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ASYMPTOTIC ANALYSIS OF THE INITIAL BOUNDARY VALUE PROBLEM FOR THE THERMOELASTIC SYSTEM IN A PERFORATED DOMAIN

BY

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Abstract. We study the initial boundary value problem for the system of thermoelasticity in a sequence of perforated cylindrical domains $Q_T^{(s)}$, $s = 1, 2, \dots$ We prove that as $s \to \infty$, the solution of the problem converges in appropriate topologies to the solution of a limit initial boundary value problem of the same type but containing some additional terms which are expressed in terms of quantities related to the geometry of $Q_T^{(s)}$. We give an explicit construction of that limit problem.

1. Introduction. Let Ω be a bounded domain in $\mathbb{R}^3$ with boundary $\partial\Omega$. For $0 < T < \infty$, we denote by Q_T the cylinder $\Omega \times (0, T)$. Let $F_i^{(s)}$, $i = 1, \dots, I(s)$, $s = 1, 2, \dots$, be a sequence of closed sets in Ω. We set $F^{(s)} = \bigcup_{i=1}^{I(s)} F_i^{(s)}$ and $\Omega^{(s)} = \Omega \setminus F^{(s)}$; the boundary $\partial\Omega^{(s)}$ of $\Omega^{(s)}$ is assumed to be sufficiently smooth (e.g., of class C^2). We later formulate some conditions on $F_i^{(s)}$ from which it follows in particular that these sets vanish as $s \to \infty$. In the cylindrical domain $Q_T^{(s)} = \Omega^{(s)} \times (0, T)$, we look for a field of displacements $u^{(s)} = u^{(s)}(x, t) = (u_1^{(s)}(x, t), u_2^{(s)}(x, t), u_3^{(s)}(x, t))$ and a field of temperatures $\theta^{(s)} = \theta^{(s)}(x, t)$ satisfying the initial boundary value problem of thermoelasticity

$$(1)\qquad \frac{\partial^2 u_l^{(s)}}{\partial t^2} - \frac{\partial}{\partial x_h}\left(a_{lj}^{hk}(x)\,\frac{\partial u_j^{(s)}}{\partial x_k}\right) + \alpha\,\frac{\partial \theta^{(s)}}{\partial x_l} = f_l^{(s)} \quad \text{in } Q_T^{(s)},\ l = 1, 2, 3,$$

$$(2)\qquad \frac{\partial \theta^{(s)}}{\partial t} - \Delta\theta^{(s)} + \alpha\,\mathrm{div}\left(\frac{\partial u^{(s)}}{\partial t}\right) = q^{(s)} \quad \text{in } Q_T^{(s)},$$

$$(3)\qquad \left.\begin{aligned} u^{(s)}(x, t) = 0 \\ \theta^{(s)}(x, t) = 0 \end{aligned}\right\} \quad \text{on } \partial\Omega^{(s)} \times (0, T),$$

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$$
\left.\begin{aligned}
u^{(s)}(x,0) &= u_0^{(s)}(x)\\
\frac{\partial u^{(s)}}{\partial t}(x,0) &= u_1^{(s)}(x)\\
\theta^{(s)}(x,0) &= \theta_0^{(s)}(x)
\end{aligned}\right\}\quad \text{in } \Omega^{(s)}, \tag{4}
$$

where $f_i^{(s)}$ is the ith component of a column vector $f^{(s)} = f^{(s)}(x,t)$ defined in Q_T, and $q^{(s)} = q^{(s)}(x,t)$ is a scalar function defined in Ω; Δ and div denote respectively the Laplace and divergence operators; $\partial\bullet$ denotes the boundary of a set $\bullet$. By ∇ or $\partial/\partial x$ we shall denote the gradient. Here and later on a pair of equal indices will mean summation from 1 to 3. We assume that the functions $a_{lj}^{hk}(x)$ are continuously differentiable in $\overline{\Omega}$ and satisfy the following conditions:

$$
a_{lj}^{hk}(x) = a_{jl}^{kh}(x) = a_{hj}^{lk}(x), \tag{5}
$$

and the condition of ellipticity:

$$
\kappa_1\eta_{lh}\eta_{lh} \le a_{lj}^{hk}(x)\eta_{lh}\eta_{jk} \le \kappa_2\eta_{lh}\eta_{lh}, \qquad x\in\overline{\Omega},\ \kappa_1,\kappa_2 = \text{const} > 0, \tag{6}
$$

where $\{\eta_{lh}\}$ is an arbitrary symmetric matrix with real entries.

In the classical elasticity theory for an isotropic material the coefficients $a_{lj}^{hk}(x)$ are given by the formula

$$
a_{lj}^{hk}(x) = \lambda\delta_{lh}\delta_{jk} + \mu(\delta_{lj}\delta_{hk} + \delta_{lk}\delta_{hj}), \tag{7}
$$

where $\lambda > 0$, $\mu > 0$ are the Lamé coefficients and δ_{lj} denotes the Kronecker symbol. In this case

$$
a_{lj}^{hk}\eta_{lh}\eta_{jk} = \lambda\eta_{hh}\eta_{ll} + 2\mu\eta_{lh}\eta_{lh},
$$

for an arbitrary symmetric real matrix $\{\eta_{lh}\}$, and as is easily seen, the ellipticity condition (6) holds with $\kappa_1 = 2\mu$ and $\kappa_2 = 2\mu + 3\lambda$. For further information on problem (1)–(4) we refer to the monographs [12] and [18].

In this work we investigate the possibilities of approximating problem (1)–(4) in the perforated cylindrical domain $Q_T^{(s)}$ by a new homogenized problem in Q_T whose solution is the limit of the sequence of vector-functions $(u^{(s)}(x,t), \theta^{(s)}(x,t))$ as $s\to\infty$. Under appropriate conditions on the geometry of the set $\Omega^{(s)}$ from which it follows in particular that the set $F^{(s)}$ vanishes as $s\to\infty$, we prove that any sequence of solutions of (1)–(4) converges in suitable topologies to a solution of an initial boundary value problem in Q_T of the same type as (1)–(4) but containing some additional terms which are expressed in terms of some quantities connected to the geometry of $\Omega^{(s)}$. This phenomenon is well known in the elliptic case (see e.g. [19] and [21] in the case of elliptic systems for the type of perforations considered here). We also refer to the important paper [11] for another approach in the linear scalar elliptic case and to [1]–[3] for the corresponding extension to the case of the stationary systems of Stokes and Navier–Stokes; this approach leads to a limit problem with an additional term which involves a Radon measure. It is worth mentioning the papers [10], [8] and [6] where the ideas of [11] are used to investigate the non-homogeneous initial boundary value problem for the wave equation and the homogeneous initial boundary value problem for the viscoelastic system respectively. The techniques that we use in the present paper are inspired from their elliptic counterpart, in particular those developed in order to treat elliptic systems as in [21]. We use an existence and uniqueness result obtained in [13]. The homogenization of the thermoelastic system in the case of rapidly oscillating coefficients in a fixed domain has been investigated previously in [17] using the techniques of asymptotic expansions as in [5] and [20].

Since the initial conditions (4) are not homogeneous (the second condition in particular), it is more convenient to use an analytic approach for the type of homogenization problem that we consider here, bypassing the reduction of the problem to a stationary one through the application of the Laplace transform. We note that even if the use of the Laplace transform turned out to be successful as in the case of homogeneous initial conditions, it would have been mainly in the derivation of the limit problem. A rigorous asymptotic analysis such as provided by the framework that we use here is beyond the reach of the Laplace transform. Furthermore our approach immediately applies to situations when the coefficients in the system are time dependent. It is well known that the Laplace transform is powerless in this case.

We note that the problem studied here is of relevance in the theory of composite materials (see e.g. [7]) and in general in the study of evolution processes taking place in strongly inhomogeneous media.

The paper is organized as follows. In the next section, we state the main assumptions on problem (1)–(4) and the perforated domain $\Omega^{(s)}$, we formulate a theorem on existence and uniqueness from [13], and we introduce some functions that are solutions of auxiliary elliptic boundary problems in the neighborhood of the sets $F_i^{(s)}$; the geometry of $\Omega^{(s)}$ is closely related to those functions whose a priori pointwise estimates play a central role in our investigations. In Section 3, we construct some asymptotic expansions with remainder terms for the solution $(u^{(s)}, \theta^{(s)})$ of problem (1)–(4) and we prove that the remainder terms converge to zero in suitable topologies, thus justifying the expansions. In Section 4, we prove our main result by constructing the initial boundary value problem satisfied by the limit of $(u^{(s)}, \theta^{(s)})$ as $s \to \infty$.

2. Preliminary results. We shall use the following function spaces as defined in [15]: $C(\cdot)$, $C^l(\cdot)$, $C_0^\infty(\cdot)$, $L^p(\cdot)$, $W_p^1(\cdot)$, $\mathring{W}_p^1(\cdot)$, $H_0^1(\cdot)$, $H^{-1}(\cdot)$, $Y(0,T,X)$, where Y and X can be one of the previous spaces; X can be a direct product of one of them, l and p can assume any non-negative integer value. We denote by X' the dual of X.

