C O L L O Q U I U M   M A T H E M A T I C U M

VOL. 110 2008 NO. 1

VAN DER CORPUT SETS IN $\mathbb{Z}^d$

BY

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Abstract. In this partly expository paper we study van der Corput sets in $\mathbb{Z}^d$, with a focus on connections with harmonic analysis and recurrence properties of measure preserving dynamical systems. We prove multidimensional versions of some classical results obtained for $d = 1$ by Kamae and M. Mendès France and by Ruzsa, establish new characterizations, introduce and discuss some modifications of van der Corput sets which correspond to various notions of recurrence, provide numerous examples and formulate some natural open questions.

Contents

2000 *Mathematics Subject Classification*: 11K06, 28D05, 37A45, 42A82.

Key words and phrases: van der Corput inequality, van der Corput set, set of recurrence, intersective set, uniform distribution, positive-definite sequences, ergodic sequences.

The first author gratefully acknowledges the support of the NSF under grant DMS-0600042.

INTRODUCTION

The main topic of our paper is the intriguing connection between positive-definite sequences, recurrence properties of measure preserving dynamical systems, and the theory of uniform distribution mod 1.

Let $(X, \mathcal{A}, \mu, T)$ be an invertible probability measure preserving dynamical system ([1]). Given a set $A \in \mathcal{A}$ with $\mu(A) > 0$, let $R_A = \{n \in \mathbb{Z},\, n \neq 0 : \mu(A \cap T^n A) > 0\}$. While the classical Poincaré recurrence theorem, which states that the set R_A is non-empty (and hence infinite), is nowadays an easy exercise, quite a few of the more subtle properties of *sets of returns* R_A and of the related sets $R_{A,\varepsilon} = \{n \in \mathbb{Z},\, n \neq 0 : \mu(A \cap T^n A) > \varepsilon\}$ are still not fully understood.

Following Furstenberg ([Fu2]), let us call a set of integers D a *set of recurrence* if for any m.p.s. $(X, \mathcal{A}, \mu, T)$ and any $A \in \mathcal{A}$ with $\mu(A) > 0$ one has $D \cap R_A \neq \emptyset$. For example, for any $k \in \mathbb{N}$, the set $k\mathbb{N}$ is a set of recurrence (just consider the system $(X, \mathcal{A}, \mu, T^k)$) and any set of recurrence has a non-empty intersection with the set $k\mathbb{N}$ (just consider a permutation of a finite set). A more general (and still rather trivial) example is provided by the set of differences $\{n_i - n_j : i > j\}$, where $(n_i)_{i \geq 1}$ is an increasing sequence of integers. (To see that this is a set of recurrence, just observe that if $\mu(A) > 0$, then the sets $T^{n_i}A$ cannot be pairwise disjoint, $\mu(X)$ being finite.) The following generalization of the Poincaré recurrence theorem obtained by Furstenberg (see [Fu1], [Fu2]) gives a much less trivial example of a set of recurrence.

([1]) Unless explicitly stated otherwise, we will assume in this paper that the measure preserving transformations we are dealing with are invertible and that invariant measures are normalized. We will write *m.p.s.* for *invertible probability measure preserving dynamical system*.

Theorem 0.1. *For any polynomial $p(n) \in \mathbb{Z}[n]$ satisfying $p(0) = 0$, for any m.p.s. $(X, \mathcal{A}, \mu, T)$ and for any $A \in \mathcal{A}$ with $\mu(A) > 0$, there exists $n \in \mathbb{N}$ such that $p(n) \neq 0$ and $\mu(A \cap T^{p(n)}A) > 0$.*

Following Ruzsa ([Ruz]), let us call a set $D \subset \mathbb{N}$ *intersective* if for any $S \subset \mathbb{N}$ of positive upper density (2) there exist $x, y \in S$ such that $x - y \in D$. It is not hard to show that a set D is intersective if and only if it is a set of *combinatorial recurrence*, that is, such that for any $S \subset \mathbb{N}$ with $\bar{d}(S) > 0$, there exists $n \in D$ such that $\bar{d}(S \cap (S - n)) > 0$. This hints that the notions of "set of recurrence" and "intersective set" are related and, indeed, it turns out that these notions coincide. (The fact that intersectivity implies measure-theoretic recurrence has been remarked by several authors, see for example [BM] and [Berg.1]. The fact that measure-theoretic recurrence implies combinatorial recurrence is a consequence of *Furstenberg's correspondence principle*, see for example [Berg.3].)

Thus, for example, Theorem 0.1 implies Sárközy's theorem ([S]), which states that for any polynomial $p(n) \in \mathbb{Z}[n]$ satisfying $p(0) = 0$ and any set $S \subset \mathbb{N}$ with $\bar{d}(S) > 0$ there exist $x, y \in S$ and $n \in \mathbb{N}$ such that $x - y = p(n)$. We remark that it was shown in [Kam-MF] that a necessary and sufficient condition for a polynomial $p(n) \in \mathbb{Z}[n]$ to satisfy the Furstenberg–Sárközy theorem is that for any positive integer k there exists an integer n such that $p(n)$ is divisible by k. Actually, Kamae and Mendès France in [Kam-MF] showed that many sets of recurrence, including the sets mentioned above, have a stronger property which they called the *van der Corput property*.

Definition. A set D of positive integers is a *van der Corput set* (or *vdC set*) if it has the following property: given a real sequence $(x_n)_{n\in\mathbb{N}}$, if all the sequences $(x_{n+d} - x_n)_{n\in\mathbb{N}}$, $d \in D$, are uniformly distributed mod 1, then the sequence $(x_n)_{n\in\mathbb{N}}$ is itself uniformly distributed mod 1.

This concept and terminology (3) come from the *van der Corput inequality*, which is presented at the beginning of the next section, and which motivates the following *van der Corput trick*: if for a given real sequence $(x_n)_{n\in\mathbb{N}}$ and any $h \in \mathbb{N}$ the sequence $(x_{n+h} - x_n)_{n\in\mathbb{N}}$ is uniformly distributed mod 1, then the sequence $(x_n)_{n\in\mathbb{N}}$ is uniformly distributed mod 1. Van der Corput's inequality and its application to uniform distribution appeared for the first time in [vdC], under the name *Dritte Haupteigenschaft* (third principal property).

(2) A subset S of $\mathbb{N}$ has *positive upper density* if

$$\bar{d}(S) := \limsup_{N\to\infty} \frac{1}{N} |S \cap \{1, \ldots, N\}| > 0.$$

(3) Ruzsa uses the name *correlative set* instead of van der Corput set.

Kamae and Mendès France showed in [Kam-MF] that every vdC set is a set of recurrence. The other implication is false: Bourgain has constructed in [Bou] an example of a set of recurrence which is not a vdC set.

