COLLOQUIUM MATHEMATICUM

VOL. 95 2003 NO. 1

COHEN–MACAULAYNESS OF MULTIPLICATION RINGS AND MODULES

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Abstract. Let R be a commutative multiplication ring and let N be a non-zero finitely generated multiplication R-module. We characterize certain prime submodules of N. Also, we show that N is Cohen–Macaulay whenever R is Noetherian.

Introduction. Throughout this paper, all rings considered will be commutative and will have non-zero identity elements. Such a ring will be denoted by R and a typical ideal of R will be denoted by $\mathfrak{a}$. There is a lot of current interest in the theory of multiplication rings and modules. Multiplication rings were introduced by W. Krull in 1926 as a generalization of Dedekind domains, and the modern concept of a multiplication module is due to Barnard. This concept has been studied in [1], [2], [3], [10] and has led to some interesting results. Let N be an R-module. Then N is said to be a *multiplication module* if every submodule of N is of the form $\mathfrak{a}N$ for some ideal $\mathfrak{a}$ of R. A *multiplication ring* is a ring in which every ideal is a multiplication module. A proper submodule P of N is said to be *prime* if whenever $rx \in P$ for $r \in R$, $x \in N$, then $x \in P$ or $r \in (P :_R N)$. (For more information about prime submodules, see [5], [11].)

Let $\mathfrak{p} \in \operatorname{Supp}(N)$. Then the *N-height* of $\mathfrak{p}$, denoted by $\operatorname{ht}_N \mathfrak{p}$, is defined to be the supremum of the lengths of chains of prime ideals of $\operatorname{Supp}(N)$ terminating with $\mathfrak{p}$. We shall say that an ideal $\mathfrak{a}$ of R is *N-proper* if $N \neq \mathfrak{a}N$, and when this is the case and R is Noetherian, we define the *N-height* of $\mathfrak{a}$ (written $\operatorname{ht}_N \mathfrak{a}$) to be

$$\inf\{\operatorname{ht}_N \mathfrak{p} : \mathfrak{p} \in \operatorname{Supp}(N/\mathfrak{a}N)\} \quad (= \inf\{\operatorname{ht}_N \mathfrak{p} : \mathfrak{p} \in \operatorname{Ass}_R(N/\mathfrak{a}N)\}).$$

Let R be a Noetherian ring and let N be a non-zero finitely generated R-module. For any N-proper ideal $\mathfrak{a}$ of R, denote by $\operatorname{grade}(\mathfrak{a}, N)$ the maximum length of all N-sequences contained in $\mathfrak{a}$. Suppose for the moment that

2000 *Mathematics Subject Classification*: 13C14, 13E05.

Research of R. Naghipour has been supported by the Research Institute for Fundamental Sciences, Tabriz, Iran.

$(R, \mathfrak{m})$ is local. Then it follows from Nakayama's Lemma that every proper ideal of R is N-proper. We say that N is a *Cohen–Macaulay module* if $\operatorname{grade}(\mathfrak{m}, N) = \operatorname{ht}_N \mathfrak{m}$.

More generally, N is said to be a *Cohen–Macaulay module* if $N_{\mathfrak{m}}$ is a Cohen–Macaulay $R_{\mathfrak{m}}$-module in the above sense for each maximal ideal $\mathfrak{m} \in \operatorname{Supp}(N)$. We refer to [6] for the basic results about Cohen–Macaulay modules. For any R-module L, we denote by $\operatorname{mAss}_R L$ the set of minimal prime ideals of $\operatorname{Ass}_R L$.

This paper is divided into two sections. In the first section we characterize certain prime submodules of a multiplication module over a (commutative) multiplication ring. In the second section we relate the notions of Cohen–Macaulay modules and multiplication modules. Indeed, we show that whenever R is a Noetherian multiplication ring and N is a non-zero finitely generated multiplication module, then N is Cohen–Macaulay.

Throughout, we shall assume that R is a multiplication ring and N is a multiplication R-module.

1. Prime submodules of a multiplication module. The main result of this section is Theorem 1.4 which provides a characterization of certain prime submodules of a multiplication module. The following lemma plays a key role in this section.