DEFINITION 1. We call the pair $(u^{(s)},\theta^{(s)})$ a *solution* of problem (1)–(4) if for all s, $u^{(s)}(x,0)=u_0^{(s)}(x)$ and $(u^{(s)},\theta^{(s)})$ satisfy the conditions

$$u^{(s)}\in L^2(0,T,(H_0^1(\Omega^{(s)}))^3),\qquad \theta^{(s)}\in L^2(0,T,H_0^1(\Omega^{(s)})),$$
$$\frac{\partial u^{(s)}}{\partial t}\in L^2(0,T,(L^2(\Omega^{(s)}))^3),\qquad \frac{\partial\theta^{(s)}}{\partial t}\in L^2(0,T,L^2(\Omega^{(s)})),$$
$$\frac{\partial^2 u^{(s)}}{\partial t^2}\in L^2(0,T,(H_0^1(\Omega^{(s)}))^{3\prime}),\qquad \theta^{(s)}\in L^2(0,T,L^2(\Omega^{(s)})),$$

and the integral identities

$$\int_{Q_T^{(s)}} -\frac{\partial u^{(s)}}{\partial t}\frac{\partial\varphi}{\partial t}\,dx\,dt + A(u^{(s)},\varphi)+C_1(\theta^{(s)},\varphi)-\int_{\Omega^{(s)}} u_1^{(s)}(x)\varphi(x,0)\,dx = \int_{Q_T^{(s)}} f^{(s)}(x,t)\varphi(x,t)\,dx\,dt, \tag{8}$$

$$\int_{Q_T^{(s)}} -\theta^{(s)}\frac{\partial\psi}{\partial t}\,dx\,dt + K(\theta^{(s)},\psi)+C_2\Big(\frac{\partial u^{(s)}}{\partial t},\psi\Big) - \int_{\Omega^{(s)}}\theta_0^{(s)}(x)\psi(x,0)\,dx = \int_{Q_T^{(s)}} q^{(s)}(x,t)\psi(x,t)\,dx\,dt, \tag{9}$$

for all vector-functions $\varphi=\varphi(x,t)=(\varphi_1,\varphi_2,\varphi_3)\in L^2(0,T,(H_0^1(\Omega^{(s)}))^3)$, with $\partial\varphi/\partial t\in L^2(0,T,(L^2(\Omega^{(s)}))^3)$, $\varphi(x,T)=0$, and scalar functions $\psi=\psi(x,t)\in L^2(0,T,H_0^1(\Omega^{(s)}))$ with $\partial\psi/\partial t\in L^2(0,T,L^2(\Omega^{(s)}))$, $\psi(x,T)=0$, where

$$\begin{aligned} A(v,\zeta)&=\int_{Q_T^{(s)}} a_{lj}^{hk}(x)\frac{\partial v}{\partial x_k}\frac{\partial\zeta_l}{\partial x_h}\,dx\,dt, & C_1(w,\zeta)&=\int_{Q_T^{(s)}}\alpha\zeta_l\frac{\partial w}{\partial x_l}\,dx\,dt,\\ K(w,\xi)&=\int_{Q_T^{(s)}}\frac{\partial w}{\partial x_l}\frac{\partial\xi}{\partial x_l}\,dx\,dt, & C_2\Big(\frac{\partial v}{\partial t},\xi\Big)&=-\int_{Q_T^{(s)}}\frac{\partial v_l}{\partial t}\frac{\partial\xi}{\partial x_l}\,dx\,dt. \end{aligned} \tag{10}$$

Throughout we assume the following conditions on the data: For any $s=1,\ldots,$

$$\begin{aligned} &f^{(s)}\in L^2(0,T,(L^2(\Omega^{(s)}))^3),\qquad q^{(s)}\in L^2(0,T,L^2(\Omega^{(s)})),\\ &u_0^{(s)}\in (H_0^1(\Omega^{(s)}))^3,\qquad u_1^{(s)}\in (L^2(\Omega^{(s)}))^3,\qquad \theta_0^{(s)}\in H_0^1(\Omega^{(s)}). \end{aligned} \tag{11}$$

REMARK 2. If $u^{(s)}$ and $\theta^{(s)}$ satisfy the conditions imposed in Definition 1 and if the above conditions on the data hold, then

$$u^{(s)} \in C([0,T],(H_0^1(\Omega^{(s)}))^3), \qquad \frac{\partial u^{(s)}}{\partial t} \in C([0,T],(L^2(\Omega^{(s)}))^3),$$
$$\theta^{(s)} \in C([0,T],L^2(\Omega^{(s)})) \cap L^2(0,T,H_0^1(\Omega^{(s)})),$$

that is, the above functions are continuous with respect to t on the interval $[0,T]$.

We have the following existence and uniqueness result which can be established by adapting the arguments used by Lions and Duvaut in [13] and [14] and based on the Galerkin method.

THEOREM 3. *Under the above conditions*, *problem* (1)–(4) *has a unique solution* $(u^{(s)},\theta^{(s)})$ *which satisfies the a priori estimates*

$$\operatorname*{ess\,sup}_{0\le t\le T}\left(\|u^{(s)}(t)\|_{(H_0^1(\Omega^{(s)}))^3} + \left\|\frac{\partial u^{(s)}(t)}{\partial t}\right\|_{(L^2(\Omega^{(s)}))^3}\right) + \left\|\frac{\partial^2 u^{(s)}(t)}{\partial t^2}\right\|_{L^2(0,T,(H_0^1(\Omega^{(s)}))^{3\prime})} \le C_1,$$

$$\operatorname*{ess\,sup}_{0\le t\le T}\|\theta^{(s)}(t)\|_{L^2(\Omega^{(s)})} + \|\theta^{(s)}\|_{L^2(0,T,H_0^1(\Omega^{(s)}))} + \left\|\frac{\partial \theta^{(s)}}{\partial t}\right\|_{L^2(0,T,H^{-1}(\Omega^{(s)}))} \le C_2,$$

where C_1 *and* C_2 *are constants independent of* s.

Extend the vector-functions $(u^{(s)},\theta^{(s)})$ to Q_T by setting them equal to zero in $F^{(s)}\times(0,T)$ and denote the resulting functions by the same symbols. From the theorem, we deduce that $u^{(s)}$, $\partial u^{(s)}/\partial t$ and $\partial^2 u^{(s)}/\partial t^2$ are bounded in $L^\infty(0,T,(H_0^1(\Omega))^3)$, $L^\infty(0,T,(L^2(\Omega))^3)$ and $L^2(0,T,(H_0^1(\Omega))^{3\prime})$, respectively. Analogously $\theta^{(s)}$ and $\partial\theta^{(s)}/\partial t$ are bounded in $L^\infty(0,T,L^2(\Omega)) \cap L^2(0,T,H_0^1(\Omega))$ and $L^2(0,T,H^{-1}(\Omega))$. Thus there exist functions u, v_1, v_2, θ and w defined in the cylinder Q_T such that

$$\begin{aligned}
& u^{(s)} \rightharpoonup u && \text{weakly* in } L^\infty(0,T,(H_0^1(\Omega))^3),\\
& \frac{\partial u^{(s)}}{\partial t} \rightharpoonup v_1 && \text{weakly* in } L^\infty(0,T,(L^2(\Omega))^3),\\
& \frac{\partial^2 u^{(s)}}{\partial t^2} \rightharpoonup v_2 && \text{weakly* in } L^2(0,T,(H_0^1(\Omega))^{3\prime}),\\
& \theta^{(s)} \rightharpoonup \theta && \text{weakly in } L^2(0,T,H_0^1(\Omega))\\
& && \qquad\text{and weakly* in } L^\infty(0,T,L^2(\Omega)),\\
& \frac{\partial \theta^{(s)}}{\partial t} \rightharpoonup w && \text{weakly in } L^2(0,T,H^{-1}(\Omega)).
\end{aligned} \tag{12}$$

We note that all the weak* convergences in (12) can be replaced by weak convergences in the corresponding spaces with L^∞ changed into L^2. It can be shown (see e.g. Evans [15, Problems 4, 5, pp. 425–426]) that $v_1 = \partial u/\partial t$, $v_2 = \partial^2 u/\partial t^2$, $w = \partial\theta/\partial t$ and that u, $\partial u/\partial t$, $\partial^2 u/\partial t^2$, θ, $\partial\theta/\partial t$ belong to the same function spaces as $u^{(s)}$, $\partial u^{(s)}/\partial t$, $\partial u^{(s)}/\partial t^2$, $\theta^{(s)}$, $\partial\theta^{(s)}/\partial t$ respectively.

Assuming that the data functions are extended to Q_T or to Ω by setting them equal to zero on $F^{(s)} \times (0,T)$ or on $F^{(s)}$, depending on the set on which they are defined, we require that the following convergences hold:

$$
\begin{aligned}
f^{(s)} &\rightharpoonup f && \text{weakly in } L^2(0,T,(L^2(\Omega))^3),\\
q^{(s)} &\rightharpoonup q && \text{weakly in } L^2(0,T,L^2(\Omega)),\\
u_0^{(s)} &\rightharpoonup u_0 && \text{weakly in } (H_0^1(\Omega))^3,\\
u_1^{(s)} &\rightharpoonup u_1 && \text{weakly in } (L^2(\Omega))^3,\\
\theta_0^{(s)} &\rightharpoonup \theta_0 && \text{weakly in } H_0^1(\Omega).
\end{aligned}
\tag{13}
$$

Our goal is to determine the initial boundary value problem satisfied by the vector-function (u,θ). To do that, we need some geometric conditions on $\Omega^{(s)}$. Let us introduce a few notations. Let $B(x,\varrho)$ be the ball of radius ϱ centered at x; $d_i^{(s)} = \min\{\varrho : F_i^{(s)} \subset B(x,\varrho)\}$; $x_i^{(s)}$ denotes the center of the ball of radius $d_i^{(s)}$ such that $F_i^{(s)} \subset \overline{B(x_i^{(s)}, d_i^{(s)})}$ (closed ball); $r_i^{(s)}$ denotes the distance between $B(x_i^{(s)}, d_i^{(s)})$ and $\bigcup_{i\neq j} B(x_j^{(s)}, d_j^{(s)}) \cup \partial\Omega$.

From now we assume that the balls $B(x_i^{(s)},1)$ lie inside Ω and that $d_i^{(s)} < 1/2$ for all i and s. Let $\dot{B}_i^{(s)} = B(x_i^{(s)},1) \setminus F_i^{(s)}$. We introduce the auxiliary vector-functions $(v_i^{r(s)}(x), \lambda_i^{(s)}(x)) \in (H^1(\dot{B}_i^{(s)}))^3 \times H^1(\dot{B}_i^{(s)})$, $r = 1,2,3$, which satisfy the following two elliptic boundary value problems:

$$
\frac{\partial}{\partial x_h}\left(a_{lj}^{hk}(x_i^{(s)})\,\frac{\partial v_{ij}^{r(s)}}{\partial x_k}\right) = 0 \qquad \text{in } \dot{B}_i^{(s)},\ l = 1,2,3, \tag{14}
$$

$$
v_i^{r(s)}(x) = e^{(r)} \qquad \text{on } \partial F_i^{(s)}, \tag{15}
$$

$$
v_i^{r(s)}(x) = 0 \qquad \text{on } \partial B(x_i^{(s)},1); \tag{16}
$$

$$
\Delta\lambda_i^{(s)} = 0 \qquad \text{in } B(x_i^{(s)},1) \setminus F_i^{(s)}, \tag{17}
$$

$$
\lambda_i^{(s)}(x) = 1 \qquad \text{on } \partial F_i^{(s)}, \tag{18}
$$

$$
\lambda_i^{(s)}(x) = 0 \qquad \text{on } \partial B(x_i^{(s)},1), \tag{19}
$$

where $e^{(r)}$ is the 3-vector whose rth component equals 1 and the remaining ones are 0, and $v_{ij}^{r(s)}$ is the jth component of $v_i^{r(s)}$. We extend the vector-function $(v_i^{r(s)}, \lambda_i^{(s)})$ to Ω, by setting it equal to $(e^{(r)},1)$ on $F_i^{(s)}$ and equal to zero outside $B(x_i^{(s)},1)$. It is well known that problems (14)–(16) (resp.