The notions introduced above are connected via the notion of positive-definiteness. Indeed, it is easy to check that the sequence $(\mu(A\cap T^nA))$ is positive-definite ([4]), which establishes the connection between sets of recurrence and properties of positive-definite sequences. As for the vdC property, let us first note that in light of Weyl's criterion (see [Ku-N]), the sequence $(x_{n+d}-x_n)_{n\in\mathbb{N}}$ is uniformly distributed mod 1 if and only if, for any $k\in\mathbb{Z}$, $k\neq 0$, one has

$$(1)\qquad \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}e^{2\pi ik(x_{n+d}-x_n)}=\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}e^{2\pi ikx_{n+d}}\overline{e^{2\pi ikx_n}}=0.$$

Now, given a bounded sequence $\alpha:\mathbb{N}\to\mathbb{C}$, it is not hard to see that for some increasing sequence of integers $(N_j)_{j\in\mathbb{N}}$ the limit

$$(2)\qquad \lim_{j\to\infty}\frac{1}{N_j}\sum_{n=1}^{N_j}\alpha(n+d)\overline{\alpha(n)}=\gamma(d)$$

exists for all $d\in\mathbb{Z}$ and that, moreover, the sequence γ is positive-definite (see [Bert]). Juxtaposing (1) and (2) we see that the vdC property is also connected to the properties of positive-definite sequences.

By the Bochner–Herglotz theorem (see for example [Rud, Subsection 1.4.3]), any positive-definite sequence φ is given by the Fourier coefficients of a positive measure ν_φ on the circle:

$$\varphi(n)=\int_{\mathbb{T}}e^{2\pi inx}\,d\nu_\varphi(x),$$

and the properties of this measure play a crucial role in verifying that certain sets are vdC and in establishing the connections between (various versions of) vdC sets and sets of recurrence (see in particular Section 3 below).

The following fact is also useful for a better understanding of the link between vdC sets and sets of recurrence. Let $D\subset\mathbb{Z}$. We prove (see Corollary 1.31) that D is a vdC set if and only if the following is true: given a bounded sequence of complex numbers $(u_n)_{n\in\mathbb{N}}$, if for all $d\in D$, the sequence $(u_{n+d}\overline{u}_n)$ converges to zero in the Cesàro sense, then the sequence (u_n) also converges to zero in the Cesàro sense. We also prove (see Theorem 3.1) that D is a set of recurrence if and only if the analogous property holds with "(u_n) is a bounded sequence of complex numbers" replaced by "(u_n) is a bounded sequence of positive real numbers".

([4]) This fact was first noticed and utilized by Khinchin in [Kh].

Driven by the desire to obtain new applications to combinatorics and to better understand the recurrence properties of measure preserving $\mathbb{Z}^d$-actions, we focus in this paper on $\mathbb{Z}^d$ versions of vdC sets. As we will see, many known properties extend from $\mathbb{Z}$ to $\mathbb{Z}^d$ with relative ease. Still, some properties turn out to be more recalcitrant and their extensions to $\mathbb{Z}^d$ demand more work.

The definition of vdC set in $\mathbb{Z}^d$ is given in Subsection 1.2. Here are some examples of facts/theorems which will be obtained in subsequent sections:

- The class of vdC sets has the Ramsey property. Namely, if D is a vdC set in $\mathbb{Z}^d$ and if $D = D_1 \cup D_2$ then at least one of the D_i is a vdC set.
- Let $p_1, \dots, p_d$ be a finite family of polynomials with integer coefficients, to which we associate the subset $S = \{(p_1(n), \dots, p_d(n)) : n \in \mathbb{N}\}$ of $\mathbb{Z}^d$. The following properties are equivalent:
  - S is a set of recurrence for $\mathbb{Z}^d$-actions ([5]).
  - S is a vdC set in $\mathbb{Z}^d$.
  - S is a set of multiple recurrence for $\mathbb{Z}$-actions ([6]).
  - For any $q \in \mathbb{N}$, there exists $n \in \mathbb{N}$ such that $p_1(n), \dots, p_d(n)$ are all divisible by q.

  Moreover these equivalent properties are also necessary and sufficient for the set S to be an *enhanced vdC set* (see Definition 3 in Subsection 2.2) and a set of *strong recurrence* (see Definition 5 in Subsection 3.1).
- Let $\mathcal{P}$ be the set of prime numbers. For any finite family $f_1, \dots, f_d$ of polynomials with integer coefficients and with zero constant terms the set $\{f_1(p-1), \dots, f_d(p-1) : p \in \mathcal{P}\}$ is a vdC set in $\mathbb{Z}^d$. (It can also be proved that it is an enhanced vdC set; see below.)
- The Cartesian product of two vdC sets is a vdC set in the corresponding product of parameter spaces.
- A subset D of $\mathbb{Z}$ is a vdC set if and only if any positive measure σ on the torus $\mathbb{T}$ such that $\sum_{d\in D} |\widehat{\sigma}(d)| < \infty$ is continuous.
- We establish a *generalized van der Corput inequality* for multiparameter sequences in a Hilbert space (Proposition 1.30).

([5]) A subset S of $\mathbb{Z}^d$ is called a *set of recurrence for $\mathbb{Z}^d$-actions* if, given any measure preserving $\mathbb{Z}^d$-action $(T_n)_{n\in\mathbb{Z}^d}$ on a probability space $(X, \mathcal{A}, \mu)$ and any $A \in \mathcal{A}$ with $\mu(A) > 0$, there exists $n \in S$, $n \neq 0$ such that $\mu(A \cap T_n A) > 0$.

([6]) A subset S of $\mathbb{Z}^d$ is called a *set of multiple recurrence for $\mathbb{Z}$-actions* if, given any m.p.s. $(X, \mathcal{A}, \mu, T)$ and any $A \in \mathcal{A}$ with $\mu(A) > 0$, there exists $(n_1, \dots, n_d) \in S \setminus \{(0, \dots, 0)\}$ such that $\mu(A \cap T^{n_1} A \cap \dots \cap T^{n_d} A) > 0$.

In order to make the paper more readable we will restrict discussion mainly to dimension $d = 2$. The reader should have no problem verifying that our proofs work for general $d \in \mathbb{N}$.

In Section 2, we introduce the notion of *enhanced vdC set*. We show that the enhanced vdC property is equivalent to the FC^+ property (which appears in [Kam-MF], with a reference to Y. Katznelson). Moreover, the enhanced vdC property is related to the notion of strong recurrence in the same way as vdC sets are related to sets of recurrence. In Subsection 2.4 we collect some natural open questions.

In Section 3 we discuss links between the recurrence and vdC properties. We also introduce and discuss the notions of *density vdC set* and *nice vdC set*.

In Section 4 we briefly discuss some modifications of the notion of vdC set which are connected to various notions of uniform distribution.

It is worth mentioning that in practically every paper in the area of ergodic Ramsey theory, some version of the van der Corput trick for sequences in Hilbert spaces is used. See for example [Fu-Kat-O], [Berg-Lei.1], [Berg-Lei-McC], [Berg-McC], [Fr-Les-Wi] dealing with multiple recurrence, and [Berg.2], [Berg-Lei.2], [Ho-Kr], [Z] and [Lei] dealing with mean convergence of multiple ergodic averages. The van der Corput trick is also useful in establishing results pertaining to pointwise convergence: see for example [Les] and [Fr].

The influence on our work of the above-mentioned paper of Kamae and Mendès France, and of the fundamental ideas developed by Ruzsa in [Ruz], cannot be overestimated.

We are especially grateful to Randall McCutcheon for numerous useful suggestions, and would like to thank Inger Håland-Knutson, Anthony Quas and Máté Wierdl for pertinent communications.

Throughout the paper, we will use the classical notation $e(t) := e^{2\pi i t}$ for $t \in \mathbb{R}$ or $t \in \mathbb{T} = \mathbb{R}/\mathbb{Z}$.

