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A BIFURCATION THEORY FOR SOME NONLINEAR ELLIPTIC EQUATIONS

BY

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Dedicated to Professor G. Santagati, with my greatest esteem, on his seventieth birthday

Abstract. We deal with the problem

$$(\mathrm{P}_\lambda) \qquad \begin{cases} -\Delta u = f(x,u) + \lambda g(x,u) & \text{in } \Omega, \\ u_{|\partial\Omega} = 0, \end{cases}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain, $\lambda \in \mathbb{R}$, and $f, g : \Omega \times \mathbb{R} \to \mathbb{R}$ are two Carathéodory functions with $f(x,0) = g(x,0) = 0$. Under suitable assumptions, we prove that there exists $\lambda^* > 0$ such that, for each $\lambda \in]0, \lambda^*[$, problem (P_λ) admits a non-zero, non-negative strong solution $u_\lambda \in \bigcap_{p\ge 2} W^{2,p}(\Omega)$ such that $\lim_{\lambda\to 0^+} \|u_\lambda\|_{W^{2,p}(\Omega)} = 0$ for all $p \ge 2$. Moreover, the function $\lambda \mapsto I_\lambda(u_\lambda)$ is negative and decreasing in $]0, \lambda^*[$, where I_λ is the energy functional related to (P_λ).

1. Introduction and statement of the result. Throughout the paper, $\Omega \subset \mathbb{R}^n$ is an open, connected, bounded set with smooth boundary, and $f, g : \Omega \times \mathbb{R} \to \mathbb{R}$ are two Carathéodory functions.

As usual, a *weak solution* of the problem

$$(\mathrm{P}_\lambda) \qquad \begin{cases} -\Delta u = f(x,u) + \lambda g(x,u) & \text{in } \Omega, \\ u_{|\partial\Omega} = 0, \end{cases}$$

where $\lambda \in \mathbb{R}$, is any $u \in W_0^{1,2}(\Omega)$ such that

$$\int_\Omega \nabla u(x) \nabla v(x)\, dx - \int_\Omega f(x,u(x))v(x)\, dx - \lambda \int_\Omega g(x,u(x))v(x)\, dx = 0$$

for all $v \in W_0^{1,2}(\Omega)$. A *strong solution* of the problem is any $u \in W_0^{1,2}(\Omega) \cap W^{2,2}(\Omega)$ which satisfies the equation almost everywhere in Ω. A *classical solution* is any $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$, zero on $\partial\Omega$, which satisfies the equation pointwise in Ω.

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If u is a strong solution of (P_λ), we also put

$$I_\lambda(u) = \frac{1}{2}\int_\Omega |\nabla u(x)|^2\,dx - \int_\Omega \Big(\int_0^{u(x)} f(x,\xi)\,d\xi\Big)\,dx - \lambda\int_\Omega\Big(\int_0^{u(x)} g(x,\xi)\,d\xi\Big)\,dx.$$

Above, of course, it is understood that the integrals which appear are well defined.

The aim of this paper is to prove the following theorem:

THEOREM 1. *Assume that*:

(i) *there is* $s > 1$ *such that*

$$\limsup_{\xi\to 0^+} \frac{\sup_{x\in\Omega}|f(x,\xi)|}{\xi^s} < \infty;$$

(ii) *there is* $q \in\,]0,1[$ *such that*

$$\limsup_{\xi\to 0^+} \frac{\sup_{x\in\Omega}|g(x,\xi)|}{\xi^q} < \infty;$$

(iii) *there are a non-empty open set* $D \subseteq \Omega$ *and a set* $B \subseteq D$ *of positive measure such that*

$$\limsup_{\xi\to 0^+} \frac{\inf_{x\in B}\int_0^\xi g(x,t)\,dt}{\xi^2} = \infty, \quad \liminf_{\xi\to 0^+} \frac{\inf_{x\in D}\int_0^\xi g(x,t)\,dt}{\xi^2} > -\infty.$$

Then, for some $\lambda^* > 0$ *and for each* $\lambda \in\,]0,\lambda^*[$, *problem* (P_λ) *admits a non-zero, non-negative strong solution* $u_\lambda \in \bigcap_{p\ge 2} W^{2,p}(\Omega)$. *Moreover*,

$$\limsup_{\lambda\to 0^+} \frac{\|u_\lambda\|_{C^1(\overline{\Omega})}}{\lambda^{q/(1-q)}} < \infty, \qquad \limsup_{\lambda\to 0^+} \frac{\|u_\lambda\|_{W^{2,p}(\Omega)}}{\lambda^{q^2/(1-q)}} < \infty$$

for all $p \ge 2$, *and the function* $\lambda \mapsto I_\lambda(u_\lambda)$ *is negative and decreasing in* $]0,\lambda^*[$. *If*, *in addition*, f, g *are continuous in* $\Omega \times [0,\infty[$ *and*

$$\liminf_{\xi\to 0^+} \frac{\inf_{x\in\Omega} g(x,\xi)}{\xi|\log\xi|^2} > -\infty,$$

then u_λ *is positive in* Ω.