(17)–(19)) are uniquely solvable; we refer for instance to Kupradze [18] (resp. Evans [15]). In particular, from [18], we see that under our smoothness conditions, $v_i^{r(s)}$ is a regular (classical) solution of (14)–(16), i.e., $v_i^{r(s)} \in (C^2(\dot{B}_i^{(s)}))^3 \cap (C^1(\overline{\dot{B}_i^{(s)}}))^3$. Analogously $\lambda_i^{(s)} \in C^2(\dot{B}_i^{(s)}) \cap C^1(\overline{\dot{B}_i^{(s)}})$.

We introduce the matrix $M(F_i^{(s)}) = \{M^{rq}(F_i^{(s)})\}_{r,q=1}^3$ where

$$M^{rq}(F_i^{(s)}) = \int\limits_{B(x_i^{(s)},1)} a_{lj}^{hk}(x_i^{(s)}) \frac{\partial v_{ij}^{r(s)}(x)}{\partial x_k} \frac{\partial v_{ij}^{q(s)}(x)}{\partial x_h}\, dx, \qquad r,q = 1,2,3,$$

and the functions $v_i^{r(s)}$ are solutions of problem (14)–(16). We also set

$$C(F_i^{(s)}) = \int\limits_{B(x_l^{(s)},1)} |\nabla \lambda_i^{(s)}(x)|^2\, dx,$$

where $\lambda_i^{(s)}$ is a solution of (17)–(19); in $C(F_i^{(s)})$ we recognize the capacity of the set $F_i^{(s)}$ (see e.g. Evans and Gariepy [16] for the definition of capacities). The following pointwise a priori estimates are particular cases of those obtained in [19, Chap. 2]:

$$|v_i^{r(s)}(x)| \le C_1 \frac{d_i^{(s)}}{|x - x_i^{(s)}|}, \qquad d_i^{(s)} < |x - x_i^{(s)}| < 1, \tag{20}$$

$$\int\limits_{B(x_i^{(s)},1)} \left| \frac{\partial v_i^{r(s)}}{\partial x} \right|^2 dx \le C_2 d_i^{(s)}, \tag{21}$$

$$|\lambda_i^{(s)}(x)| \le C_3 \frac{d_i^{(s)}}{|x - x_i^{(s)}|}, \qquad d_i^{(s)} < |x - x_i^{(s)}| < 1, \tag{22}$$

$$\int\limits_{B(x_i^{(s)},1)} |\nabla \lambda_i^{(s)}|^2\, dx \le C_4 d_i^{(s)}, \tag{23}$$

where the constants C_l are independent of s.

We require the following hypotheses on the perforated domain $\Omega^{(s)}$:

(H1) $$d_i^{(s)} \le A_1 r_i^{(s)}, \qquad \lim_{s\to\infty} \max_{1\le i\le I(s)} \{r_i^{(s)}\} = 0.$$

(H2) $$\sum_{i=1}^{I(s)} \frac{[d_i^{(s)}]^2}{[r_i^{(s)}]^3} \le A_2.$$

(H3) There exist a 3×3 matrix-function $m(x) = \{m_{rq}(x)\}_{r,q=1}^3$ with bounded entries and a bounded function $c(x)$ such that for all sets $G \subset \Omega$,

$$\lim_{s\to\infty} \sum_{i\in I(G)} M(F_i^{(s)}) = \int_G m(x)\,dx,$$

$$\lim_{s\to\infty} \sum_{i\in I(G)} C(F_i^{(s)}) = \int_G c(x)\,dx,$$

where $I(G)$ is the set of indices i such that $F_i^{(s)} \subset G$.

Our main result is

THEOREM 4. *Let conditions* (5), (6), (13), (H1), (H2) *and* (H3) *be satisfied and let* $(u^{(s)}, \theta^{(s)})$ *be the sequence of solutions of problem* (1)–(4) *in the sense of Definition* 1 *such that* $\partial^2 u^{(s)}/\partial t^2$ *and* $\partial\theta^{(s)}/\partial t$ *are uniformly bounded in* $L^2(0,T,(L^2(\Omega))^3)$ *and in* $L^2(0,T,L^2(\Omega))$, *respectively. Then* $(u^{(s)}, \theta^{(s)})$ *weakly converges in the sense of* (12) *to a weak solution of the initial boundary value problem*

$$\frac{\partial^2 u_l}{\partial t^2} - \frac{\partial}{\partial x_h}\left(a_{lj}^{hk}(x)\,\frac{\partial u_j}{\partial x_k}\right) + \alpha\,\frac{\partial \theta}{\partial x_l} + m_{lj}(x)u_j = f_l \quad \text{in } Q,\ \ l = 1,2,3, \tag{24}$$

$$\frac{\partial \theta}{\partial t} - \Delta\theta + \alpha\,\mathrm{div}\left(\frac{\partial u}{\partial t}\right) + c(x)\theta = q, \quad \text{in } Q, \tag{25}$$

$$\left.\begin{aligned} u(x,t) &= 0\\ \theta(x,t) &= 0 \end{aligned}\right\} \quad \text{on } \partial\Omega \times (0,T) \tag{26}$$

$$\left.\begin{aligned} u(x,0) &= u_0(x)\\ \frac{\partial u}{\partial t}(x,0) &= u_1(x)\\ \theta(x,0) &= \theta_0(x) \end{aligned}\right\} \quad \text{in } \Omega. \tag{27}$$

Furthermore $(u^{(s)}, \theta^{(s)})$ *strongly converges to* (u,θ) *in* $L^2(0,T,(\mathring{W}_p^1(\Omega))^3) \times L^2(0,T,\mathring{W}_p^1(\Omega))$ *for all* $1 < p < 2$.

REMARK 5. The limit problem (24)–(27) differs from the original problem (1)–(4) by some additional terms containing the functions $m(x)$ and $c(x)$ from hypothesis (H3). In certain cases these functions can be derived explicitly.

Let us cover $\mathbb{R}^3$ with a sequence of cubes $K(x_i^{(s)}, \varepsilon)$ of edge ε centered at the points $x_i^{(s)}$. We remove from each $K(x_i^{(s)}, \varepsilon)$ a ball $B_i^{(s)} = B(x_i^{(s)}, d_i^{(s)})$, $i = 1, \ldots, s$, with $2d_i^{(s)} \equiv d < \varepsilon/3$, such that $\varepsilon \to 0$ as $s \to \infty$. Thus the points $x_i^{(s)}$ form a periodic 3-dimensional lattice of period ε. Let $\Omega \subset \mathbb{R}^3$ be a domain whose boundary does not intersect any $B_i^{(s)}$. Let $\Omega^{(s)} = \Omega \setminus \bigcup_i B_i^{(s)}$, $s = 1, 2, \ldots$ We consider problem (1)–(4) in $\Omega^{(s)} \times [0,T]$ and problems (14)–(19) with $B(x_i^{(s)}, 1)$ (resp. $F_i^{(s)}$) replaced by $B(x_i^{(s)}, \varepsilon/2)$

(resp. $B_i^{(s)}$) and such that the coefficients in (14) are given by formula (7). Choosing $d = \varepsilon^3$, it can be shown that conditions (H1) and (H2) of Theorem 4 are satisfied and $c(x) = c = 2\pi$. Furthermore

$$m(x) = C(\mu, \lambda) a(x) E,$$

where E is the 3×3 unit matrix, C is a constant depending on μ and λ and

$$\int_G a(x)\, dx = \lim_{s\to\infty} \sum_{i\in I(G)} d_i^{(s)}.$$

We refer to [19, Chap.1 (pp. 55–56), Chap. 2 (pp. 167–169)] and to [6] for details and references.

3. Asymptotic expansion of $(u^{(s)}, \theta^{(s)})$**.** In this section we construct an asymptotic expansion with remainder term for a solution $(u^{(s)}, \theta^{(s)})$ of problem (1)–(4). We justify the asymptotic expansion by proving that the remainder term converges to zero in suitable topologies. As a corollary we get the second assertion of Theorem 4. We start by introducing appropriate test functions.