1. VAN DER CORPUT SETS IN $\mathbb{Z}^d$

In this section we develop a theory of van der Corput sets in the multidimensional lattice $\mathbb{Z}^d$, which is parallel to the known theory in $\mathbb{Z}$ (see [Kam-MF], [Ruz], [Mo]). As we have already mentioned in the introduction, we limit our presentation to the case $d = 2$. Definitions, results and arguments in this section follow the one-dimensional case, except at one point: in order to obtain a *generalized van der Corput inequality*, Ruzsa uses in [Ruz] a theorem of Fejér stating that any positive trigonometric polynomials in one variable is the square modulus of another trigonometric polynomial; this fact is no longer true for trigonometric polynomials of several variables, hence we are forced to use a different argument to derive the generalized van der Corput inequality in the multidimensional case (cf. Subsection 1.4).

1.1. Van der Corput's inequality and van der Corput's principle

1.1.1. *Van der Corput's inequality in $\mathbb{Z}^2$.* For $a, b, c, d \in \mathbb{Z}$, we write $(a, b) \leq (c, d)$ if $a \leq c$ and $b \leq d$. (Similarly for $<$, $\geq$ and $>$.) We write 0 for $(0, 0) \in \mathbb{Z}^2$.

THEOREM 1.1. *Let $N = (N_1, N_2) \in \mathbb{N}^2$, and $(u_n)_{0<n\leq N}$ be a finite family of complex numbers indexed by $([1, N_1] \times [1, N_2]) \cap \mathbb{Z}^2$. For $h = (h_1, h_2) \in \mathbb{Z}^2$, define*

$$\gamma(N, h) := \sum_{\substack{0<n\leq N \\ 0<n+h\leq N}} u_{n+h}\overline{u}_n.$$

For any $H = (H_1, H_2) \in \mathbb{N}^2$, we have

$$\Big|\sum_{0<n\leq N} u_n\Big|^2 \leq [] \frac{(N_1 + H_1)(N_2 + H_2)}{H_1^2 H_2^2} \times \sum_{-H<h<H} (H_1 - |h_1|)(H_2 - |h_2|)\gamma(N, h).$$

The above inequality is usually applied in the following form:

$$\Big|\sum_{0<n\leq N} u_n\Big|^2 \leq \frac{(N_1 + H_1)(N_2 + H_2)}{H_1 H_2} \sum_{-H<h<H} |\gamma(N, h)|. \tag{3}$$

(The proof of Theorem 1.1 is an elementary application of Cauchy's inequality. It is a particular case of the calculations presented in Subsection 1.1.3.)

1.1.2. *Van der Corput's principle in $\mathbb{Z}^2$.* Let $(u_n)_{n\in\mathbb{N}^2}$ be a family of complex numbers. Starting from inequality (3), dividing by $(N_1 N_2)^2$, then letting N_1 and N_2 go to infinity, we deduce that, for any $H \in \mathbb{N}^2$,

$$\limsup_{N_1,N_2\to\infty} \left|\frac{1}{N_1 N_2} \sum_{0<n\leq N} u_n\right|^2 \leq \frac{1}{H_1 H_2} \sum_{-H<h<H} \limsup_{N_1,N_2\to\infty} \frac{1}{N_1 N_2} |\gamma(N, h)|.$$

As a direct consequence we obtain the following proposition.

PROPOSITION 1.2. *If $(u_n)_{n\in\mathbb{N}^2}$ is a family of complex numbers such that*

$$\inf_{H>0} \frac{1}{H_1 H_2} \sum_{-H<h<H} \limsup_{N_1,N_2\to\infty} \frac{1}{N_1 N_2} |\gamma(N, h)| = 0$$

then

$$\lim_{N_1,N_2\to\infty} \frac{1}{N_1N_2} \sum_{0<n\le N} u_n = 0.$$

We use the following notion of uniform distribution for a family indexed by $\mathbb{N}^2$.

DEFINITION 1. A family $(x_n)_{n\in\mathbb{N}^2}$ of real numbers is *uniformly distributed mod* 1 (*u.d. mod* 1) if for any continuous function f on $\mathbb{R}$, invariant under translations by elements of $\mathbb{Z}$, we have

$$\lim_{N_1,N_2\to\infty} \frac{1}{N_1N_2} \sum_{0<n\le N} f(x_n) = \int_{[0,1]} f(t)\,dt. \tag{4}$$

Other useful notions of uniform distribution can be introduced: for example, one can replace in (4) the averages $\left(\frac{1}{N_1N_2}\sum_{0<n\le N}\cdots\right)_{N_1,N_2\to\infty}$ by $\left(\frac{1}{(N_1-M_1)(N_2-M_2)}\sum_{M\le n<N}\cdots\right)_{N_1-M_1,N_2-M_2\to\infty}$; this leads to the notion of *well distributed sequences*. Or, one can consider averages defined by a given Følner sequence. We postpone remarks on these variations to Section 4.

Note that since property (4) has an asymptotic nature, it makes sense even if the entries in the sequence (x_n) are defined only for indices $n = (n_1, n_2)$ for n_1, n_2 large enough. We tacitly utilize this observation in the formulation of Corollary 1.3 below and throughout the paper.

Let us recall the classical Weyl criterion for uniform distribution (see [We], [Ku-N]). A family $(x_n)_{n\in\mathbb{N}^2}$ of real numbers is u.d. mod 1 if and only if, for any $k \in \mathbb{Z}\setminus\{0\}$,

$$\lim_{N_1,N_2\to\infty} \frac{1}{N_1N_2} \sum_{0<n\le N} e(kx_n) = 0.$$

As in dimension 1, van der Corput's principle in $\mathbb{Z}^d$ has a useful corollary pertaining to uniform distribution.

COROLLARY 1.3. *Let* $(x_n)_{n\in\mathbb{N}^2}$ *be a family of real numbers. If for any* $h \in \mathbb{Z}^2\setminus\{0\}$ *the family* $(x_{n+h}-x_n)_{n\in\mathbb{N}^2}$ *is u.d. mod* 1, *then the family* $(x_n)_{n\in\mathbb{N}^2}$ *is u.d. mod* 1.

When we apply Proposition 1.2 to prove Corollary 1.3, we see that it is sufficient to let only one of H_1, H_2 go to infinity. The following definition will allow us to give a more general version of this corollary.

Let D be a subset of $\mathbb{Z}^2$. We define

$$\delta(D) := \sup_{H_1,H_2\ge 0} \frac{1}{(2H_1+1)(2H_2+1)} \operatorname{card}(D\cap[-H_1,H_1]\times[-H_2,H_2]).$$

(Note that $\delta(D)$ is *not* the ordinary notion of density, which corresponds to $\limsup_{\min\{H_1,H_2\}\to\infty}$.)