Before giving the proof of Theorem 1, we make some remarks on it.

First of all, we observe that it is a bifurcation result. In fact, once we observe that (by (i) and (ii)) 0 is a solution of (P_λ) for each λ, this means, in particular, that $\lambda = 0$ is a bifurcation point for problem (P_λ), in the sense that, for each $p \ge 2$, $(0,0)$ belongs to the closure in $W^{2,p}(\Omega)\times\mathbb{R}$ of the set

$$\{(u,\lambda) \in W^{2,p}(\Omega)\times\,]0,\infty[\; : u \text{ is a strong solution of } (\mathrm{P}_\lambda),\ u \ne 0,\ u \ge 0\}.$$

Among the known results, the one which is closest to Theorem 1 is certainly Theorem 2.1 of [1].

Indeed, the latter, relating to the specific problem

$$\begin{cases} -\Delta u = u^s + \lambda u^q & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u_{|\partial\Omega} = 0, \end{cases}$$

with $0 < q < 1 < s$, ensures the existence of $\lambda_0 > 0$ such that for each $\lambda \in]0, \lambda_0[$, the problem admits a classical minimal solution u_λ with $I_\lambda(u_\lambda) < 0$. Moreover, $\lim_{\lambda\to 0^+} \sup_\Omega |u_\lambda| = 0$ and the function $\lambda \mapsto u_\lambda(x)$ is increasing for each $x \in \Omega$. Finally, for $\lambda = \lambda_0$ there is a weak solution, while for $\lambda > \lambda_0$ there is no classical solution. In Remark 2.5 of [1], the authors observe that the result still holds if one replaces u^q with any concave function that behaves like u^q near $u = 0$, and u^s with any superlinear function that behaves like u^s near $u = 0$ and near $u = \infty$. We wish to stress that this remark concerns all the qualitative aspects of the result. In particular, in the approach of [1], concavity plays an essential role also in the proof that $I_\lambda(u_\lambda) < 0$. However, if one restricts oneself only to the solvability of the problem for each $\lambda > 0$ small enough, then the method of sub- and super-solutions as exploited in Lemma 3.1 of [1] can be readily applied under much more general assumptions which meet those of Theorem 1. Here is the statement one can obtain in this way:

THEOREM A. *Besides conditions* (i) *and* (ii) *of Theorem* 1, *assume that*

(iii′) $$\lim_{\xi\to 0^+} \frac{\inf_{x\in\Omega} g(x,\xi)}{\xi} = \infty.$$

Then, for some $\lambda^* > 0$ *and for each* $\lambda \in]0, \lambda^*[$, *problem* (P_λ) *admits a positive weak solution* $u_\lambda \in L^\infty(\Omega)$, *and* $\lim_{\lambda\to 0^+} \|u_\lambda\|_{L^\infty(\Omega)} = 0$.

Thus, Theorem 1 ensures not only that the conclusion of Theorem A holds, but also that the function $\lambda \mapsto I_\lambda(u_\lambda)$ is negative and decreasing, even in the presence of condition (iii) which, of course, is much less restrictive than (iii′).

It is clear that the superiority of Theorem 1 over Theorem A is maximum in the cases when (iii) holds, while (iii′) is violated. For instance, we have the following examples of application of Theorem 1:

PROPOSITION 1. *Let* $0 < q < 1 < s$ *and let* α, β *be two bounded and locally Hölder continuous functions on* Ω. *Assume that*

(∗) $$0 \le \inf_\Omega \beta, \qquad 0 < \sup_\Omega \beta.$$

Then, for some $\lambda^* > 0$ and for each $\lambda \in]0, \lambda^*[$, the problem

$$\begin{cases} -\Delta u = \alpha(x)u^s + \lambda\beta(x)u^q & \text{in } \Omega, \\ u_{|\partial\Omega} = 0, \end{cases}$$

admits a positive classical solution $u_\lambda \in \bigcap_{p\geq 2} W^{2,p}(\Omega)$. Moreover,