Let

$$\varrho_i^{(s)} = \max\left\{\left(1 + \frac{1}{2A_1}\right) d_i^{(s)}, \frac{1}{2A_4}\,(r_i^{(s)})^3 \ln^2 r_i^{(s)}\right\},$$

where A_1 is the constant from hypothesis (H1) and

$$A_4 = \max_{0<t\le \operatorname{diam}\Omega} \{t^2 \ln^2 t\}.$$

It is easy to show that $\varrho_i^{(s)} \le d_i^{(s)} + r_i^{(s)}/2$ and that $B(x_i^{(s)}, d_i^{(s)} + r_i^{(s)}/2) \cap B(x_j^{(s)}, d_j^{(s)} + r_j^{(s)}/2) = \emptyset$ for $i \ne j$. Let θ_1 and θ_2 be such that $0 < \theta_2 < \theta_1 < 1$. We consider the functions $\psi_i^{(s)} \in C_0^\infty(\mathbb{R}^3)$ such that $0 \le \psi_i^{(s)}(x) \le 1$, $\psi_i^{(s)}(x) = 0$ for $|x - x_i^{(s)}| \ge \theta_1 \varrho_i^{(s)}$, $\psi_i^{(s)}(x) = 1$ for $|x - x_i^{(s)}| \le \theta_2 \varrho_i^{(s)}$, $|\partial\psi_i^{(s)}/\partial x| \le C/\varrho_i^{(s)}$; C is a constant independent of s. We define $D_i^{(s)} = B(x_i^{(s)}, \theta_1 \varrho_i^{(s)})$ and

$$I_s' = \left\{i = 1, \dots, I(s) : \left(1 + \frac{1}{2A_1}\right) d_i^{(s)} \ge \frac{1}{2A_4}\,(r_i^{(s)})^3 \ln^2 r_i^{(s)}\right\},$$

$$I_s'' = \left\{i = 1, \dots, I(s) : \left(1 + \frac{1}{2A_1}\right) d_i^{(s)} < \frac{1}{2A_4}\,(r_i^{(s)})^3 \ln^2 r_i^{(s)}\right\}.$$

Simple calculations show the following relations (see [21]):

$$\lim_{s\to\infty} \sum_{i\in I_s'} d_i^{(s)} = 0, \tag{28}$$

$$\lim_{s\to\infty} \sum_{i\in I_s''} [\varrho_i^{(s)}]^3 = 0, \tag{29}$$

$$\sum_{i=1}^{I(s)} d_i^{(s)} < K_1, \tag{30}$$

$$\sum_{i=1}^{I(s)} [r_i^{(s)}]^3 < K_2, \tag{31}$$

where K_1 and K_2 are some constants independent of s.

By Remark 2 and the density result [9, Proposition 3.60], since $u \in L^\infty(0,T,(H_0^1(\Omega))^3) \cap C([0,T],(L_2(\Omega))^3)$ and $\theta \in L^2(0,T,H_0^1(\Omega)) \cap C([0,T], L^2(\Omega))$, there exist sequences of functions u_m and θ_m, $m = 1,2,\dots$, such that $u_m \in C^\infty([0,T],(C_0^\infty(\Omega))^3)$ and $\theta_m \in C^\infty([0,T], C_0^\infty(\Omega))$ and

$$\begin{aligned} u_m &\to u && \text{strongly in } C([0,T],(L^2(\Omega))^3),\\ u_m &\to u && \text{strongly in } L^\infty(0,T,(H_0^1(\Omega))^3),\\ \frac{\partial u_m}{\partial t} &\to \frac{\partial u}{\partial t} && \text{strongly in } C([0,T],L^2(\Omega)),\\ \theta_m &\to \theta && \text{strongly in } C([0,T],L^2(\Omega)),\\ \theta_m &\to \theta && \text{strongly in } L^2(0,T,H_0^1(\Omega)). \end{aligned} \tag{32}$$

In what follows we denote by C_m constants depending on m and independent of s.

We seek a solution $(u^{(s)},\theta^{(s)})$ of (1)–(4) in the form of the asymptotic expansions

$$u^{(s)}(x,t) = u_m(x,t) - H_{1s}(x,t) - H_{2s}(x,t) + R_{1s}(x,t), \tag{33}$$

$$\theta^{(s)}(x,t) = \theta_m(x,t) - H_{3s}(x,t) - H_{4s}(x,t) + R_{2s}(x,t), \tag{34}$$

where

$$H_{1s}(x,t) = \sum_{r=1}^{3}\sum_{i\in I_s'} v_i^{r(s)}(x) u_{mr}(x,t)\psi_i^{(s)}(x),$$

$$H_{2s}(x,t) = \sum_{r=1}^{3}\sum_{i\in I_s''} v_i^{r(s)}(x) u_{mr}(x,t)\psi_i^{(s)}(x),$$

$$H_{3s}(x,t) = \sum_{r=1}^{3}\sum_{i\in I_s'} \lambda_i^{(s)}(x) \theta_m(x,t)\psi_i^{(s)}(x),$$

$$H_{4s}(x,t) = \sum_{r=1}^{3}\sum_{i\in I_s''} \lambda_i^{(s)}(x) \theta_m(x,t)\psi_i^{(s)}(x),$$

$v_i^{r(s)}$, $\lambda_i^{(s)}$ are respectively the solutions of (14)–(16) and (17)–(19), u_{mr} is the rth component of u_m, R_{1s} and R_{2s} are the remainder terms; it is understood that the functions H_{ks} $(k = 1,\dots,4)$ and R_{ks} $(k = 1,2)$ depend on m but the omission in the notation is for sake of simplicity.

We establish the following corrector results which justify the expansions (33).

THEOREM 6. *Let the assumptions of Theorem* 4 *be satisfied. Then*

$$H_{1s} \to 0 \quad \text{strongly in } L^2(0,T,(H_0^1(\Omega^{(s)}))^3), \tag{35}$$

$$H_{2s} \to 0 \quad \text{strongly in } L^2(0,T,(W_p^1(\Omega^{(s)}))^3) \text{ for all } p \in (1,2), \tag{36}$$

$$\frac{\partial H_{1s}}{\partial t}, \frac{\partial H_{2s}}{\partial t} \to 0 \quad \text{strongly in } L^2(0,T,(L^2(\Omega))^3). \tag{37}$$

Proof. STEP 1. We show that

$$\|H_{1s}\|_{L^2(0,T,(H_0^1(\Omega^{(s)}))^3)} \to 0 \quad \text{as } s \to \infty. \tag{38}$$

We have

$$\|H_{1s}\|^2_{L^2(0,T,(H_0^1(\Omega^{(s)}))^3)} \le 2(I_{1s} + I_{2s} + I_{3s}), \tag{39}$$

where

$$I_{1s} = \sum_{r=1}^{3} \sum_{i \in I'_s} \int_0^T \int_{D_i^{(s)}} \left| \frac{\partial v_i^{r(s)}}{\partial x} \right|^2 |u_{mr}|^2 \, dx\, dt,$$

$$I_{2s} = \sum_{r=1}^{3} \sum_{i \in I'_s} \int_0^T \int_{D_i^{(s)}} \left| \frac{\partial u_{mr}}{\partial x} \right|^2 |v_i^{r(s)}|^2 \, dx\, dt,$$

$$I_{3s} = \sum_{r=1}^{3} \sum_{i \in I'_s} \int_0^T \int_{D_i^{(s)}} |v_i^{r(s)}|^2 |u_{mr}|^2 \left| \frac{\partial \psi_i^{(s)}}{\partial x} \right|^2 dx\, dt.$$

By (21) and the boundedness of u_m, we have

$$I_{1s} \le C_m \sum_{i \in I'_s} d_i^{(s)}. \tag{40}$$

Next by the boundedness of the gradient of u_m, Poincaré's inequality and inequality (21), we have

$$I_{2s} \le C_m \sum_{i \in I'_s} d_i^{(s)}. \tag{41}$$

Since $v_i^{r(s)} \in C^2(\dot{B}_i^{(s)}) \cap C^1(\overline{\dot{B}_i^{(s)}})$, it follows that $v_i^{r(s)}$ is bounded on $B(x_i^{(s)},1)$ thanks to its extension by $e^{(r)}$ in $F_i^{(s)}$. Thus, by the definition of $\psi_i^{(s)}$ and boundedness of $v_i^{r(s)}$,

$$(42)\quad I_{3s} \le C_m \sum_{r=1}^{3} \sum_{i\in I'_s} [\varrho_i^{(s)}]^{-2} \int_0^T \int_{D_i^{(s)}} |v_i^{r(s)}|^2\,dx \le C_m \sum_{i\in I'_s} \varrho_i^{(s)} \le C_m \sum_{i\in I's} d_i^{(s)}.$$

By (39)–(42), passing to the limit as $s\to\infty$ and using the fact that C_m is independent of s, we deduce from (28) that (38) holds. This proves (35).

STEP 2. We show that

$$(43)\qquad \lim_{s\to\infty} \|H_{2s}\|_{L^2(0,T,(\overset{\circ}{W}{}_p^1(\Omega^{(s)}))^3)} = 0.$$

We begin by establishing the inequality

$$(44)\qquad \|H_{2s}\|_{(H_0^1(\Omega))^3} < C_m, \quad \text{with } C_m \text{ independent of } t.$$

Arguing as in Step 1, we have

$$\begin{aligned}
\|H_{2s}\|^2_{(H_0^1(\Omega))^3} &\le \sum_{r=1}^{3} \sum_{i\in I''_s} \left\| \frac{\partial}{\partial x}(v_i^{r(s)} u_{mr} \psi_i^s) \right\|^2_{(L^2(\Omega))^3} \\
&\le C \sum_{r=1}^{3} \sum_{i\in I''_s} \Bigg\{ \int_{D_i^{(s)}} \left|\frac{\partial v_i^{r(s)}}{\partial x}\right|^2 |u_{mr}|^2\,dx + \int_{D_i^{(s)}} \left|\frac{\partial u_{mr}}{\partial x}\right|^2 |v_i^{r(s)}|^2\,dx \\
&\quad + \int_{D_i^{(s)}} \left|\frac{\partial \psi_i^{(s)}}{\partial x}\right|^2 |v_i^{r(s)}|^2 |u_{mr}|^2\,dx \Bigg\} \\
&\le C_m \sum_{r=1}^{3} \sum_{i\in I''_s} \Bigg\{ \int_{D_i^{(s)}} \left|\frac{\partial v_i^{r(s)}}{\partial x}\right|^2 dx + \int_{D_i^{(s)}} \left|\frac{\partial u_{kr}}{\partial x}\right|^2 dx \\
&\quad + [\varrho_i^{(s)}]^{-2} \int_{D_i^{(s)}\setminus B(x_i^{(s)},\theta_2\varrho_i^{(s)})} |v_i^{r(s)}|^2\,dx \Bigg\} \\
&\le C_m \Bigg\{ \sum_{i\in I''_s} d_i^{(s)} + \sum_{i\in I''_s} [\varrho_i^{(s)}]^3 + \int_{\bigcup_{i\in I''_s} D_i^{(s)}} \left|\frac{\partial u_{mr}}{\partial x}\right|^2 dx \Bigg\},
\end{aligned}$$

where we have used the properties of $\psi_i^{(s)}$ and u_m and inequalities (20) and (21). Thanks to (29) the second term in the last bracket converges to zero, the third term converges to zero by absolute continuity of integrals since $|\bigcup_{i\in I''_s} D_i^{(s)}| \to 0$, while the first term is bounded by (30). This proves (44).