COROLLARY 1.4. *Let $(x_n)_{n\in\mathbb{N}^2}$ be a family of real numbers, and $D \subset \mathbb{Z}^2 \setminus \{0\}$. If $\delta(D) = 1$ and if, for any $d \in D$, the family $(x_{n+d} - x_n)$ is u.d. mod 1, then the family (x_n) is u.d. mod 1.*

Proof. There exists a sequence $(H^{(k)})$ (with $H^{(k)} := (H_1^{(k)}, H_2^{(k)})$) in $(\mathbb{N} \cup \{0\})^2$ such that

$$\lim_{k\to\infty} \frac{1}{(2H_1^{(k)}+1)(2H_2^{(k)}+1)} \operatorname{card}(D \cap [-H_1^{(k)}, H_1^{(k)}] \times [-H_2^{(k)}, H_2^{(k)}]) = 1.$$

Let $(u_n)_{n\in\mathbb{N}^2}$ be a family of complex numbers of modulus 1 such that, for any $d \in D$,

$$\lim_{N_1,N_2\to\infty} \frac{1}{N_1N_2} \sum_{0<n\leq N} u_{n+d}\overline{u}_n = 0.$$

For any $d \in D$, we have

$$\lim_{N_1,N_2\to\infty} \frac{1}{N_1N_2} \gamma(N,d) = 0.$$

We deduce from van der Corput's inequality that

$$\left|\frac{1}{N_1N_2} \sum_{0<n\leq N} u_n\right|^2 \leq \frac{(N_1+H_1+1)(N_2+H_2+1)}{N_1N_2(H_1+1)(H_2+1)} \sum_{-H\leq d\leq H} \frac{1}{N_1N_2} |\gamma(N,d)|.$$

Using the fact that $|\gamma(N,d)| \leq N_1N_2$, we obtain

$$\limsup_{N_1,N_2\to\infty} \left|\frac{1}{N_1N_2} \sum_{0<n\leq N} u_n\right|^2 \leq \frac{1}{(H_1+1)(H_2+1)} \operatorname{card}(D^c \cap [-H_1, H_1] \times [-H_2, H_2]).$$

The right hand side of the last inequality goes to zero along the sequence $(H^{(k)})$. This argument can be applied to $u_n = e(kx_n)$ (no matter how x_n is defined for $n \in \mathbb{Z}^2 \setminus \mathbb{N}^2$) for any choice of $k \in \mathbb{Z}$, $k \neq 0$. Thus, the result follows from Weyl's criterion. ∎

EXAMPLE. If, for any positive integer j, the family $(x_{n+(j,0)} - x_n)$ is u.d. mod 1, then the family (x_n) is u.d. mod 1.

EXAMPLE. The first application of van der Corput's inequality was to Weyl's equidistribution theorem for polynomial sequences ([We], [vdC]). The two-parameter version of this theorem says the following: if $P \in \mathbb{R}[X, Y]$ is a real polynomial in two variables and if at least one coefficient of a non-constant monomial in P is irrational, then the family $(P(n_1,n_2))_{(n_1,n_2)\in\mathbb{N}^2}$ is uniformly distributed mod 1. (This result has a straightforward generalization to polynomials in more than two variables.) This multiparameter equidistribution theorem is a direct consequence of either Corollary 1.3, or Corollary 1.4 applied to the sets $D = 0\times\mathbb{N}$ and $D = \mathbb{N}\times 0$.

1.1.3. *An abstract version of van der Corput's principle*

Proposition 1.5. *Let $(G,\cdot)$ be a group, and E, D two finite subsets of G. Let u be a complex-valued function defined on E. Then*

$$\Big|\sum_{n\in E} u(n)\Big|^2 \le \frac{|E\cdot D^{-1}|}{|D|} \sum_{d\in D\cdot D^{-1}} \Big| \sum_{\substack{n\in E\\ n\in E\cdot d^{-1}}} u(n\cdot d)\overline{u(n)}\Big|. \tag{5}$$

Proof. Define $u(n)$ to be zero if $n\notin E$. We have

$$\Big|\sum_{n\in E} u(n)\Big|^2 = \Big|\frac{1}{|D|}\sum_{d\in D}\sum_{n\in E\cdot d^{-1}} u(n\cdot d)\Big|^2 = \Big|\frac{1}{|D|}\sum_{n\in E\cdot D^{-1}}\sum_{d\in D} u(n\cdot d)\Big|^2.$$

Using Cauchy's inequality, we obtain

$$\Big|\sum_{n\in E} u(n)\Big|^2 \le \frac{|E\cdot D^{-1}|}{|D|^2}\sum_{n\in G}\Big|\sum_{d\in D} u(n\cdot d)\Big|^2,$$

and this last expression is equal to

$$\begin{aligned}
&\frac{|E\cdot D^{-1}|}{|D|^2}\sum_{d,d'\in D}\sum_{n\in G} u(n\cdot d)\overline{u(n\cdot d')}\\
&\qquad\qquad = \frac{|E\cdot D^{-1}|}{|D|^2}\sum_{d'\in D}\sum_{d\in D\cdot d'^{-1}}\sum_{n\in G} u(n\cdot d)\overline{u(n)}\\
&\qquad\qquad \le \frac{|E\cdot D^{-1}|}{|D|^2}\sum_{d'\in D}\sum_{d\in D\cdot D^{-1}}\Big|\sum_{n\in G} u(n\cdot d)\overline{u(n)}\Big|. \quad\blacksquare
\end{aligned}$$

Note that inequality (5) contains inequality (3) as a special case corresponding to

$$G=\mathbb{Z}^2,\qquad E=([1,N_1]\times[1,N_2])\cap\mathbb{Z}^2,\qquad D=([1,H_1]\times[1,H_2])\cap\mathbb{Z}^2.$$

Remark 1.6. The vdC inequality that has been stated above for a family of complex numbers can be extended verbatim to any family of vectors in a complex linear space equipped with a scalar product. This fact is very useful in many applications to mean convergence theorems or recurrence theorems in ergodic theory (see for example Lemma A6 and the references in [Berg-McC]).

1.2. Van der Corput sets

1.2.1. *Definition*

DEFINITION 2. A subset D of $\mathbb{Z}^2 \setminus \{0\}$ is a *van der Corput set* (vdC set) if for any family $(u_n)_{n\in\mathbb{Z}^2}$ of complex numbers of modulus 1 such that

$$\forall d \in D, \qquad \lim_{N_1,N_2\to\infty} \frac{1}{N_1N_2} \sum_{0\le n<(N_1,N_2)} u_{n+d}\overline{u}_n = 0$$

we have

$$\lim_{N_1,N_2\to\infty} \frac{1}{N_1N_2} \sum_{0\le n<(N_1,N_2)} u_n = 0. \tag{6}$$

Equivalently, D is a vdC set if any family $(x_n)_{n\in\mathbb{N}^2}$ of real numbers having the property that for all $d \in D$ the family $(x_{n+d} - x_n)_{n\in\mathbb{N}^2}$ is u.d. mod 1, is itself u.d. mod 1.

(A natural Cesàro summation method is also given by "bilateral averages". One obtains an equivalent definition of vdC set if we replace in Definition 2 sums $\sum_{0\le n<(N_1,N_2)}$ by sums $\sum_{(-N_1,-N_2)<n<(N_1,N_2)}$. See Section 4.)

EXAMPLE 1.7. If $\delta(D) = 1$, then D is a vdC set (see Corollary 1.4).

Note that various modifications of the notion of uniform distributions (for example, considering other types of averages) lead, generally speaking, to different notions of vdC set. See Section 4 for some remarks and open questions.