LEMMA 1.1. *Suppose that* $\mathfrak{m}$ *is a maximal ideal of* R. *Then*, *for each integer* n, *the factor module* $\mathfrak{m}^n N/\mathfrak{m}^{n+1}N$ *is simple.*

Proof. Let M be a submodule of N such that $\mathfrak{m}^{n+1}N \subsetneq M \subseteq \mathfrak{m}^n N$. We show that $M = \mathfrak{m}^n N$. In view of [1, Corollary 1.4], there exists an ideal $\mathfrak{a}$ of R such that $M = \mathfrak{a}\mathfrak{m}^n N$. Because $\mathfrak{m}^{n+1}N \subsetneq M$, we have $\mathfrak{a} \not\subseteq \mathfrak{m}$, and so $\mathfrak{a}+\mathfrak{m} = R$. Consequently, $\mathfrak{m}^n N = \mathfrak{m}^n(\mathfrak{a}+\mathfrak{m})N = \mathfrak{m}^n\mathfrak{a}N + \mathfrak{m}^{n+1}N$. Therefore $M = \mathfrak{m}^n N$, as desired. ∎

PROPOSITION 1.2. *Suppose that* $\mathfrak{m}$ *is a maximal ideal of* R. *Then*, *for every positive integer* n, *the factor module* $\mathfrak{m}N/\mathfrak{m}^{n+1}N$ *has a unique composition series*

$$\mathfrak{m}N/\mathfrak{m}^{n+1}N \subseteq \mathfrak{m}^2N/\mathfrak{m}^{n+1}N \subseteq \ldots \subseteq \mathfrak{m}^n N/\mathfrak{m}^{n+1}N.$$

Proof. Let n be an arbitrary positive integer. We may assume that $\mathfrak{m}^2 N \neq \mathfrak{m}N$. Then there are $a \in \mathfrak{m}$ and $y \in N$ such that $ay \notin \mathfrak{m}^2 N$. By Lemma 1.1, $aN + \mathfrak{m}^2 N = \mathfrak{m}N$. Now, it is easily seen that $aN + \mathfrak{m}^n N = \mathfrak{m}N$. Accordingly, for all positive integers k with $k \leq n$, we have

$$(*) \qquad a^k N + \mathfrak{m}^n N = \mathfrak{m}^k N.$$

Now, to prove the assertion, let M be a submodule of N such that $\mathfrak{m}^{n+1}N \subseteq M \subseteq \mathfrak{m}N$. Let i be the greatest positive integer such that $M \subseteq \mathfrak{m}^i N$. If $i = n$, the result follows from Lemma 1.1.

Hence let $i < n$. Suppose that $x \in M \backslash \mathfrak{m}^{i+1}N$. Then $x \notin \mathfrak{m}^{n+1}N$. From $(*)$ we have $\mathfrak{m}^i N = a^i N + \mathfrak{m}^{n+1}N$, so there exist $y \in N$ and $z \in \mathfrak{m}^{n+1}N$ such that $x = a^i y + z$. Accordingly $a^i y \notin \mathfrak{m}^{i+1}N$, which implies that $ay \in \mathfrak{m}N \backslash \mathfrak{m}^2 N$. Consequently, by Lemma 1.1, $\mathfrak{m}N = \mathfrak{m}^2 N + Ray$. Also, we have $\mathfrak{m}^{i-1}N = \mathfrak{m}^n N + a^{i-1}N$. Therefore

$$\begin{aligned}\mathfrak{m}^i N &= \mathfrak{m}(\mathfrak{m}^n N + a^{i-1}N) = \mathfrak{m}^{n+1}N + a^{i-1}\mathfrak{m}N \\ &= \mathfrak{m}^{n+1}N + a^{i-1}(\mathfrak{m}^2 N + Ray) \subseteq \mathfrak{m}^{i+1}N + Ra^i y \subseteq \mathfrak{m}^i N.\end{aligned}$$