$$\limsup_{\lambda\to 0^+} \frac{\|u_\lambda\|_{C^1(\overline{\Omega})}}{\lambda^{q/(1-q)}} < \infty, \qquad \limsup_{\lambda\to 0^+} \frac{\|u_\lambda\|_{W^{2,p}(\Omega)}}{\lambda^{q^2/(1-q)}} < \infty$$

for all $p \geq 2$, and the function

$$\lambda \mapsto \frac{1}{2}\int_\Omega |\nabla u_\lambda(x)|^2\,dx - \frac{1}{s+1}\int_\Omega \alpha(x)|u_\lambda(x)|^{s+1}\,dx - \frac{\lambda}{q+1}\int_\Omega \beta(x)|u_\lambda(x)|^{q+1}\,dx$$

is negative and decreasing in $]0, \lambda^*[$.

Note a remarkable improvement with respect to the version of Proposition 1 one would get by applying Theorem A. In this case, in fact, condition $(*)$ should be replaced by $\inf_\Omega \beta > 0$.

PROPOSITION 2. *Let $\varphi \in C^2([0,\infty[)$ be bounded together with φ' and φ'', and let $a, \mu, s \in \mathbb{R}$ with $a > 0$ and $s > 1$. Then, for some $\lambda^* > 0$ and for each $\lambda \in]0, \lambda^*[$, the problem*

$$\begin{cases} -\Delta u = \mu u^s + \lambda[(\varphi'(|\log u|^2) - a)\log u + \varphi(|\log u|^2) - a/2]u & \text{in } \Omega, \\ u_{|\partial\Omega} = 0, \end{cases}$$

admits a positive classical solution $u_\lambda \in C^2(\overline{\Omega})$. Moreover, for each $r > 0$ and $p \geq 2$,

$$\limsup_{\lambda\to 0^+} \frac{\|u_\lambda\|_{W^{2,p}(\Omega)}}{\lambda^r} < \infty$$

and the function

$$\lambda \mapsto \frac{1}{2}\int_\Omega |\nabla u_\lambda(x)|^2\,dx - \frac{\mu}{s+1}\int_\Omega |u_\lambda(x)|^{s+1}\,dx - \frac{\lambda}{2}\int_\Omega |u_\lambda(x)|^2(\varphi(|\log u_\lambda(x)|^2) - a\log u_\lambda(x))\,dx$$

is negative and decreasing in $]0, \lambda^*[$.

The proof of Proposition 2 is given in Section 3. In view of the above discussion, Proposition 2 is particularly interesting when the set $\{\xi > 0 : \varphi'(\xi) \geq a\}$ is unbounded.

On the other hand, from the comparison with Theorem 2.1 of [1], an open question arises: under the assumptions of Theorem 1, does problem

(P_λ) admit a non-zero, non-negative, minimal solution for each $\lambda > 0$ small enough? We conjecture that the answer is negative.

Finally, we point out that our proof of Theorem 1 is genuinely variational. Precisely, it comes from combining, in a careful way, a truncation and bootstrap argument (inspired by [3]) with the general approach to finding local minima proposed in [5].

2. Proof of Theorem 1. First of all, observe that, by (i) and (ii), there are $\alpha, L > 0$, with $\alpha \le 1$, such that

$$|f(x,\xi)| \le L|\xi|^s \quad \text{and} \quad |g(x,\xi)| \le L|\xi|^q$$

for every $x \in \Omega$, $\xi \in [0,\alpha]$. Of course, if $n \ge 3$, it is not restrictive to assume that $s \le (n+2)/(n-2)$. Next, define $f_0, g_0 : \Omega \times \mathbb{R} \to \mathbb{R}$ as follows:

$$f_0(x,\xi) = \begin{cases} f(x,\alpha) & \text{if } \xi > \alpha, \\ f(x,\xi) & \text{if } \xi \in [0,\alpha], \\ 0 & \text{if } \xi < 0, \end{cases} \qquad g_0(x,\xi) = \begin{cases} g(x,\alpha) & \text{if } \xi > \alpha, \\ g(x,\xi) & \text{if } \xi \in [0,\alpha], \\ 0 & \text{if } \xi < 0. \end{cases}$$

Of course, we have

$$|f_0(x,\xi)| \le L\min\{|\xi|^s, |\xi|\} \tag{1}$$

and

$$|g_0(x,\xi)| \le L|\xi|^q \tag{2}$$

for every $x \in \Omega$, $\xi \in \mathbb{R}$. For simplicity, denote by E the space $W_0^{1,2}(\Omega)$ equipped with the norm

$$\|u\| = \Big(\int_\Omega |\nabla u(x)|^2\,dx\Big)^{1/2}.$$

For each $u \in E$, put

$$\Phi(u) = -\int_\Omega \Big(\int_0^{u(x)} g_0(x,\xi)\,d\xi\Big)\,dx,$$

$$\Psi(u) = \int_\Omega |\nabla u(x)|^2\,dx - 2\int_\Omega\Big(\int_0^{u(x)} f_0(x,\xi)\,d\xi\Big)\,dx.$$