Next since for all t, $H_{2s}(x,t)=0$ outside $\bigcup_{i\in I''_s} D_i^{(s)}$, we get

$$\|H_{2s}\|^2_{(\overset{\circ}{W}{}^1_p(\Omega))^3} \le C_m \|H_{2s}\|^2_{(H^1_0(\Omega))^3} \Big\{ \sum_{i\in I''_s} [\varrho_i^{(s)}]^3 \Big\}^{2/p-1}.$$

Integrating both sides over t from 0 to T, passing to the limit as $s\to\infty$, and using (29) and (44), we get (43) thanks to the fact that C_m is independent of s. This proves the relation (36). Using the same arguments as in the proofs of the two previous assertions of the theorem we easily get (37). The theorem is proved. ∎

The next result concerns the behavior of the remainder term in the asymptotic expansion (33).

THEOREM 7. *Let the conditions of Theorem* 4 *be satisfied. Then*

(45) $$R_{1s}(x,t)\to 0 \quad \textit{strongly in } L^2(0,T,(H^1_0(\Omega))^3),$$

(46) $$\frac{\partial R_{1s}}{\partial t}\to 0 \quad \textit{weakly in } L^2(0,T,(L^2(\Omega))^3).$$

Proof. The statement (46) is a straightforward consequence of the asymptotic expansion (33), and the convergences (12) and (37). By (33), (12) and Theorem 6, we see that $R_{1s}\in L^2(0,T,(H^1_0(\Omega^{(s)}))^3)$ and it converges weakly to zero in $L^2(0,T,(H^1_0(\Omega^{(s)}))^3)$. Since $\partial R_{1s}/\partial t$ converges weakly to zero in $L^2(0,T,(L^2(\Omega^{(s)}))^3)$, it follows from Aubin's Theorem [4] (see also [9, p. 61]) that R_{1s} strongly converges to zero in $L^2(0,T,(L^2(\Omega^{(s)}))^3)$. Multiplying both sides of the system (1) by R_{1s} and integrating over $Q_T^{(s)}$, we get

(47) $$\int_{Q_T^{(s)}} \frac{\partial^2 u^{(s)}}{\partial t^2} R_{1s}\,dx\,dt + A(u^{(s)},R_{1s}) + C_1(\theta^{(s)},R_{1s}) = \int_{Q_T^{(s)}} f(x,t)R_{1s}(x,t)\,dx\,dt.$$

It is obvious that strong convergence of R_{1s} to zero in $L^2(0,T,(L^2(\Omega^{(s)}))^3)$ implies

(48) $$\lim_{s\to\infty}\bigg(|C_1(\theta^{(s)},R_{1s})| + \bigg|\int_{Q_T^{(s)}} f(x,t)R_{1s}(x,t)\,dx\,dt\bigg| + \bigg|\int_{Q_T^{(s)}} \frac{\partial^2 u^{(s)}}{\partial t^2} R_{1s}\,dx\,dt\bigg|\bigg)=0;$$

for the estimation of the last term on the left-hand side of this equality we have used the assumption on uniform boundedness of $\partial^2 u^{(s)}/\partial t^2$ in $L^2(0,T,(L^2(\Omega))^3)$. We now proceed to the estimation of the second term on the left-hand side of (47). Since $v_i^{r(s)}$ is a weak solution of problem (14)–(16), it satisfies the integral identity

$$\int\limits_{B(x_i^{(s)},1)\setminus F_i^{(s)}} a_{lj}^{hk}(x)\,\frac{\partial v_{ij}^{r(s)}}{\partial x_k}\,\frac{\partial \varphi_l}{\partial x_h}\,dx = 0 \tag{49}$$

for all $\varphi(x) = (\varphi_1, \varphi_1, \varphi_3) \in (H_0^1(B(x_i^{(s)},1) \setminus F_i^{(s)}))^3$.

We write

$$A(u^{(s)}, R_{1s}) = A(u, R_{1s}) - A(H_{1s}, R_{1s}) - A(H_{2s}, R_{1s}) + A(R_{1s}, R_{1s}). \tag{50}$$

Since R_{1s} converges weakly to zero in $L^2(0,T,(H_0^1(\Omega))^3)$ and H_{1s} converges strongly to zero in $L^2(0,T,(H_0^1(\Omega))^3)$, it follows that $A(u,R_{1s})$ and $A(H_{1s},R_{1s})$ converge to zero as $s \to \infty$. We now estimate the third term on the right-hand side of (50). We write

$$A(H_{2s}, R_{1s}) = J_{1s} - J_{2s} + J_{3s}, \tag{51}$$

where

$$J_{1s} = \int\limits_0^T \Big[\sum_{r=1}^3 \sum_{i\in I_s''} \int\limits_{D_i^{(s)}\setminus F_i^{(s)}} a_{lj}^{hk}(x)\,\frac{\partial v_{ij}^{r(s)}}{\partial x_k}\,\frac{\partial (R_{l1s}u_{mr}\psi_i^{(s)})}{\partial x_h}\,dx\Big]\,dt,$$

$$J_{2s} = \int\limits_0^T \sum_{r=1}^3 \sum_{i\in I_s''} \int\limits_{D_i^{(s)}} a_{lj}^{hk}(x) R_{l1s}\,\frac{\partial v_{ij}^{r(s)}}{\partial x_k}\,\frac{\partial (u_{mr}\psi_i^{(s)})}{\partial x_h}\,dx\,dt,$$

$$J_{3s} = \int\limits_0^T \sum_{r=1}^3 \sum_{i\in I_s''} \int\limits_{D_i^{(s)}} a_{lj}^{hk}(x) v_{ij}^{r(s)}\,\frac{\partial (u_{mr}\psi_i^{(s)})}{\partial x_k}\,\frac{\partial R_{l1s}}{\partial x_h}\,dx\,dt,$$

and R_{l1s} is the lth component of R_{1s}; we have used the fact that $v_i^{r(s)} = e^{(r)}$ on $F_i^{(s)}$. For s sufficiently large we may assume that $\theta_1\varrho_i^{(s)} < 1$. Thus $\psi_i^{(s)}(x) = 0$ outside $B(x_i^{(s)},1)$. Hence the integral in J_{1s} reduces to an integral over $B(x_i^{(s)},1) \setminus F_i^{(s)}$. Since $R_{l1s}u_{mr}\psi_i^{(s)} \in H_0^1(B(x_i^{(s)},1) \setminus F_i^{(s)})$, thanks to the integral identity (49), we see that $J_{1s} = 0$.

We shall need the following Poincaré inequality proved in [22, Lemma 1.4, Chap. 8]: Let $K(\varrho_1,\varrho_2)$ be the ring $\{x : 0 \le \varrho_1 < |x| < \varrho_2 \le a\}$; then for any function $f \in H^1(B(0,a))$ we have

$$\int\limits_{K(\varrho_1,\varrho_2)} |f(x)|^2\,dx \le c[\varrho_2^2 - \varrho_1^2] \int\limits_{B(0,a)} \left|\frac{\partial f}{\partial x}\right|^2 dx + c\,\frac{\varrho_2^3 - \varrho_1^3}{a^3} \int\limits_{K(a/2,a)} |f(x)|^2\,dx. \tag{52}$$

From now on we assume that s is so large that $\theta_1 \varrho_i^{(s)} < 1$. For $0 < \mu < 1$, let

$$B_\mu = \{x \in \overline{B(x_i^{(s)}, 1)} : 0 \le |v_i^{r(s)}(x)| \le \mu\}.$$

We define

$$v_{i\mu}^{(s)}(x) = \begin{cases} v_i^{r(s)}(x) & \text{if } x \in B_\mu, \\ \mu e^{(r)} & \text{if } x \notin B_\mu. \end{cases}$$

It is clear that $\varphi(x) = v_{i\mu}^{r(s)}(x) - \mu e^{(r)} v_i^{r(s)}(x) \in (H_0^1(B(x_i^{(s)}, 1) \setminus F_i^{(s)}))^3$, since $v_i^{r(s)}(x) = e^{(r)}$ on $F_i^{(s)}$, i.e., $|v_i^{r(s)}(x)| = 1$ on $\partial F_i^{(s)}$ and $v_{i\mu}^{(s)}(x) = \mu e^{(r)}$, thus $\varphi(x) = 0$ on $\partial F_i^{(s)}$; analogously $v_i^{(s)}(x) = 0$ on $\partial B(x_i^{(s)}, 1)$, hence $v_{i\mu}^{(s)}(x) = v_i^{r(s)}(x)$, i.e., $\varphi(x) = 0$ on $\partial B(x_i^{(s)}, 1)$. Hence substituting this function in the integral identity (49) we get, after standard calculations with the use of the ellipticity condition (6),

$$\int_{B_\mu} \left|\frac{\partial v_i^{r(s)}}{\partial x}\right|^2 dx \le C\mu \int_{B(x_i^{(s)},1)} \left|\frac{\partial v_i^{r(s)}}{\partial x}\right|^2 dx.$$

Applying inequality (21) to the right-hand side we get

$$\int_{B_\mu} \left|\frac{\partial v_i^{r(s)}}{\partial x}\right|^2 dx \le C\mu d_i^{(s)}. \tag{53}$$

We have

$$J_{2s} \le J'_{2s} + J''_{2s}.$$

where

$$J'_{2s} = \sum_{r=1}^{3} \sum_{i \in I''_s} \int_0^T \int_{D_i^{(s)}} |R_{1s}| \left|\frac{\partial u_{mr}}{\partial x}\right| |\psi_i^{(s)}| \left|\frac{\partial v_i^{r(s)}}{\partial x_k}\right| dx,$$

$$J''_{2s} = \sum_{r=1}^{3} \sum_{i \in I''_s} \int_0^T \int_{D_i^{(s)}} |R_{1s}|\, |u_{mr}| \left|\frac{\partial \psi_i^{(s)}}{\partial x}\right| \left|\frac{\partial v_i^{r(s)}}{\partial x_k}\right| dx.$$

Since $v_i^{r(s)} \in C^2(\dot{B}_i^{(s)}) \cap C^1(\overline{\dot{B}_i^{(s)}})$ it is clear that $|\partial v_i^{r(s)}/\partial x_k|$ is bounded. Thus by Hölder's inequality we have

$$\begin{aligned} J'_{2s} &\le C \sum_{r=1}^{3} \sum_{i \in I''_s} \left[\int_0^T \int_{D_i^{(s)}} |R_{1s}|^2\, dx\, dt\right]^{1/2} \left[\int_0^T \int_{D_i^{(s)}} \left|\frac{\partial u_{mr}}{\partial x}\right|^2 \left|\frac{\partial v_i^{r(s)}}{\partial x_k}\right|^2 dx\, dt\right]^{1/2} \\ &\le C \left[\int_{Q_T} |R_{1s}|^2\, dx\, dt\right]^{1/2} \left[\int_{Q_T} \left|\frac{\partial u_m}{\partial x}\right|^2 dx\, dt\right]^{1/2}. \end{aligned}$$

As $R_{1s} \to 0$ strongly in $L^2(0,T,(L^2(\Omega))^3)$ and the second integral is finite, we obtain $J'_{2s} \to 0$ as $s \to \infty$.