Hence, we have

$$\mathfrak{m}^i (N/R(a^i y)) = \mathfrak{m}^{i+1}(N/R(a^i y)).$$

This implies that

$$\mathfrak{m}^i (N/R(a^i y)) = \mathfrak{m}^{i+1}(N/R(a^i y)) = \ldots = \mathfrak{m}^{n+1}(N/R(a^i y)).$$

So

$$\mathfrak{m}^i N = \mathfrak{m}^{n+1}N + R(a^i y) = \mathfrak{m}^{n+1}N + Rx \subseteq M,$$

and the result follows. ∎

PROPOSITION 1.3. *Suppose that* $\mathfrak{m}$ *is a maximal ideal of* R *such that* $\mathfrak{m}^n N \neq \mathfrak{m}^{n+1}N$ *for all* $n \in \mathbb{N}$. *Then* $P := \bigcap_{n \geq 1} \mathfrak{m}^n N$ *is a prime submodule of* N.

Proof. Suppose that, on the contrary, $rx \in P$ for some $x \in N \backslash P$ and $r \in R \backslash (P :_R N)$ (note that $P \neq N$). Then there is an integer $i \geq 0$ such that $x \in \mathfrak{m}^i N \backslash \mathfrak{m}^{i+1}N$. On the other hand, as $r \notin (P :_R N) = \bigcap_{n \geq 1}(\mathfrak{m}^n N :_R N)$, there exists an integer $j \geq 0$ such that $r \in (\mathfrak{m}^j N :_R N) \backslash (\mathfrak{m}^{j+1}N :_R N)$. So, there exists $y \in N$ such that $ry \in \mathfrak{m}^j N \backslash \mathfrak{m}^{j+1}N$. Hence by Lemma 1.1, $\mathfrak{m}^i N = \mathfrak{m}^{i+1}N + Rx$ and $\mathfrak{m}^j N = \mathfrak{m}^{j+1}N + R(ry)$. So, we will have

$$\begin{aligned}\mathfrak{m}^{i+j}N &= \mathfrak{m}^i(\mathfrak{m}^{j+1}N + R(ry)) = \mathfrak{m}^{i+j+1}N + \mathfrak{m}^i(ry) \\ &\subseteq \mathfrak{m}^{i+j+1}N + r\mathfrak{m}^{i+1}N + R(rx) \subseteq \mathfrak{m}^{i+j+1}N + R(rx).\end{aligned}$$

Using this, together with $\mathfrak{m}^{i+j}N \nsubseteq \mathfrak{m}^{i+j+1}N$, we deduce that $rx \notin \mathfrak{m}^{i+j+1}N$. Consequently, $rx \notin P$, which is a contradiction. ∎

We are now ready to state and prove the main result of this section, which is a characterization of prime submodules of N.

THEOREM 1.4. *Suppose that* $\mathfrak{m}$ *is a maximal ideal of* R. *Consider the following conditions*:

(i) $\mathfrak{m}^n N \neq \mathfrak{m}^{n+1}N$ *for all* $n \in \mathbb{N}$;

(ii) *The submodule* $\mathfrak{m}N$ *is prime and it contains properly only one prime submodule of* N.

Then (i) *always implies* (ii), *and the converse holds whenever N is finitely generated.*

Proof. (i)⇒(ii). It follows from [4, Lemma 1] that $\mathfrak{m}N$ is a prime submodule. Now let Q be a prime submodule of N which is contained properly in $\mathfrak{m}N$. Note that, by 1.3, there exists such a prime submodule. Since $\mathfrak{m}N$ is a multiplication module, by [1, Corollary 1.4], there exists an ideal $\mathfrak{a}$ of R such that $Q = \mathfrak{a}\mathfrak{m}N$. On the other hand, because $\mathfrak{m} \not\subseteq (Q :_R N)$ and Q is a prime submodule, we have $\mathfrak{a}N \subseteq Q$ and so $Q \subseteq \mathfrak{m}Q$. Now it is easily seen that $Q \subseteq \bigcap_{n\geq 1} \mathfrak{m}^n N$.