1.2.2. *Spectral characterization.* If σ is a finite measure on the 2-torus $\mathbb{T}^2$, we define its Fourier transform $\widehat{\sigma}$ by $\widehat{\sigma}(n) = \int_{\mathbb{T}^2} e(n_1x_1 + n_2x_2)\,d\sigma(x_1, x_2)$ for any $n = (n_1, n_2) \in \mathbb{Z}^2$.

THEOREM 1.8. *Let $D \subset \mathbb{Z}^2 \setminus \{0\}$. The following statements are equivalent*:

- (S1) *D is a van der Corput set.*
- (S2) *If σ is a positive measure on $\mathbb{T}^2$ such that $\widehat{\sigma}(d) = 0$ for all $d \in D$, then $\sigma(\{(0,0)\}) = 0$.*
- (S3) *If σ is a positive measure on $\mathbb{T}^2$ such that $\widehat{\sigma}(d) = 0$ for all $d \in D$, then σ is continuous.*

(We prove later (Subsection 1.5) that (S1)–(S3) are equivalent to the following property: any positive measure σ on the 2-torus $\mathbb{T}^2$ such that $\sum_{d\in D} |\widehat{\sigma}(d)| < \infty$ is continuous.)

The equivalence of (S2) and (S3) is clear, since a translation of a measure does not change the modulus of its Fourier coefficients. For a one-dimensional space of parameters the implication (S2)⇒(S1) is proved in [Kam-MF] and the implication (S1)⇒(S2) can be found in [Ruz].

LEMMA 1.9. *Let $(u_n)_{n\in\mathbb{Z}^2}$ be a bounded family of complex numbers and $(N^{(j)})_{j\in\mathbb{N}} = ((N_1^{(j)}, N_2^{(j)}))_{j\in\mathbb{N}}$ be a sequence in $\mathbb{N}^2$ such that $\min(N_1^{(j)}, N_2^{(j)}) \to \infty$ as $j \to \infty$. If, for all $h \in \mathbb{Z}^2$,*

$$\gamma(h) := \lim_{j\to\infty} \frac{1}{N_1^{(j)} N_2^{(j)}} \sum_{0\le n<N^{(j)}} u_{n+h}\overline{u}_n$$

exists, then there exists a positive measure σ on the 2-torus $\mathbb{T}^2$ such that, for all $h \in \mathbb{Z}^2$,

$$\widehat{\sigma}(h) = \gamma(h)$$

and this measure satisfies

$$\limsup_{j\to\infty} \frac{1}{N_1^{(j)} N_2^{(j)}} \Big| \sum_{0\le n<N^{(j)}} u_n \Big| \le \sqrt{\sigma(\{(0,0)\})}.$$

Sketch of proof. We write $x = (x_1, x_2)$, $n = (n_1, n_2)$, etc.

The family $(\gamma_h)_{h\in\mathbb{Z}^2}$ is positive-definite and the Bochner–Herglotz theorem guarantees the existence of the desired positive measure σ (see for example [Rud, Subsection 1.4.3]). This measure is the weak limit of the sequence of absolutely continuous measures $(\sigma_{N^{(j)}})$ where σ_N has density

$$g_N(x) := \frac{1}{N_1 N_2} \Big| \sum_{0\le n<N} u_n e(-n_1x_1 - n_2x_2) \Big|^2$$

with respect to Lebesgue measure dx_1dx_2.

We define

$$h_N(x) := \frac{1}{N_1 N_2} \Big| \sum_{0\le n<N} e(-n_1x_1 - n_2x_2) \Big|^2.$$

The sequence of measures with density h_N converges weakly to the Dirac delta measure at $(0,0)$, denoted by δ.

We follow the method of [Co-Kam-MF], in particular their Theorem 2, which utilizes the connection between the *affinity* ([7]) of two probability measures and weak convergence. Denoting by $\varrho(\mu,\nu)$ the affinity of two

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([7]) Let μ and ν be two probability measures on $\mathbb{T}^2$. The *affinity* $\varrho(\mu,\nu)$ is defined as

$$\varrho(\mu,\nu) = \int_{\mathbb{T}^2} \left(\frac{d\mu}{dm}\right)^{1/2} \left(\frac{d\nu}{dm}\right)^{1/2} dm,$$

where m is any measure with respect to which both μ and ν are absolutely continuous. Note that affinity is also called the *Hellinger integral* by probabilists. It is proved in [Co-Kam-MF] that if (μ_n) and (ν_n) are two weakly convergent sequences of probability measures, then

$$\limsup_{n\to\infty} \varrho(\mu_n,\nu_n) \le \varrho(\lim \mu_n, \lim \nu_n).$$

probability measures on $\mathbb{T}^2$, we have

$$\varrho(g_N(x)dx, h_N(x)dx) = \int_{\mathbb{T}^2} \sqrt{g_N(x)h_N(x)}\,dx_1\,dx_2,$$

$$\varrho(\sigma,\delta) = \sqrt{\sigma(\{(0,0)\})},$$

and

$$\limsup_{j\to\infty} \int_{\mathbb{T}^2} \sqrt{g_{N^{(j)}}(x)h_{N^{(j)}}(x)}\,dx_1\,dx_2 \le \sqrt{\sigma(\{(0,0)\})}.$$

The conclusion of the lemma then follows from the inequality

$$\frac{1}{N_1^{(j)}N_2^{(j)}}\Big|\sum_{0\le n<N^{(j)}} u_n\Big| \le \int_{\mathbb{T}^2} \sqrt{g_{N^{(j)}}(x)h_{N^{(j)}}(x)}\,dx_1\,dx_2. \;\blacksquare$$

Proof of Theorem 1.8. Let us first prove that (S2)⇒(S1). Let $(u_n)_{n\in\mathbb{Z}^2}$ be a bounded family of complex numbers such that, for all $d \in D$,

$$\lim_{N_1,N_2\to\infty} \frac{1}{N_1N_2} \sum_{0\le n<(N_1,N_2)} u_{n+d}\overline{u}_n = 0.$$

There exists a sequence $(N^{(j)})_{j\in\mathbb{N}}$ in $\mathbb{N}^2$ such that:

- $\min(N_1^{(j)}, N_2^{(j)}) \to \infty$;
- $\displaystyle\lim_{j\to\infty} \frac{1}{N_1^{(j)}N_2^{(j)}}\Big|\sum_{0\le n<N^{(j)}} u_n\Big| = \limsup_{N_1,N_2\to\infty} \frac{1}{N_1N_2}\Big|\sum_{0\le n<N} u_n\Big|$;
- $\displaystyle\forall h\in\mathbb{Z}^2, \quad \gamma(h) := \lim_{j\to\infty} \frac{1}{N_1^{(j)}N_2^{(j)}} \sum_{0\le n<N^{(j)}} u_{n+h}\overline{u}_n$ exists.

The map γ is the Fourier transform of a positive measure σ on the 2-torus. We have $\widehat{\sigma}(d) = 0$ for all $d \in D$. By condition (S2), the measure σ has no point mass at $(0,0)$, and, using Lemma 1.9, we conclude that the family $(u_n)_{n\in\mathbb{Z}^2}$ converges to zero in the sense of (6). We have proved that D is a vdC set.

Following Ruzsa ([Ruz]), we will use a probabilistic argument to prove that (S1)⇒(S2). The next two lemmas are routine variations on the theme of the law of large numbers.