First of all, note that, since f_0, g_0 are bounded, the functionals Φ, Ψ turn out to be well defined, continuous and Gateaux differentiable in E. Moreover, by the Rellich–Kondrashov theorem, Φ is sequentially weakly continuous and Ψ is sequentially weakly lower semicontinuous. By (1) and by the Sobolev embedding theorem, for some constant $c > 1$ and for all $u \in E$, we have

$$\Psi(u) \ge \int_\Omega |\nabla u(x)|^2\,dx - 2L\int_\Omega |u(x)|^{s+1}\,dx \ge \|u\|^2(1 - c\|u\|^{s-1}).$$

From this, since $s > 1$, we get

$$\inf_{r \le \|u\| \le (2c)^{1/(1-s)}} \Psi(u) \ge r^2/2 \tag{3}$$

for all $r \in]0, (2c)^{1/(1-s)}[$.

We now prove that

$$\liminf_{\|u\| \to 0^+} \frac{\Phi(u)}{\Psi(u)} = -\infty. \tag{4}$$

To this end, we use condition (iii). So, fix a sequence $\{\xi_k\}$ in $]0, 1[$, converging to 0, and constants $\delta \in]0, \alpha]$ and Λ in such a way that

$$\lim_{k \to \infty} \frac{\inf_{x \in B} \int_0^{\xi_k} g(x,t)\,dt}{\xi_k^2} = \infty$$

and

$$\inf_{x \in D} \int_0^{\xi} g(x,t)\,dt \ge \Lambda \xi^2$$

for all $\xi \in [0, \delta]$. Next, fix a set $C \subset B$ of positive measure and a function $v \in E$ such that $v(x) \in [0,1]$ for all $x \in \Omega$, $v(x) = 1$ for all $x \in C$ and $v(x) = 0$ for all $x \in \Omega \setminus D$. Finally, fix $Q > 0$ and M satisfying

$$Q < \frac{M \operatorname{meas}(C) + \Lambda \int_{D \setminus C} |v(x)|^2\,dx}{\|v\|^2 + \frac{2L}{s+1} \int_D |v(x)|^{s+1}\,dx}.$$

Then there is $\nu \in \mathbb{N}$ such that $\xi_k < \delta$, $\Psi(\xi_k v) > 0$ (recall (3)) and

$$\inf_{x \in B} \int_0^{\xi_k} g(x,t)\,dt \ge M \xi_k^2$$

for all $k > \nu$. Taking into account (1) and that $\xi_k < 1$, for each $k > \nu$ we have

$$\begin{aligned} -\frac{\Phi(\xi_k v)}{\Psi(\xi_k v)} &\ge \frac{\int_C (\int_0^{\xi_k} g_0(x,t)\,dt)\,dx + \int_{D \setminus C} (\int_0^{\xi_k v(x)} g_0(x,t)\,dt)\,dx}{\xi_k^2 \|v\|^2 + \frac{2L}{s+1} \xi_k^{s+1} \int_D |v(x)|^{s+1}\,dx} \\ &\ge \frac{M \operatorname{meas}(C) + \Lambda \int_{D \setminus C} |v(x)|^2\,dx}{\|v\|^2 + \frac{2L}{s+1} \int_D |v(x)|^{s+1} dx} > Q. \end{aligned}$$

Since Q could be arbitrarily large, it follows that

$$\lim_{k \to \infty} -\frac{\Phi(\xi_k v)}{\Psi(\xi_k v)} = \infty$$

from which (4) clearly follows.

Now, for each $\varrho > 0$, we denote by X_ϱ the closed ball in E, centred at 0, of radius ϱ. Note that, by (4), one has $\inf_{X_\varrho} \Phi < 0$. Put

$$\gamma = \sup_{\varrho>0} \frac{-\inf_{X_\varrho} \Phi}{\varrho^{q+1}}.$$

By (2), it follows that $\gamma < \infty$. So, we have

$$\frac{\varrho^2}{-\inf_{X_\varrho} \Phi} \geq \frac{1}{\gamma}\, \varrho^{1-q} \tag{5}$$