We now estimate J''_{2s}. Since by (20),

$$\mu_i^{(s)} = \sup_{D_i^{(s)} \setminus B(x_i^{(s)}, \theta_2 \varrho_i^{(s)})} |v_i^{r(s)}(x)| \le C \frac{d_i^{(s)}}{\varrho_i^{(s)}}, \tag{54}$$

we have by Hölder's inequality, the definition of $\psi_i^{(s)}$ and boundedness of u_m,

$$J''_{2s} \le C_m \sum_{r=1}^{3} \sum_{i \in I''_s} \Big[[\varrho_i^{(s)}]^{-2} \int_{B_{\mu_i^{(s)}}} \Big| \frac{\partial v_i^{r(s)}}{\partial x} \Big|^2 dx \Big]^{1/2} \times \Big[\int_0^T \int_{D_i^{(s)}} |R_{1s}|^2 \, dx \, dt \Big]^{1/2}.$$

By (53), (54) and (52) (with $\varrho_1 = 0$, $\varrho_2 = \theta_1 \varrho_i^{(s)}$, and $a = a_i^{(s)} = d_i^{(s)} + r_i^{(s)}/2$) we get

$$\begin{aligned} J''_{2s} &\le C_m \sum_{i \in I''_s} [[\varrho_i^{(s)}]^{-2} \mu_i^{(s)} d_i^{(s)}]^{1/2} \\ &\quad \times \Big[[\varrho_i^{(s)}]^2 \int_0^T \int_{B(x_i^{(s)}, a_i^{(s)})} \Big| \frac{\partial R_{1s}}{\partial x} \Big|^2 dx \, dt \\ &\quad + \frac{[\varrho_i^{(s)}]^3}{[r_i^{(s)}/2 + d_i^{(s)}]^3} \int_0^T \int_{B(x_i^{(s)}, a_i^{(s)})} |R_{1s}|^2 \, dx \, dt \Big]^{1/2} \\ &\le C_m \Big[\sum_{i \in I''_s} \frac{[d_i^{(s)}]^2}{\varrho_i^{(s)}} \Big]^{1/2} \Big[\sum_{i \in I''_s} \int_0^T \int_{B(x_i^{(s)}, a_i^{(s)})} \Big| \frac{\partial R_{1s}}{\partial x} \Big|^2 dx \, dt \Big]^{1/2} \\ &\quad + C_m \Big[\sum_{i \in I''_s} \frac{[d_i^{(s)}]^2}{[r_i^{(s)}]^3} \Big]^{1/2} \Big[\sum_{i \in I''_s} \int_0^T \int_{B(x_i^{(s)}, a_i^{(s)})} |R_{1s}|^2 \, dx \, dt \Big]^{1/2}. \end{aligned} \tag{55}$$

By definition of I''_s we have

$$\sum_{i \in I''_s} \frac{[d_i^{(s)}]^2}{\varrho_i^{(s)}} \le \sup \Big\{ \frac{1}{\ln^2 r_i^{(s)}} \Big\} \sum_{i=1}^{I(s)} \frac{[d_i^{(s)}]^2}{[r_i^{(s)}]^3}.$$

By hypothesis (H2) the sum on the right-hand side is uniformly bounded.

Thus letting $s \to \infty$ on both sides we see that

$$\lim_{s\to\infty} \sum_{i\in I''_s} \frac{[d_i^{(s)}]^2}{\varrho_i^{(s)}} = 0. \tag{56}$$

As the balls $B(x_i^{(s)}, a_i^{(s)})$ do not intersect, we easily infer that the right-hand side of (55) converges to zero. This implies that $J''_{2s} \to 0$ as $s \to \infty$. Hence we have proved that $J_{2s} \to 0$ as $s \to \infty$.

We proceed to the estimation of J_{3s}. The properties of u_m and $v_i^{r(s)}$ lead to

$$\begin{aligned} J_{3s} &\le \Big[\sum_{r=1}^{3}\sum_{i\in I''_s} \int_{D_i^{(s)}} |v_i^{r(s)}|^2\,dx + [\varrho_i^{(s)}]^{-2} \int_{D_i^{(s)}\setminus B(x_i^{(s)},\theta_2\varrho_i^{(s)})} |v_i^{r(s)}|^2\,dx\Big]^{1/2} \\ &\quad\times \bigg(\int_{Q_T} \left|\frac{\partial R_{1s}}{\partial x}\right|^2 dx\,dt\bigg)^{1/2} \\ &\le C_m\Big[\sum_{i\in I''_s}[\varrho_i^{(s)}]^3\Big]^{1/2} + C_m\Big[\sum_{i\in I''_s}[\varrho_i^{(s)}]^{-2} \int_{D_i^{(s)}\setminus B(x_i^{(s)},\theta_2\varrho_i^{(s)})} \frac{[d_i^{(s)}]^2}{|x-x_i^{(s)}|^2}\,dx\Big]^{1/2} \\ &\le C_m\Big[\sum_{i\in I''_s}[\varrho_i^{(s)}]^3\Big]^{1/2} + C_m\Big[\sum_{i\in I''_s}\frac{[d_i^{(s)}]^2}{\varrho_i^{(s)}}\Big]^{1/2}. \end{aligned}$$

Here we have used inequality (20). By a passage to the limit as $s \to \infty$ and using (56) and (29) we get $\lim_{s\to\infty} J_{3s} = 0$.

Recapitulating we deduce from (51) that $\lim_{s\to\infty} A(H_{2s}, R_{1s}) = 0$. Hence from (47), (48) and (50), we conclude that

$$\lim_{s\to\infty} A(R_{1s}, R_{1s}) = 0.$$

Thus

$$\lim_{s\to\infty} \int_{Q_T^{(s)}} \left|\frac{\partial R_{1s}}{\partial x}\right|^2 dx\,dt = 0.$$

This proves (45). Theorem 7 is proved. ■

In analogy with Theorems 6 and 7 we get the following corrector result justifying the asymptotic expansion (34).

THEOREM 8. *Let the conditions of Theorem 4 be satisfied. Then for the functions H_{3s} and H_{4s} from (34) the following convergences hold:*

$$\lim_{s\to\infty} \|H_{3s}\|_{L^2(0,T,H_0^1(\Omega))} = 0,$$
$$\lim_{s\to\infty} \|H_{4s}\|_{L^2(0,T,\mathring{W}_p^1(\Omega^{(s)}))} = 0, \qquad p \in (1,2).$$

Furthermore

$$R_{2s}(x,t) \to 0 \quad \text{strongly in } L^2(0,T,H_0^1(\Omega)).$$

REMARK 9. Thanks to Theorems 6, 7 and 8, and the relations (32), we see that $(u^{(s)},\theta^{(s)})$ strongly converges to (u,θ) in $L^2(0,T,(\mathring{W}_p^1(\Omega))^3)\times L^2(0,T,\mathring{W}_p^1(\Omega))$ for all $1<p<2$. Hence the last assertion of Theorem 4 is proved.

4. Derivation of the limit problem. Let $g(x,t)=(g_1(x,t),g_2(x,t),g_3(x,t))$ and $h(x,t)$ be arbitrary functions which belong respectively to $C^\infty(0,T,(C_0^\infty(\Omega))^3)$ and $C^\infty(0,T,C_0^\infty(\Omega))$ such that $g(x,T)=0$ and $h(x,T)=0$. We consider the sequences of functions

$$g_s(x,t)=g(x,t)-G_{1s}(x,t)-G_{2s}(x,t), \tag{57}$$

$$h_s(x,t)=h(x,t)-G_{3s}(x,t)-G_{4s}(x,t), \tag{58}$$

where

$$G_{1s}(x,t)=\sum_{r=1}^{3}\sum_{i\in I_s'} v_i^{r(s)}(x)g_r(x,t)\psi_i^{(s)}(x),$$

$$G_{2s}(x,t)=\sum_{r=1}^{3}\sum_{i\in I_s''} v_i^{r(s)}(x)g_r(x,t)\psi_i^{(s)}(x),$$

$$G_{3s}(x,t)=\sum_{r=1}^{3}\sum_{i\in I_s'} v_i^{r(s)}(x)h(x,t)\psi_i^{(s)}(x),$$

$$G_{4s}(x,t)=\sum_{r=1}^{3}\sum_{i\in I_s''} v_i^{r(s)}(x)h(x,t)\psi_i^{(s)}(x),$$

where $\psi_i^{(s)}$ are the functions defined in the previous section. It is easy to verify that g_s and h_s vanish on $F_i^{(s)}\times(0,T]$, $g_s\in L^2(0,T,(H_0^1(\Omega))^3)$, $\partial g_s/\partial t\in L^2(0,T,(H_0^1(\Omega))^3)$, $\partial^2 g_s/\partial t^2\in L^2(0,T,(L^2(\Omega))^3)$, $h_s\in L^2(0,T,H_0^1(\Omega))$, $\partial h_s/\partial t\in L^2(0,T,L^2(\Omega))$. Using the same arguments as in the previous section when proving Theorems 6 and 7, we have the following relations for $p\in(1,2)$:

$$\begin{aligned}
&\lim_{s\to\infty}\|G_{1s}\|_{L^2(0,T,(H_0^1(\Omega))^3)}=0,\\
&\lim_{s\to\infty}\|G_{2s}\|_{L^2(0,T,(W_p^1(\Omega))^3)}=0,\\
&\lim_{s\to\infty}\left\|\frac{\partial G_{ls}}{\partial t}\right\|_{L^2(0,T,(L^2(\Omega))^3)}=0,\qquad l=1,2,\\
&\lim_{s\to\infty}\|G_{3s}\|_{L^2(0,T,H_0^1(\Omega))}=0,\\
&\lim_{s\to\infty}\|G_{4s}\|_{L^2(0,T,W_p^1(\Omega))}=0.
\end{aligned} \tag{59}$$