Lemma 1.10. *Let $(\theta(n))_{n\in\mathbb{N}^2}$ be an i.i.d. family of random variables with values in the 2-torus $\mathbb{T}^2$. Write $\theta(n) = (\theta_1(n),\theta_2(n))$. Define a family of complex random variables $(Y(n))_{n\in\mathbb{N}^2}$ by*

$$Y(n_1,n_2) := e(r_1\theta_1(m_1,m_2) + r_2\theta_2(m_1,m_2))$$

if $n_i = m_i^2 + r_i$, with $0 \le r_i \le 2m_i$, $i = 1,2$. Then, almost surely,

$$\lim_{N_1,N_2\to\infty} \frac{1}{N_1N_2} \sum_{0<n\le N} Y(n) = \mathbb{P}(\theta = 0).$$

Lemma 1.11. *Let* $(X(n))_{n\in\mathbb{N}^2}$ *be an i.i.d. family of bounded complex random variables. Define a new family* $(Z(n))_{n\in\mathbb{N}^2}$ *of complex random variables by*

$$Z(n_1,n_2) := X(m_1,m_2)$$

if $n_i = m_i^2 + r_i$ *with* $0 \le r_i \le 2m_i$, $i = 1,2$. *Then, almost surely,*

$$\lim_{N_1,N_2\to\infty} \frac{1}{N_1N_2} \sum_{0<n\le N} Z(n) = \mathbb{E}[X].$$

Let us briefly explain how (S1)⇒(S2) follows from these lemmas. Suppose that a vdC set $D \subset \mathbb{Z}^2$ and a measure σ on $\mathbb{T}^2$ are given. We suppose that the Fourier transform of σ is null on D. Without loss of generality, we can suppose that σ is a probability measure, and we consider a family of random variables $(\theta(n))_{n\in\mathbb{N}^2}$ independent and of law σ. We define, as in Lemma 1.10, a family of complex random variables $(Y(n))_{n\in\mathbb{N}^2}$. A slight modification ([8]) of Lemma 1.11 gives us the following result: for all $h \in \mathbb{Z}^2$, almost surely,

$$\lim_{N_1,N_2\to\infty} \frac{1}{N_1N_2} \sum_{0<n\le N} Y(n+h)\overline{Y(n)} = \mathbb{E}[e(h_1\theta_1 + h_2\theta_2)].$$

This last quantity is exactly $\widehat{\sigma}(h)$ and, by hypothesis, it is null for $h \in D$. Since D is a vdC set, we conclude that

$$\lim_{N_1,N_2\to\infty} \frac{1}{N_1N_2} \sum_{0<n\le N} Y(n) = 0.$$

By Lemma 1.10, this means that $\mathbb{P}(\theta = 0) = 0$, i.e. $\sigma(\{(0,0)\}) = 0$. ∎

1.2.3. *Some corollaries.* Here are some direct applications of the spectral characterization.

Corollary 1.12 (Ramsey property; cf. [Ruz, Corollary 1]). *If* $D = D_1 \cup D_2$ *is a vdC set in* $\mathbb{Z}^2$, *then at least one of the sets* D_1 *or* D_2 *is a vdC set.* (*In particular, if* D *is a vdC set in* $\mathbb{Z}^2$ *and* E *is a finite subset of* D, *then* $D \setminus E$ *is still a vdC set in* $\mathbb{Z}^2$.)

Proof. If σ_1 and σ_2 are positive measures on $\mathbb{T}^2$ such that $\widehat{\sigma}_i$ is null on D_i, then the Fourier transform of their convolution $\sigma_1 \star \sigma_2$ vanishes on $D_1 \cup D_2$. And $\sigma_1 \star \sigma_2(\{0\}) \ge \sigma_1(\{0\}) \times \sigma_2(\{0\})$. ∎

If $\mathcal{F}$ is a family of subsets of $\mathbb{Z}^2$, we denote by $\mathcal{F}^*$ its dual family, that is, the family of all sets $G \subset \mathbb{Z}^2$ such that $G \cap F \neq \emptyset$ for all $F \in \mathcal{F}$. The Ramsey property described in Corollary 1.12 has a remarkable consequence for the family of vdC* sets: if A is a vdC set and B is a vdC* set, then $A \cap B$ is a vdC set; this implies that the family of vdC* sets is stable with respect to finite intersections, hence is a filter.

([8]) Details are provided after Lemma 2.4 in Section 2.2.

Corollary 1.13 (Sets of differences). *If I is an infinite subset of $\mathbb{Z}^2$, then the set of differences $D := \{n - m : n, m \in I \text{ and } n \neq m\}$ is a vdC set.*

Proof. Suppose that σ is a probability measure on $\mathbb{T}^2$ whose Fourier transform vanishes on D. This means that the characters $x \mapsto e(n \cdot x)$ with $n \in I$ form an orthonormal family in $L^2(\sigma)$. For any finite subset J of I, we have

$$(\operatorname{card} J)^2 \sigma(\{(0,0)\}) \le \int_{\mathbb{T}^2} \Big| \sum_{n \in J} e(n \cdot x) \Big|^2 \, d\sigma(x) = \operatorname{card} J.$$

This implies that σ has no point mass at zero. ∎

Remark 1.14. The above proof gives, in fact, more: *any set D which contains sets of differences of arbitrarily large finite sets is a vdC set.*

Corollary 1.15 (Linear transformations of vdC sets). *Let d and e be positive integers, and let L be a linear transformation from $\mathbb{Z}^d$ into $\mathbb{Z}^e$ (i.e. an $e \times d$ matrix with integer entries).*

(1) *If D is a vdC set in $\mathbb{Z}^d$ and if $0 \notin L(D)$, then $L(D)$ is a vdC set in $\mathbb{Z}^e$.*
(2) *Let $D \subset \mathbb{Z}^d$. If the linear map L is one-to-one, and if $L(D)$ is a vdC set in $\mathbb{Z}^e$, then D is a vdC set in $\mathbb{Z}^d$.*

Proof. Let D be vdC set in $\mathbb{Z}^d$ and σ a positive measure on the e-torus such that $\widehat{\sigma}$ vanishes on $L(D)$. Let us denote by ${}^tL$ the map from $\mathbb{T}^e$ into $\mathbb{T}^d$ defined by $k \cdot {}^tL(x) = L(k) \cdot x$ for $k \in \mathbb{Z}^d$ and $x \in \mathbb{T}^e$. Denoting by σ' the image of σ under the linear transformation ${}^tL$, we see that, for all $k \in \mathbb{Z}^d$, $\widehat{\sigma'}(k) = \widehat{\sigma}(L(k))$. Hence the Fourier transform $\widehat{\sigma'}$ vanishes on the vdC set D. The measure σ' has no mass at zero, and hence σ also has no mass at zero. This proves the first assertion.

Suppose now that L is one-to-one and that $L(D)$ is a vdC set in $\mathbb{Z}^e$. Consider the lattice $L(\mathbb{Z}^d)$ in $\mathbb{Z}^e$. By a classical lemma (see for example [G, Exercise 8 of Chapter 31]), there exist $n_1, \dots, n_e$ in $\mathbb{Z}^e$ and positive integers $p_1, \dots, p_d$ such that $\mathbb{Z}^e = \mathbb{Z}n_1 + \dots + \mathbb{Z}n_e$ and $L(\mathbb{Z}^d) = p_1\mathbb{Z}n_1 + \dots + p_d\mathbb{Z}n_d$. This allows us to view $L(D)$ as a vdC set in $\mathbb{Z}^d \simeq \mathbb{Z}n_1 + \dots + \mathbb{Z}n_d$ and L as an endomorphism of $\mathbb{Z}^d$.