for all $\varrho > 0$. Next, fix λ satisfying

$$0 < \lambda \leq \overline{\lambda}, \tag{6}$$

where

$$\overline{\lambda} = \frac{1}{8} \min\left\{\frac{1}{\gamma}\,(2c)^{(1-q)/(1-s)}, -\frac{1}{\inf_{X_1} \Phi}\right\},$$

the constant c being that in (3). Also, put

$$\varrho_\lambda = (8\gamma\lambda)^{1/(1-q)}. \tag{7}$$

So, in particular, we have

$$\varrho_\lambda \leq (2c)^{1/(1-s)}. \tag{8}$$

Since E is reflexive, X_{ϱ_λ} is sequentially weakly compact. Thus, since $\Phi + \frac{1}{2\lambda}\Psi$ is sequentially weakly lower semicontinuous, there is $u_\lambda \in X_{\varrho_\lambda}$ such that

$$\Phi(u_\lambda) + \frac{1}{2\lambda}\,\Psi(u_\lambda) = \inf_{u\in X_{\varrho_\lambda}} \left(\Phi(u) + \frac{1}{2\lambda}\,\Psi(u)\right).$$

We claim that

$$\Psi(u_\lambda) < -4\lambda \inf_{X_{\varrho_\lambda}} \Phi. \tag{9}$$

Arguing by contradiction, assume that $\Psi(u_\lambda) \geq -4\lambda \inf_{X_{\varrho_\lambda}} \Phi$. Then, taking into account that $\inf_{X_{\varrho_\lambda}} \Phi < 0$, we would have

$$\begin{aligned}\Phi(u_\lambda) - 2\inf_{X_{\varrho_\lambda}} \Phi &= \Phi(u_\lambda) + \frac{1}{2\lambda}\left(-4\lambda \inf_{X_{\varrho_\lambda}} \Phi\right) \leq \Phi(u_\lambda) + \frac{1}{2\lambda}\,\Psi(u_\lambda) \\ &\leq \Phi(0) + \frac{1}{2\lambda}\,\Psi(0) = 0 < \inf_{X_{\varrho_\lambda}} \Phi - 2\inf_{X_{\varrho_\lambda}} \Phi \leq \Phi(u_\lambda) - 2\inf_{X_{\varrho_\lambda}} \Phi,\end{aligned}$$

which is absurd.

Now, observe that, due to (4), there is a sequence $\{v_k\}$ in $X_{\varrho_\lambda} \setminus \{0\}$ such that $\lim_{k\to\infty} \Phi(v_k)/\Psi(v_k) = -\infty$. Hence, for k large enough, we have

$$\frac{\Phi(v_k)}{\Psi(v_k)} < -\frac{1}{2\lambda}$$

and so (by (3) and (8))

$$\Phi(v_k) + \frac{1}{2\lambda}\Psi(v_k) < 0 = \Phi(0) + \frac{1}{2\lambda}\Psi(0).$$

This means that

$$\inf_{X_{\varrho_\lambda}} \left(\Phi + \frac{1}{2\lambda}\Psi\right) < 0. \tag{10}$$

Hence, $u_\lambda \neq 0$. Next, from (5) and (7), we get

$$\varrho_\lambda^2 \geq -\frac{1}{\gamma} \inf_{X_{\varrho_\lambda}} \Phi \varrho_\lambda^{1-q} = -8\lambda \inf_{X_{\varrho_\lambda}} \Phi.$$

Consequently,

$$(-8\lambda \inf_{X_{\varrho_\lambda}} \Phi)^{1/2} \leq \varrho_\lambda.$$

From (3) and (8), we infer that for each $u \in X_{\varrho_\lambda}$ satisfying

$$(-8\lambda \inf_{X_{\varrho_\lambda}} \Phi)^{1/2} \leq \|u\|$$

one has

$$\Psi(u) \geq -4\lambda \inf_{X_{\varrho_\lambda}} \Phi.$$

Hence, in view of (9), since $u_\lambda \in X_{\varrho_\lambda}$, one has

$$\|u_\lambda\| < (-8\lambda \inf_{X_{\varrho_\lambda}} \Phi)^{1/2}. \tag{11}$$

From this, in particular, it follows that u_λ is a local minimum in E of the functional $\Phi + \frac{1}{2\lambda}\Psi$, and hence

$$\Phi'(u_\lambda) + \frac{1}{2\lambda}\Psi'(u_\lambda) = 0.$$

This means that

$$\int_\Omega \nabla u_\lambda(x) \nabla v(x)\, dx - \int_\Omega f_0(x, u_\lambda(x)) v(x) dx - \lambda \int_\Omega g_0(x, u_\lambda(x)) v(x)\, dx = 0 \tag{12}$$

for all $v \in E$.

