N}(s, s'),\ v'\mathrm{N}(s, s')v_n,\ v'\mathrm{N}(s', s)v_n\}. \end{aligned} \tag{28}$$

The second equality follows from the fact that all three elements in $\mathrm{T}(w_{(1)}, w_{(2)})$ have unique last letters. From (28) we get

$$\mathrm{T}^3(w_{(1)}, w_{(2)}) = T(v'\mathrm{c}(s, s')\mathrm{N}(s, s')) = \{v'\mathrm{N}(s, s'), v'\mathrm{N}(s', s)\}, \tag{29}$$

because $v'\mathrm{N}(s, s')v_n$ and $v'\mathrm{N}(s', s)v_n$ have unique last letters. Summarizing, we have shown that $\operatorname{card}\{\mathrm{T}^j(w)\} \le 3$ for $j \le k = \mathrm{m}(a, b) + 2$ and $\mathrm{T}^k(w) = \mathrm{T}(v'\mathrm{c}(s, s')\mathrm{N}(s, s'))$. We now apply the induction hypothesis to conclude that $\operatorname{card}\{\mathrm{T}^j(w)\} \le 3$ for $j \le \ell(w)$.

The estimate for the distance $\ell(u^{-1}u')$ can be derived easily from the proof. ∎

Remark. The statement is not true if we allow $\mathrm{m}(a, b) = \mathrm{m}(a, c) = \mathrm{m}(a, b) = 3$. Indeed, the Cayley graph of the group generated by a, b and c yields a hexagonal tiling of the plane. Then one can easily find two vertices

x and y for which there are arbitrarily many intermediate vertices at a given distance from x and y.

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