In particular,

$$
\begin{aligned}
&g_s(x,t) \rightharpoonup g(x,t) && \text{weakly in } L^2(0,T,(H_0^1(\Omega))^3),\\
&g_s(x,t) \to g(x,t) && \text{strongly in } L^2(0,T,(W_p^1(\Omega))^3),\\
&\frac{\partial g_s(x,t)}{\partial t} \to \frac{\partial g(x,t)}{\partial t} && \text{strongly in } L^2(0,T,(L^2(\Omega))^3),\\
&h_s(x,t) \rightharpoonup h(x,t) && \text{weakly in } L^2(0,T,H_0^1(\Omega)),\\
&h_s(x,t) \to h(x,t) && \text{strongly in } L^2(0,T,W_p^1(\Omega)),
\end{aligned} \tag{60}
$$

for $p \in (1,2)$.

Substituting g_s and h_s as test functions in (8) and (9) respectively we get

$$
\int_{Q_T} -\frac{\partial u^{(s)}}{\partial t}\,\frac{\partial g_s}{\partial t}\,dx\,dt + A(u^{(s)},g_s) + C_1(\theta^{(s)},g_s) - \int_\Omega u_1^{(s)}(x)g_s(x,0)\,dx = \int_{Q_T} f^{(s)}(x,t)g_s(x,t)\,dx\,dt, \tag{61}
$$

and

$$
\int_{Q_T} -\theta^{(s)}\frac{\partial h_s}{\partial t}\,dx\,dt + K(\theta^{(s)},h_s) + C_2\left(\frac{\partial u^{(s)}}{\partial t},h_s\right) - \int_\Omega \theta_0^{(s)}(x)h_s(x,0)\,dx = \int_{Q_T} q^{(s)}(x,t)h_s(x,t)\,dx\,dt, \tag{62}
$$

where K and C_2 are defined in (10). By (59), (60), (12) and (13), we get as $s \to \infty$ the following convergences:

$$
\int_{Q_T} -\frac{\partial u^{(s)}}{\partial t}\,\frac{\partial g_s}{\partial t}\,dx\,dt - \int_\Omega u_1^{(s)}(x)g_s(x,0)\,dx \to \int_{Q_T} -\frac{\partial u}{\partial t}\,\frac{\partial g}{\partial t}\,dx\,dt - \int_\Omega u_1(x)g(x,0)\,dx, \tag{63}
$$

$$
C_1(\theta^{(s)},g_s) \to C_1(\theta,g_s), \tag{64}
$$

$$
\int_{Q_T} f^{(s)}(x,t)g_s(x,t)\,dx\,dt \to \int_{Q_T} f(x,t)g(x,t)\,dx\,dt, \tag{65}
$$

$$
\int_{Q_T} -\theta^{(s)}\,\frac{\partial h_s}{\partial t}\,dx\,dt - \int_\Omega \theta_0^{(s)}(x)h_s(x,0)\,dx \to \int_{Q_T} -\theta\,\frac{\partial h}{\partial t}\,dx\,dt - \int_\Omega \theta_0(x)h(x,0)\,dx, \tag{66}
$$

$$C_2\left(\frac{\partial u^{(s)}}{\partial t}, h_s\right) \to C_2\left(\frac{\partial u}{\partial t}, h\right), \tag{67}$$

$$\int\limits_{Q_T} q^{(s)}(x,t)h_s(x,t)\,dx\,dt \to \int\limits_{Q_T} q(x,t)h(x,t)\,dx\,dt. \tag{68}$$

We now estimate $A(u^{(s)}, g^{(s)})$ and $K(\theta^{(s)}, h_s)$. Since these two expressions are similar in form, we restrict ourselves to $A(u^{(s)}, g^{(s)})$. By (57) we have

$$A(u^{(s)}, g_s) = A(u^{(s)}, g) - A(u^{(s)}, G_{1s}) - A(u^{(s)}, G_{2s}). \tag{69}$$

From (59) and the weak* convergence of $u^{(s)}$ to u in $L^\infty(0, T, (H_0^1(\Omega))^3)$ as $s \to \infty$, it follows that

$$A(u^{(s)}, g) \to A(u, g), \qquad A(u^{(s)}, G_{1s}) \to 0. \tag{70}$$

For similar reasons,

$$A(u^{(s)}, G_{2s}) = \beta_s + A(H_{2s}, G_{2s}), \tag{71}$$

where $\lim_{s\to\infty} \beta_s = 0$. Let us introduce the matrices $A^{kh}(x) = \{a_{lj}^{hk}(x)\}_{l,j=1}^3$, $k, h = 1, 2, 3$. Thus

$$\begin{aligned} A(H_{2s}, G_{2s}) &= \sum_{i\in I_s''} \int\limits_{Q_T^{(s)}} A^{kh}(x) \frac{\partial}{\partial x_k}(v_i^{r(s)}(x) u_{mr}(x,t) \psi_i^{(s)}(x)) \\ &\qquad \times \frac{\partial}{\partial x_h}(v_i^{q(s)}(x) g_q(x,t) \psi_i^{(s)}(x))\,dx\,dt \\ &= J_{1s} + J_{2s} + J_{3s} + J_{4s}, \end{aligned}$$

where

$$\begin{aligned} J_{1s} &= \sum_{i\in I_s''} \int\limits_{Q_T} A^{kh}(x_i^{(s)})\, \frac{\partial v_i^{r(s)}}{\partial x_k} \\ &\qquad \times \frac{\partial}{\partial x_h}(v_i^{q(s)}(x) u_{mr}(x,t) g_q(x,t) [\psi_i^{(s)}(x)]^2)\,dx\,dt, \\ J_{2s} &= \sum_{i\in I_s''} \int\limits_{Q_T} A^{kh}(x) v_i^{r(s)}(x) \\ &\qquad \times \frac{\partial}{\partial x_k}(u_{mr}(x,t)\psi_i^{(s)}(x))\, \frac{\partial}{\partial x_h}(v_i^{q(s)}(x) g_q(x,t) \psi_i^{(s)}(x))\,dx\,dt, \\ J_{3s} &= -\sum_{i\in I_s''} \int\limits_{Q_T} A^{kh}(x) v_i^{q(s)}(x) g_q(x,t) \psi_i^{(s)}(x) \\ &\qquad \times \frac{\partial v_i^{r(s)}(x)}{\partial x_h}\, \frac{\partial}{\partial x_k}(u_{mr}(x,t) \psi_i^{(s)}(x))\,dx\,dt, \end{aligned}$$

$$J_{4s} = \sum_{i \in I_s''} \int_{D_i^{(s)}} [A^{kh}(x) - A^{kh}(x_i^{(s)})] \frac{\partial v_i^{r(s)}}{\partial x_k} \times \frac{\partial}{\partial x_h} (v_i^{q(s)}(x) u_{mr}(x,t) g_q(x,t) [\psi_i^{(s)}(x)]^2) \, dx \, dt.$$

Let us estimate J_{2s}. We note that it is a sum of expressions of the type

$$\sum_{i \in I_s''} \int_{Q_T} A^{kh}(x) v_i^{r(s)} \frac{\partial^{\alpha_1} u_{mr}}{\partial x_k^{\alpha_1}} \frac{\partial^{\alpha_2} \psi_i^{(s)}}{\partial x_k^{\alpha_2}} \frac{\partial^{\alpha_3} v_i^{q(s)}}{\partial x_h^{\alpha_3}} \frac{\partial^{\alpha_4} g_q}{\partial x_h^{\alpha_4}} \frac{\partial^{\alpha_5} \psi_i^{(s)}}{\partial x_h^{\alpha_5}} \, dx \, dt, \tag{72}$$

with $\alpha_1 + \alpha_2 = 1$, $\alpha_3 + \alpha_4 + \alpha_5 = 1$, $\alpha_l \in \{0, 1\}$. We estimate one of these expressions, say

$$J_{2s}^1 = \sum_{i \in I_s''} \int_{Q_T} A^{kh}(x) v_i^{r(s)} u_{mr} \psi_i^{(s)} \frac{\partial \psi_i^{(s)}}{\partial x_k} \frac{\partial v_i^{q(s)}}{\partial x_h} g_q \, dx \, dt.$$

Since u_{mr} is bounded, we see that

$$J_{2s}^1 \le C \sum_{i \in I_s''} \int_{\Omega} |v_i^{r(s)}| \left| \frac{\partial \psi_i^{(s)}}{\partial x_k} \right| \left| \frac{\partial v_i^{q(s)}}{\partial x_h} \right| dx \, dt.$$