Now, to complete the proof, we have to show that $Q = \bigcap_{n\geq 1} \mathfrak{m}^n N$. Since $R/(Q :_R N)$ is a multiplication domain, it is easy to see that every non-zero ideal of $R/(Q :_R N)$ is invertible. Hence $R/(Q :_R N)$ is a Dedekind domain and therefore, by [6, Theorems 8.10 and 11.6], it follows that $\bigcap_{n\geq 1}(\mathfrak{m}/(Q :_R N))^n = 0$. We can now use [1, Theorem 1.6 (i)] to deduce that $\bigcap_{n\geq 1}\mathfrak{m}^n(N/Q) = 0$. This completes the proof of (ii).

Finally, assume that N is finitely generated and that (ii) holds. We show that (i) is true. Let Q be the (unique) prime submodule of N which is contained properly in $\mathfrak{m}N$. Suppose the contrary, i.e. there exists $n \in \mathbb{N}$ such that $\mathfrak{m}^n N = \mathfrak{m}^{n+1}N$. Then the ideal $(Q :_R N)$ is properly contained in $\mathfrak{m}$ and $(\mathfrak{m}/(Q :_R N))^n N/Q = (\mathfrak{m}/(Q :_R N))^{n+1}N/Q$. Now, Nakayama's Lemma (see [6, Theorem 2.2]) and the fact that $R/(Q :_R N)$ is a Noetherian domain provide a contradiction. ∎

COROLLARY 1.5. *Suppose that $\mathfrak{m}$ is a maximal ideal of R such that $\mathfrak{m}^n N \neq \mathfrak{m}^{n+1}N$ for all $n \in \mathbb{N}$. Let P be a prime submodule of N such that $P \subsetneq \mathfrak{m}N$. Then $P = \bigcap_{n\geq 1}\mathfrak{m}^n N$ and $P = \mathfrak{m}P$.*

Proof. The only non-obvious point is to prove that $\mathfrak{m}P = P$. By [1, Corollary 1.4] there exists an ideal $\mathfrak{a}$ of R such that $P = \mathfrak{a}\mathfrak{m}N$. Now, because $\mathfrak{m} \not\subseteq (P :_R N)$, it follows that $\mathfrak{a}N \subseteq P$. Consequently, $P \subseteq \mathfrak{m}P$, as desired. ∎

COROLLARY 1.6. *Suppose, in addition, that R is Noetherian and N is finitely generated. Let $\mathfrak{m}$ be a maximal ideal of R, and let P be a prime submodule of N with $P \subsetneq \mathfrak{m}N$. Then $P = \bigcap_{n\geq 1}\mathfrak{m}^n N$ and $P = \mathfrak{m}P$.*

Proof. By [7, Result 2], we have $\mathfrak{m}^n N \neq \mathfrak{m}^{n+1}N$ for all $n \in \mathbb{N}$. The claim therefore follows from Corollary 1.5. ∎

COROLLARY 1.7. *Let $(R, \mathfrak{m})$ be a Noetherian local ring and N a finitely generated R-module with $\operatorname{ht}_N \mathfrak{m} \geq 1$. Then the zero submodule of N is prime.*

Proof. Since $\operatorname{ht}_N \mathfrak{m} \geq 1$, by Nakayama's Lemma, we have $\mathfrak{m}^n N \neq \mathfrak{m}^{n+1}N$ for all $n \in \mathbb{N}$. Now the result follows from Corollary 1.5 and Krull's Intersection Theorem. ∎

2. Multiplication and Cohen–Macaulay modules. Before stating the next proposition which plays a key role in the proof of the main result of this section, we fix a notation, employed by P. Schenzel in [9] in the case $N = R$.