We claim that u_λ is non-negative in Ω. Assume the contrary. Then, by the continuity of u_λ (see below), the set $A = \{x \in \Omega : u_\lambda(x) < 0\}$ is non-empty and open. Of course, $u_{\lambda|A} \in W_0^{1,2}(A)$, and (by (12)), for each $v \in C_0^\infty(A)$, one has

$$\int_A \nabla u_\lambda(x) \nabla v(x)\, dx = 0.$$

By density, this equality actually holds for each $v \in W_0^{1,2}(A)$, and so, in particular, $\int_A |\nabla u_\lambda(x)|^2\,dx = 0$, which is absurd.

Next, since f_0, g_0 are bounded, from standard regularity results ([2, Theorems 8.8 and 8.12 and Lemmas 9.16 and 9.17]), it follows that, for each $p > 1$, u_λ belongs to $W^{2,p}(\Omega)$, one has

$$-\Delta u_\lambda(x) = f_0(x, u_\lambda(x)) + \lambda g_0(x, u_\lambda(x)) \tag{13}$$

for almost every $x \in \Omega$, and there exists some constant c_p independent of λ such that

$$\|u_\lambda\|_{W^{2,p}(\Omega)} \le c_p\Big(\int_\Omega |f_0(x, u_\lambda(x)) + \lambda g_0(x, u_\lambda(x))|^p\,dx\Big)^{1/p}.$$

Then, in view of (1), (2) and (6), taking into account that $q < 1$, by the Hölder inequality, we have

$$\|u_\lambda\|_{W^{2,p}(\Omega)} \le c'_p(\|u_\lambda\|_{L^p(\Omega)} + \|u_\lambda\|^q_{L^p(\Omega)}) \tag{14}$$

where

$$c'_p = c_p L \max\{1, \overline{\lambda}(\mathrm{meas}(\Omega))^{(1-q)/p}\}.$$

We now claim that there is a constant c'' independent of λ such that

$$\|u_\lambda\|_{C^1(\overline{\Omega})} \le c''(\|u_\lambda\| + \|u_\lambda\|^q). \tag{15}$$

The basic fact is that $W^{2,t}(\Omega)$ is continuously embedded in $C^1(\overline{\Omega})$ for each $t > n$. So, if $n = 1$, then (15) follows directly from (14) for $p = 2$. If $n = 2$, the same happens by taking $p = 3$ and observing that $W^{1,2}(\Omega)$ is continuously embedded in $L^3(\Omega)$. If $n > 2$, since $W^{2,p}(\Omega)$ (resp. $W^{2,n/2}(\Omega)$) is continuously embedded in $L^{np/(n-2p)}(\Omega)$ for $p < n/2$ (resp. in $L^r(\Omega)$ for each $r \ge 1$), we use (14) iteratively starting from $p = 3/2$. We thus get (15) after a finite number of steps.

Now, putting together (5), (7), (11) and (15), and recalling that $\|u_\lambda\| \le 1$ (by (6)), we get

$$\begin{aligned} \|u_\lambda\|_{C^1(\overline{\Omega})} &\le 2c''\|u_\lambda\|^q < 2c''(8\gamma(8\gamma\lambda)^{(q+1)/(1-q)}\lambda)^{q/2} \\ &\le 2c''(8\gamma)^{q/(1-q)}\lambda^{q/(1-q)}. \end{aligned} \tag{16}$$

Therefore, if $\lambda < \lambda^*$ with $\lambda^* \le \overline{\lambda}$ small enough, then $\|u_\lambda\|_{C^1(\overline{\Omega})} \le \alpha$, and hence $f_0(x, u_\lambda(x)) = f(x, u_\lambda(x))$, $g_0(x, u_\lambda(x)) = g(x, u_\lambda(x))$ for all $x \in \Omega$. So, in view of (13), u_λ is a non-zero, non-negative strong solution of problem (P_λ), and, by (14) and (16), one has

$$\limsup_{\lambda\to 0^+} \frac{\|u_\lambda\|_{C^1(\overline{\Omega})}}{\lambda^{q/(1-q)}} < \infty, \qquad \limsup_{\lambda\to 0^+} \frac{\|u_\lambda\|_{W^{2,p}(\Omega)}}{\lambda^{q^2/(1-q)}} < \infty$$

for all $p > 1$. Now, let $0 < \lambda' < \lambda'' < \lambda^*$. Then, since $\varrho_{\lambda'} < \varrho_{\lambda''}$ and $\Psi(u_{\lambda'}) > 0$, we have

$$\Phi(u_{\lambda''}) + \frac{1}{2\lambda''}\Psi(u_{\lambda''}) \le \Phi(u_{\lambda'}) + \frac{1}{2\lambda''}\Psi(u_{\lambda'}) < \Phi(u_{\lambda'}) + \frac{1}{2\lambda'}\Psi(u_{\lambda'}).$$