Let σ' be a positive measure on the d-torus such that $\widehat{\sigma'}$ vanishes on D. The linear map ${}^tL$ from $\mathbb{T}^d$ into $\mathbb{T}^d$ is finite-to-one and onto. Since it is onto, it has a right inverse and we can see σ' as the image of a positive measure σ on the d-torus, under the map ${}^tL$. The Fourier transform $\widehat{\sigma}$ vanishes on $L(D)$, hence the measure σ is continuous. Since ${}^tL$ is finite-to-one, we conclude that the measure σ' is also continuous. This proves the second assertion. ∎

Corollary 1.16 (Lattices are vdC*). *If G is any d-dimensional lattice in $\mathbb{Z}^d$, and if D is a vdC set in $\mathbb{Z}^d$, then $G \cap D$ is a vdC set in $\mathbb{Z}^d$.*

Proof. To begin, we remark that if G is a lattice in $\mathbb{Z}^d$, and if $z \in \mathbb{Z}^d$, $z \notin G$, then the translate $z + G$ is not a vdC set in $\mathbb{Z}^d$ (test the definition of a vdC set on the indicator function of the set G). Since G is a d-dimensional lattice in $\mathbb{Z}^d$, there exist finitely many points $z_1, \ldots, z_k$ in $\mathbb{Z}^d$ and outside G such that

$$\mathbb{Z}^d = G \cup \bigcup_{i=1}^{k} (z_i + G).$$

Let D be a vdC set in $\mathbb{Z}^d$. We have

$$D = (G \cap D) \cup \bigcup_{i=1}^{k} (z_i + G) \cap D.$$

Since none of the sets $(z_i + G) \cap D$ is vdC, Corollary 1.12 tells us that the set $(G \cap D)$ is vdC. ∎

Remark 1.17. As a consequence of the last two statements, we note the following fact, which is a direct extension of Corollary 2 in [Ruz]:

Let L be a one-to-one linear transformation from $\mathbb{Z}^d$ into itself; let D be a vdC set in $\mathbb{Z}^d$; the set of $n \in \mathbb{Z}^d$ such that $L(n) \in D$ is vdC in $\mathbb{Z}^d$.

Indeed, by Corollary 1.16, $D \cap L(\mathbb{Z}^d)$ is a vdC set in $\mathbb{Z}^d$ and, by Corollary 1.15, its inverse image under L is a vdC set.

The spectral characterization also implies that various formulations of the vdC property, associated to different averaging methods, are in fact equivalent (see Section 4).

1.3. The Kamae–Mendès France criterion

1.3.1. *The criterion.* Let $D \subset \mathbb{Z}^2$ and let P be a real trigonometric polynomial on $\mathbb{T}^2$. We say that *the spectrum of P is contained in D* if P is a linear combination of the characters $(x_1, x_2) \mapsto e(d_1x_1 + d_2x_2)$ with $(d_1, d_2) \in \pm D$. In the case of a one-dimensional space of parameters, the following proposition appears in Ruzsa's article [Ruz], with the same proof.

Proposition 1.18. *A subset D of $\mathbb{Z}^2 \setminus \{0\}$ is a van der Corput set if and only if for all $\varepsilon > 0$, there exists a real trigonometric polynomial P on the 2-torus $\mathbb{T}^2$ whose spectrum is contained in D and which satisfies $P(0) = 1$, $P \geq -\varepsilon$.*

Proof. Assume that such a trigonometric polynomial exists. Let σ be a positive measure on $\mathbb{T}^2$ whose Fourier transform $\widehat{\sigma}$ is null on D. Then

$$\int_{\mathbb{T}^2} P \, d\sigma = 0.$$

But from $P(0)=1$ and $P \geq -\varepsilon$ we deduce that

$$\int_{\mathbb{T}^2} P\,d\sigma \geq \sigma(\{0\}) - \varepsilon\sigma(\mathbb{T}^2 \setminus \{0\}).$$

Thus necessarily $\sigma(\{0\})=0$, and we deduce from Theorem 1.8 that D is a vdC set.

For the proof of the converse implication, we follow Ruzsa's argument ([Ruz, Section 5]). We will write $m \cdot x := m_1x_1 + m_2x_2$ if $x=(x_1,x_2) \in \mathbb{T}^2$ and $m=(m_1,m_2) \in \mathbb{Z}^2$.

Suppose that D is a subset of $\mathbb{Z}^2$ and that there exists $0<\varepsilon<1$ such that, for any real trigonometric polynomial P with spectrum in D and such that $P(0)=1$, we have $\min(P+\varepsilon) \leq 0$. In the Banach space $\mathcal{C}_{\mathbb{R}}(\mathbb{T}^2)$ of real continuous functions on $\mathbb{T}^2$, equipped with the uniform norm, we consider the set $\mathcal{F}$ of strictly positive functions and the set $\mathcal{Q}$ of real trigonometric polynomials P, with spectrum in D and such that $P(0)=1$. By hypothesis, the convex sets $\mathcal{F}$ and $\varepsilon+\mathcal{Q}$ are disjoint. By the Hahn–Banach theorem, there exists a non-zero real-valued continuous linear functional L on $\mathcal{C}_{\mathbb{R}}(\mathbb{T}^2)$ which takes non-negative values on $\mathcal{F}$ and non-positive values on $\varepsilon+\mathcal{Q}$. Let us denote by σ the measure on $\mathbb{T}^2$ associated to L by the Riesz representation theorem: $L(f)=\int_{\mathbb{T}^2} f\,d\sigma$ for all $f \in \mathcal{C}_{\mathbb{R}}(\mathbb{T}^2)$. Since $L \geq 0$ on $\mathcal{F}$, this measure is positive and we can assume that it is normalized. Let $m,n \in \pm D$. If $P \in \mathcal{Q}$, then, for all $\lambda \in \mathbb{R}$, the function $x \mapsto \varepsilon + P + \lambda(\cos 2\pi(m\cdot x) - \cos 2\pi(n \cdot x))$ is still in $\varepsilon+\mathcal{Q}$. This implies that $\int \cos 2\pi(m\cdot x)\,d\sigma(x) = \int \cos 2\pi(n\cdot x)\,d\sigma(x)$. Similarly, for all $\lambda \in \mathbb{R}$, the function $x \mapsto \varepsilon + P + \lambda \sin 2\pi(m\cdot x)$ is still in $\varepsilon+\mathcal{Q}$, and this implies that $\int \sin 2\pi(m\cdot x)\,d\sigma(x)=0$.