REMARK 2.1. Let S be a multiplicatively closed subset of R. For a submodule M of N, we use $S(M)$ to denote the submodule $\bigcup_{s\in S}(M :_N s)$. Note that, whenever R is Noetherian ring and N is finitely generated, the primary decomposition of $S(M)$ consists of the intersection of all primary components of M whose associated prime ideals do not meet S. In other words

$$\mathrm{Ass}_R(N/S(M)) = \{\mathfrak{p} \in \mathrm{Ass}_R(N/M) : \mathfrak{p} \cap S = \emptyset\}.$$

PROPOSITION 2.2. *Suppose N is finitely generated and M is a submodule of N. Then $S(M) = M$, where $S = R\setminus \bigcup\{\mathfrak{p} \in \operatorname{Spec} R : \mathfrak{p}$ is minimal over $\mathrm{Ann}_R(N/M)\}$.*

Proof. By passing to N/M we may assume that $M = 0$. We will show that $S(0) = 0$. Suppose that $S(0) \neq 0$ and look for a contradiction. Let x be a non-zero element of $S(0)$. Let $\mathfrak{m}$ be a minimal prime ideal over $\mathrm{Ann}_R(x)$. Then there exists a minimal prime ideal $\mathfrak{p}$ over $\mathrm{Ann}_R(N)$ such that $\mathfrak{p} \subseteq \mathfrak{m}$. Clearly $\mathfrak{p} \neq \mathfrak{m}$. As in the proof of Theorem 1.4, one can see that $R/\mathfrak{p}$ is a Dedekind (Noetherian) domain, and so $\dim R/\mathfrak{p} \leq 1$. Hence $\mathfrak{m}$ must be a maximal ideal. On the other hand, in view of [7, Result 2], $\mathfrak{p}N \neq \mathfrak{m}N$. Consequently, in view of [7, Lemma 3] and Corollary 1.5, we see that $\mathfrak{p}N = \bigcap_{n\geq 1}\mathfrak{m}^n N$ and $\mathfrak{p}N = \mathfrak{m}\mathfrak{p}N$. Moreover, because $\mathrm{Ann}_R(x) \subseteq \mathfrak{m}$, there is an ideal $\mathfrak{a}$ of R such that $\mathrm{Ann}_R(x) = \mathfrak{a}\mathfrak{m}$.

First, we treat the case where $\mathfrak{a} \subseteq \mathrm{Ann}_R(x)$. Then $\mathrm{Ann}_R(x) = \mathfrak{m}\,\mathrm{Ann}_R(x)$. It follows that $\mathrm{Ann}_R(x)N \subseteq \bigcap_{n\geq 1}\mathfrak{m}^n N$, and therefore $\mathrm{Ann}_R(x)N \subseteq \mathfrak{p}N$. Consequently, $\mathrm{Ann}_R(x) \subseteq \mathfrak{p}$ by [7, Result 2]; this contradicts the fact that $\mathfrak{m}$ is minimal over $\mathrm{Ann}_R(x)$.

Next, we treat the case in which $\mathfrak{a} \not\subseteq \mathrm{Ann}_R(x)$. Since $\mathfrak{p}N$ is a $\mathfrak{p}$-prime submodule of N (see [7, Lemma 3]), it is easy to see that $\mathfrak{a}x \subseteq \mathfrak{p}N$. Consequently, there is an ideal $\mathfrak{b}$ of R such that $\mathfrak{a}x = \mathfrak{b}\mathfrak{p}N$. Hence $\mathfrak{a}x = \mathfrak{b}\mathfrak{p}N = \mathfrak{b}\mathfrak{p}\mathfrak{m}N = \mathfrak{m}\mathfrak{a}x = \mathrm{Ann}_R(x)x = 0$, a contradiction. Therefore $S(0) = 0$. ∎

COROLLARY 2.3. *Suppose that R is Noetherian and N is finitely generated. Then for any submodule M of N, $\mathrm{Ass}_R(N/M) = \mathrm{mAss}_R(N/M)$.*

Proof. This follows immediately from Proposition 2.2 and Remark 2.1. ∎

We are now ready to state the main result of this section.

THEOREM 2.4. *Suppose that R is Noetherian and N is a non-zero finitely generated R-module. Then N is a Cohen–Macaulay module. In particular, every multiplication Noetherian ring is Cohen–Macaulay.*

Proof. This follows from Corollary 2.3 and [8, Proposition 2.2]. ■

Acknowledgments. The authors are deeply grateful to the referee for his careful reading of the manuscript and helpful suggestions. Also, we would like to thank Dr. K. Divaani-Aazar for careful reading of the first draft.

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Received 5 March 2002;
revised 21 August 2002 (4178)