For each $\lambda \in]0, \lambda^*[$, we have

$$I_\lambda(u_\lambda) = \lambda\biggl(\Phi(u_\lambda) + \frac{1}{2\lambda}\Psi(u_\lambda)\biggr).$$

Then, recalling (10), we conclude that the function $\lambda \mapsto I_\lambda(u_\lambda)$ is negative and decreasing in $]0, \lambda^*[$.

Finally, assume the additional hypotheses to prove that u_λ is positive. Of course, we can assume that $\alpha < 1/e$ and that

$$g(x,\xi) \ge -L\xi|\log\xi|^2$$

for all $x \in \Omega$ and $\xi \in]0, \alpha]$. Put

$$h(\xi) = \begin{cases} L(1+\lambda^*)\xi|\log\xi|^2 & \text{if } \xi \in]0,\alpha], \\ 0 & \text{if } \xi = 0, \\ L(1+\lambda^*)\alpha|\log\alpha|^2 & \text{if } \xi > \alpha. \end{cases}$$

Recalling (1), for $\lambda \in]0, \lambda^*[$, we have

$$f_0(x,\xi) + \lambda g_0(x,\xi) \ge -L\xi - \lambda L\xi|\log\xi|^2 > -L(1+\lambda)\xi|\log\xi|^2$$

for all $x \in \Omega$ and $\xi \in]0, \alpha]$. Consequently,

$$f_0(x,\xi) + \lambda g_0(x,\xi) \ge -h(\xi) \tag{17}$$

for all $x \in \Omega$ and $\xi \ge 0$. Clearly,

$$\int_0^1 (\xi h(\xi))^{-1/2}\, d\xi = (L(1+\lambda^*))^{-1/2}\int_0^1 \frac{1}{\xi|\log\xi|}\, d\xi = \infty. \tag{18}$$

Now, in view of (12), (17) and (18), the positivity of u_λ in Ω is ensured by Theorem 3 of [4] (see also [6]). The proof is complete. ∎

3. Remarks. With obvious changes in the above proof, we also obtain

THEOREM 2. *Assume that*:

(i_1) *there is* $s > 1$ *such that*

$$\limsup_{\xi\to 0^-} \frac{\sup_{x\in\Omega}|f(x,\xi)|}{|\xi|^s} < \infty;$$

(ii_1) *there is* $q \in]0,1[$ *such that*

$$\limsup_{\xi\to 0^-} \frac{\sup_{x\in\Omega}|g(x,\xi)|}{|\xi|^q} < \infty;$$

(iii$_1$) *there are a non-empty open set $D \subseteq \Omega$ and a set $B \subseteq D$ of positive measure such that*

$$\limsup_{\xi\to 0^-} \frac{\inf_{x\in B}\int_0^\xi g(x,t)\,dt}{\xi^2} = \infty, \qquad \liminf_{\xi\to 0^-} \frac{\inf_{x\in D}\int_0^\xi g(x,t)\,dt}{\xi^2} > -\infty.$$

Then, for some $\lambda^* > 0$ and for each $\lambda \in\,]0,\lambda^*[$, problem (P_λ) *admits a non-zero, non-positive strong solution $u_\lambda \in \bigcap_{p\geq 2} W^{2,p}(\Omega)$. Moreover,*

$$\limsup_{\lambda\to 0^+} \frac{\|u_\lambda\|_{C^1(\overline{\Omega})}}{\lambda^{q/(1-q)}} < \infty, \qquad \limsup_{\lambda\to 0^+} \frac{\|u_\lambda\|_{W^{2,p}(\Omega)}}{\lambda^{q^2/(1-q)}} < \infty$$

for all $p \geq 2$, and the function $\lambda \mapsto I_\lambda(u_\lambda)$ is negative and decreasing in $]0,\lambda^*[$.