We assume that s is so large that $\theta_1 \varrho_i^{(s)} < 1$. We have $\psi_i^{(s)}(x) \equiv 1$ in $B(x_i^{(s)}, \theta_2 \varrho_i^{(s)})$ and $\psi_i^{(s)}(x) = 0$ for $|x - x_i^{(s)}| > \theta_1 \varrho_i^{(s)}$, thus

$$\begin{aligned} J_{2s}^1 &\le C \sum_{i \in I_s''} \int_0^T \int_{D_i^{(s)} \setminus B(x_i^{(s)}, \theta_2 \varrho_i^{(s)})} |v_i^{r(s)}| \left| \frac{\partial \psi_i^{(s)}}{\partial x_k} \right| \left| \frac{\partial v_i^{q(s)}}{\partial x_h} \right| dx \, dt \\ &\le CT \sum_{i \in I_s''} [\varrho_i^{(s)}]^{-1} \Big(\int_{D_i^{(s)} \setminus B(x_i^{(s)}, \theta_2 \varrho_i^{(s)})} |v_i^{r(s)}|^2 \, dx \Big)^{1/2} \\ &\quad \times \Big(\int_{B(x_i^{(s)}, 1)} \left| \frac{\partial v_i^{r(s)}}{\partial x} \right|^2 dx \Big)^{1/2} \\ &\le C \sum_{i \in I_s''} [d_i^{(s)}]^{1/2} \frac{d_i^{(s)}}{[\varrho_i^{(s)}]^{1/2}} \le C \sum_{i \in I_s''} d_i^{(s)} \sum_{i \in I_s''} \frac{[d_i^{(s)}]^2}{\varrho_i^{(s)}} \\ &\le C \sup \left\{ \frac{1}{\ln^2 r_i^{(s)}} \right\} \sum_{i \in I_s''} d_i^{(s)} \sum_{i \in I_s''} \frac{[d_i^{(s)}]^2}{[r_i^{(s)}]^3}. \end{aligned}$$

The last two factors are bounded by hypothesis (H2) and the relation (30). Thus by letting $s \to \infty$, we get

$$\lim_{s \to \infty} J_{2s}^1 = 0.$$

The other expressions in (72), as well as J_{3s} and J_{4s} are proved to converge to zero analogously; we use the fact that the entries of A^{hk} are continuous. Hence

$$\lim_{s\to\infty} J_{2s} = \lim_{s\to\infty} J_{3s} = \lim_{s\to\infty} J_{4s} = 0.$$

It remains to estimate J_{1s}. We have

$$J_{1s} = J_{1s}^1 + J_{1s}^2,$$

where

$$J_{1s}^1 = \sum_{i\in I_s''} \int_{Q_T} A^{kh}(x_i^{(s)}) \frac{\partial v_i^{r(s)}}{\partial x_k} \frac{\partial}{\partial x_h}(v_i^{q(s)} u_{mr} g_q(\psi_i^{(s)} - 1)(\psi_i^{(s)} + 1))\,dx\,dt,$$

$$J_{1s}^2 = \sum_{i\in I_s''} \int_{Q_T} A^{kh}(x_i^{(s)}) \frac{\partial v_i^{r(s)}}{\partial x_k} \frac{\partial}{\partial x_h}(v_i^{q(s)} u_{mr} g_q)\,dx\,dt.$$

The function to which $\partial/\partial x_h$ is applied in the expression of J_{1s}^1 and which we denote by $\Phi_i^{(s)}(x,t)$ is equal to zero on $F_i^{(s)}$ and outside $D_i^{(s)} \subset B(x_i^{(s)},1)$, thus

$$J_{1s}^1 = \sum_{i\in I_s''} \int_0^T \Bigg[\int_{B(x_i^{(s)},1)\setminus F_i^{(s)}} A^{kh}(x) \frac{\partial v_i^{r(s)}(x)}{\partial x_k} \frac{\partial \Phi_i^{(s)}(x,t)}{\partial x_h}\,dx \Bigg]\,dt.$$

By the definition of $v_i^{r(s)}$, we see that $J_{1s}^1 = 0$.

Next

$$J_{1s}^2 = \sum_{i\in I_s''} \int_{Q_T} A^{kh}(x_i^{(s)}) \frac{\partial v_i^{r(s)}}{\partial x_k} \frac{\partial v_i^{q(s)}}{\partial x_h} u_{mr} g_q\,dx\,dt$$
$$+ \sum_{i\in I_s''} \int_{Q_T} A^{kh}(x_i^{(s)}) v_i^{q(s)} \frac{\partial v_i^{r(s)}}{\partial x_k} \frac{\partial [u_{mr} g_q]}{\partial x_h}\,dx\,dt.$$

Arguments similar to those used previously show that the second term in the above sum converges to zero as $s\to\infty$. Let us estimate the first term. We call it $J_{1s}^{2\prime}$. We can write

$$J_{1s}^{2\prime} = \sum_{i\in I_s} \int_{Q_T} A^{kh}(x_i^{(s)}) \frac{\partial v_i^{r(s)}}{\partial x_k} \frac{\partial v_i^{q(s)}}{\partial x_h} u_{mr} g_q\,dx\,dt + \delta_s,$$

where $\delta_s \to 0$ as $s\to\infty$. Here we have used the fact that the sum of the integrals in $J_{1s}^{2\prime}$ over $i\in I_s'$ converges to zero as $s\to\infty$ in order to rewrite the sum over all $i\in I_s$. We cover Ω with a system of sufficiently smooth disjoint sets G_l, $l=1,\dots,L$, such that $\operatorname{diam} G_l < 2$, $\Omega = \bigcup_l G_l$. We denote by $I_s(G_l)$ the set of indices $i=1,\dots,I(s)$ such that $F_i^{(s)} \subseteq G_l$. Since $\operatorname{diam} F_i^{(s)}$ vanishes as $s\to\infty$, for sufficiently large s the G_l's can be chosen such that

$\bigcup_l I_s(G_l) = \{1, \dots, I(s)\}$, i.e., all $F_i^{(s)}$ lie inside some G_l. Let $m_{rq}(x)$ be the entries of the matrix $m(x)$ defined in hypothesis (H3). For large s, we have

$$J_{1s}^{2\prime} = \int_0^T \int_\Omega m_{rq}(x) u_r g_q(x,t)\, dx\, dt$$
$$+ \sum_{l=1}^{L} \int_0^T \Big\{ \sum_{i\in I(G_l)} \int_{B(x_i^{(s)},1)} A^{kh}(x_i^{(s)}) \frac{\partial v_i^{r(s)}}{\partial x_k} \frac{\partial v_i^{q(s)}}{\partial x_h}\, dx$$
$$- \int_{G_l} m_{rq}(x)\, dx \Big\} u_{mr} g_q(x,t)\, dt$$
$$+ \sum_{l=1}^{L} \int_0^T \int_{G_l} m_{rq}(x)(u_{mr} - u_r) g_q(x,t)\, dx\, dt + \delta_s.$$

Taking into account hypothesis (H3), using the fact that u_{mr} converges to u_r strongly in $L^2(Q_T)$, we see by passing to the limit as $s, m \to \infty$ that the last three terms on the right-hand side of the above equality converge to zero. Thus we have

$$\lim_{s\to\infty} J_{1s}^{2\prime} = \int_0^T \int_\Omega m_{rq}(x) u_r g_q\, dx\, dt.$$

Combining the above convergences, we conclude from (71) that

$$\lim_{s\to\infty} A(u^{(s)}, G_{2s}) = \int_{Q_T} m_{rq}(x) u_r(x,t) g_q(x,t)\, dx\, dt.$$

Hence from (61), (63)–(70) we find that (u, θ) satisfies the integral identity

$$\int_{Q_T} -\frac{\partial u}{\partial t}\frac{\partial g}{\partial t}\, dx\, dt + \int_{Q_T} A^{kh}(x) \frac{\partial u}{\partial x_k}\frac{\partial g}{\partial x_h}\, dx\, dt + \int_{Q_T} m(x) u g\, dx\, dt$$
$$+ \int_{Q_T} \alpha g_l \frac{\partial \theta}{\partial x_l}\, dx\, dt - \int_\Omega u_1(x) g(x,0)\, dx = \int_{Q_T} f g\, dx\, dt. \tag{73}$$

Analogously it is shown that (u, θ) also satisfies the integral identity

$$\int_{Q_T} -\theta \frac{\partial h}{\partial t}\, dx\, dt + \int_{Q_T} \frac{\partial \theta}{\partial x_l}\frac{\partial h}{\partial x_l}\, dx\, dt + \int_{Q_T} \frac{\partial}{\partial x_l}\Big(\frac{\partial u_l}{\partial t}\Big) h\, dx\, dt$$
$$+ \int_{Q_T} c(x)\theta h\, dx\, dt - \int_\Omega \theta_0(x) h(x,0)\, dx = \int_{Q_T} q h\, dx\, dt. \tag{74}$$

Finally let us show that (u, θ) satisfies the initial conditions (27). We limit ourselves to the verification of the first two conditions. We follow [15, Section 7.2]. We consider an arbitrary vector-function $v \in C^2(0, T, (H_0^1(\Omega))^3)$ such that $v(x,0) = v(x,T) = 0$. Let $\langle\cdot,\cdot\rangle$ denote the duality pairing between $(H_0^1(\Omega))^3$ and $((H_0^1(\Omega))^3)'$. By integrating by parts twice we have

$$\int_0^T \left\langle u^{(s)}(t), \frac{d^2v}{dt}(t)\right\rangle dt = -\left\langle u^{(s)}(0), \frac{dv}{dt}(0)\right\rangle + \left\langle \frac{du^{(s)}}{dt}(0), v(0)\right\rangle + \int_0^T \left\langle \frac{d^2u^{(s)}}{dt^2}(t), v(t)\right\rangle dt, \tag{75}$$

$$\int_0^T \left\langle u(t), \frac{d^2v}{dt}(t)\right\rangle dt = -\left\langle u(0), \frac{dv}{dt}(0)\right\rangle + \left\langle \frac{du}{dt}(0), v(0)\right\rangle r + \int_0^T \left\langle \frac{d^2u}{dt^2}(t), v(t)\right\rangle dt. \tag{76}$$

Passing to the limit in (75) and taking account of the relations (13) and (12), we deduce from the resulting equation and (76) that

$$-\left\langle u(0), \frac{dv}{dt}(0)\right\rangle + \left\langle \frac{du}{dt}(0), v(0)\right\rangle = -\left\langle u_0, \frac{dv}{dt}(0)\right\rangle + \langle u_1, v(0)\rangle.$$

Since $\frac{dv}{dt}(0)$ and $v(0)$ are arbitrary it follows that $u(x,0) = u_0(x)$, and $\frac{du}{dt}(x,0) = u_1(x)$. Thus u satisfies the first two initial conditions in (27). Analogous arguments show that θ satisfies the third condition in (27). Since g and h are arbitrary test functions and (u,θ) satisfies the integral identities (73) and (74) we conclude that (u,θ) is a weak solution of problem (24)–(27). This completes the proof of Theorem 4.

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