So, putting together Theorems 1 and 2, we get

THEOREM 3. *Assume that:*

(i$_2$) *there is $s > 1$ such that*

$$\limsup_{\xi\to 0} \frac{\sup_{x\in\Omega}|f(x,\xi)|}{|\xi|^s} < \infty;$$

(ii$_2$) *there is $q \in\,]0,1[$ such that*

$$\limsup_{\xi\to 0} \frac{\sup_{x\in\Omega}|g(x,\xi)|}{|\xi|^q} < \infty;$$

(iii$_2$) *there are a non-empty open set $D \subseteq \Omega$ and a set $B \subseteq D$ of positive measure such that*

$$\limsup_{\xi\to 0^-} \frac{\inf_{x\in B}\int_0^\xi g(x,t)\,dt}{\xi^2} = \limsup_{\xi\to 0^+} \frac{\inf_{x\in B}\int_0^\xi g(x,t)\,dt}{\xi^2} = \infty,$$

$$\liminf_{\xi\to 0} \frac{\inf_{x\in D}\int_0^\xi g(x,t)\,dt}{\xi^2} > -\infty.$$

Then, for some $\lambda^* > 0$ and for each $\lambda \in\,]0,\lambda^*[$, problem (P_λ) *admits a non-zero, non-negative strong solution $u_\lambda \in \bigcap_{p\geq 2} W^{2,p}(\Omega)$ and a non-zero, non-positive strong solution $v_\lambda \in \bigcap_{p\geq 2} W^{2,p}(\Omega)$. Moreover,*

$$\limsup_{\lambda\to 0^+} \frac{\max\{\|u_\lambda\|_{C^1(\overline{\Omega})}, \|v_\lambda\|_{C^1(\overline{\Omega})}\}}{\lambda^{q/(1-q)}} < \infty,$$

$$\limsup_{\lambda\to 0^+} \frac{\max\{\|u_\lambda\|_{W^{2,p}(\Omega)}, \|v_\lambda\|_{W^{2,p}(\Omega)}\}}{\lambda^{q^2/(1-q)}} < \infty$$

for all $p \geq 2$, and the functions $\lambda \mapsto I_\lambda(u_\lambda)$, $\lambda \mapsto I_\lambda(v_\lambda)$ are negative and decreasing in $]0,\lambda^*[$.

Remark 1. Assume that the assumptions of Theorem 1 are satisfied. In addition, suppose that there exists $\eta > 0$ such that the functions f, g are locally Hölder continuous in $\Omega \times [0,\eta]$. Then each u_λ is a classical solution of problem (P_λ). If f, g are Hölder continuous in $\Omega \times [0,\eta]$, we even have $u_\lambda \in C^2(\overline{\Omega})$.

To see this, we can assume $\sup_\Omega u_\lambda \le \eta$. Since u_λ is Lipschitzian in Ω and Ω is bounded, the composite function $x \mapsto f(x, u_\lambda(x)) + \lambda g(x, u_\lambda(x))$ is then locally Hölder continuous in Ω (it turns out to be Hölder continuous in Ω when so f, g are in $\Omega \times [0,\eta]$). Now, our claim follows directly from Theorem 9.19 of [2].

Remark 2. Clearly, Remark 1 applies to Proposition 1.

Proof of Proposition 2. Apply Theorem 1 taking $f(\xi) = \mu\xi^s$ for all $\xi \ge 0$ and

$$g(\xi) = \begin{cases} [(\varphi'(|\log\xi|^2) - a)\log\xi + \varphi(|\log\xi|^2) - a/2]\xi & \text{if } \xi > 0, \\ 0 & \text{if } \xi = 0. \end{cases}$$

So, f, g are continuous, and (i), (ii) (with any $q \in]0,1[$) are clearly satisfied. For $\xi > 0$, we have

$$\int_0^\xi g(t)\,dt = \frac{1}{2}\xi^2(\varphi(|\log\xi|^2) - a\log\xi).$$

Hence, since $a > 0$ and φ is bounded, (iii) also holds. Furthermore, since φ' is bounded, we have

$$\liminf_{\xi\to 0^+} \frac{g(\xi)}{\xi|\log\xi|} > -\infty$$

and hence, *a fortiori*,

$$\liminf_{\xi\to 0^+} \frac{g(\xi)}{\xi|\log\xi|^2} > -\infty.$$

Finally, since φ'' is bounded, for each $\alpha \in]0,1[$, we have

$$\lim_{\xi\to 0^+} (g'(\xi) + \alpha\xi^{\alpha-1}) = \infty, \qquad \lim_{\xi\to 0^+} (g'(\xi) - \alpha\xi^{\alpha-1}) = -\infty.$$

Hence, in a (right, bounded) neighbourhood of 0, the function $\xi \mapsto g(\xi) + \xi^\alpha$ is increasing and the function $\xi \mapsto g(\xi) - \xi^\alpha$ is decreasing. Of course, this implies that the function g (as well as f, of course) is Hölder continuous, with exponent α, in that neighbourhood. Now, the conclusion follows directly from Theorem 1 jointly with Remark 1. ∎

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