We define $r := \int \cos 2\pi(m\cdot x)\,d\sigma(x)$ for $m \in \pm D$. If $P \in \mathcal{Q}$, we have

$$\int_{\mathbb{T}^2} (\varepsilon+P)\,d\sigma \leq 0$$

and, writing

$$P(x) = \sum_{m \in \pm D} a_m \cos 2\pi(m\cdot x) + b_m \sin 2\pi(m \cdot x),$$

we have

$$\int_{\mathbb{T}^2} P\,d\sigma = r \sum_{m\in\pm D} a_m = rP(0) = r.$$

Hence $r \leq -\varepsilon < 0$. Denoting by δ the Dirac mass at 0, we consider a new probability measure σ' defined by

$$\sigma' := \frac{1}{1-r}\,(\sigma - r\delta).$$

We have $\sigma'(\{0\}) \geq -r/(1-r) > 0$. But this probability satisfies $\widehat{\sigma'}(m)=0$ for all $m \in D$, and, using Theorem 1.8, we conclude that D is not a vdC set. ■

1.3.2. *Application to polynomial sequences and sequences of shifted primes.* The following proposition is the two-dimensional extension of Example 3 in [Kam-MF].

PROPOSITION 1.19. *Let $D \subset \mathbb{Z}^2$. For each $q \in \mathbb{N}$, define*

$$D_q := \{(d_1, d_2) \in D : q! \text{ divides } d_1 \text{ and } d_2\}.$$

Suppose that, for every q, there exists a sequence $(h^{q,n})_{n\in\mathbb{N}}$ in D_q such that, for every $x = (x_1, x_2) \in \mathbb{R}^2$, if x_1 or x_2 is irrational, the sequence $(h^{q,n} \cdot x)_{n\in\mathbb{N}}$ is uniformly distributed mod 1. Then D is a vdC set.

Proof. Define a family of trigonometric polynomials with spectrum contained in D, by the formula

$$P_{q,N}(x) := \frac{1}{N}\sum_{n=1}^{N} e(h^{q,n} \cdot x), \tag{7}$$

where q and N are positive integers and $x \in \mathbb{R}^2$. By hypothesis, if $x \notin \mathbb{Q}^2$ then $\lim_{N\to\infty} P_{q,N}(x) = 0$. For each q, there exists a subsequence $(P_{q,N'})$ which is pointwise convergent to a function g_q. For all $x \in \mathbb{Q}^2$, we have $g_q(x) = 1$ for all large enough q, and for all $x \notin \mathbb{Q}^2$, we have $g_q(x) = 0$. The sequence (g_q) is pointwise convergent to the characteristic function of $\mathbb{Q}^2$. Consider now a positive measure σ on $\mathbb{T}^2$ whose Fourier transform $\widehat{\sigma}$ vanishes on D. We have $\int P_{q,N}\, d\sigma = 0$ for all q, N. Applying the dominated convergence theorem twice, we conclude that $\sigma(\mathbb{Q}^2) = 0$. In particular $\sigma(\{0\}) = 0$, and we are done. ∎

A sequence $(d_n)_{n\in\mathbb{N}}$ in $\mathbb{Z}^2$ will be called a *vdC sequence* if the set of its values $\{d_n : n \in \mathbb{N}\}$ is a vdC set.

The (d-dimensional version of the) following proposition extends Theorem 4.2 in [Berg.1].

PROPOSITION 1.20. *Let p_1 and p_2 be two polynomials with integer coefficients. The sequence $(p_1(n), p_2(n))_{n\in\mathbb{N}}$ is a vdC sequence in $\mathbb{Z}^2$ if and only if for all positive integers q, there exists $n \geq 1$ such that q divides $p_1(n)$ and $p_2(n)$.*

Note that the divisibility condition is satisfied if p_1 and p_2 have zero constant term.

Proof of Proposition 1.20. By Corollary 1.16, the divisibility condition is necessary for the sequence $(p_1(n), p_2(n))$ to be vdC. Let us prove that this condition is sufficient. We distinguish two cases: either p_1 and p_2 are proportional, or not.

In the first case, there exists a polynomial $p \in \mathbb{Z}[X]$ and integers a, b such that $p_1 = ap$ and $p_2 = bp$. The polynomial p satisfies the divisibility property, which ensures that $(p(n))$ is a vdC sequence in $\mathbb{Z}$ (this is a direct consequence of the one-dimensional version of Proposition 1.19, cf. [Kam-MF]). By the first statement of Corollary 1.15, this implies that $(ap(n), bp(n))$ is a vdC sequence in $\mathbb{Z}^2$.

Consider now the second case, in which polynomials p_1 and p_2 are not proportional. Let q be a positive integer and $(x_1, x_2) \in \mathbb{R}^2 \setminus \mathbb{Q}^2$; there exists $n \geq 1$ such that $q! \,|\, p_1(n)$ and $q! \,|\, p_2(n)$; for all $k \in \mathbb{Z}$, we have $q! \,|\, p_1(n+kq!)$ and $q! \,|\, p_2(n+kq!)$. We claim that the sequence

$$(p_1(n+kq!)x_1 + p_2(n+kq!)x_2)_{k\in\mathbb{N}} \tag{8}$$

is uniformly distributed mod 1. This fact implies, by Proposition 1.19, that $(p_1(n), p_2(n))$ is a vdC sequence in $\mathbb{Z}^2$. To prove the claim, we consider first the case when 1, x_1 and x_2 are linearly independent over $\mathbb{Q}$; in this case the sequence (8) is u.d. mod 1 by Weyl's theorem. Consider now the case in which 1, x_1 and x_2 are linearly dependent over $\mathbb{Q}$ and x_1 is irrational; in this case we have $x_2 = rx_1 + s$ with $r, s \in \mathbb{Q}$, and, if q has been chosen large enough, the sequence (8) has (mod 1) the form

$$((p_1(n+kq!) + rp_2(n+kq!))x_1)_{k\in\mathbb{N}};$$

we conclude once more by Weyl's theorem since the polynomial $p_1 + rp_2$ is not constant. Finally, if x_1 is rational, then x_2 is irrational and the argument is similar. ∎

Remark 1.21 (see Appendix). There exist pairs of polynomials p_1, p_2 satisfying:

- For all integers a and b and for all positive integers q, there exists n such that $q \,|\, ap_1(n) + bp_2(n)$ (hence $(ap_1(n) + bp_2(n))_{n\in\mathbb{N}}$ is a vdC sequence in $\mathbb{Z}$).
- There exists a positive integer q such that for no n are the numbers $p_1(n)$ and $p_2(n)$ simultaneously multiples of q (hence $(p_1(n), p_2(n))_{n\in\mathbb{N}}$ is not a vdC sequence in $\mathbb{Z}^2$).

Let $\mathcal{P}$ be the set of prime numbers. It is shown in [Kam-MF] that $\mathcal{P} - 1$ and $\mathcal{P} + 1$ are vdC sets, and that no other translate of $\mathcal{P}$ is a vdC set. This can be extended to polynomials along $\mathcal{P} - 1$ and $\mathcal{P} + 1$, and to the multidimensional setting. For example, we have the following result.

Proposition 1.22. *Let f, g be two (non-zero) polynomials with integer coefficients and zero constant term. The set $\{(f(p-1), g(p-1)) : p \in \mathcal{P}\}$ is a vdC set in $\mathbb{Z}^2$.*

The proof of this proposition relies on Proposition 1.19 and on the following Vinogradov type theorem.

Theorem 1.23. *Let q be a positive integer and h be a real polynomial such that the polynomial $h - h(0)$ has at least one irrational coefficient.*

The sequence $(h(p))$ *is uniformly distributed mod* 1, *where* p *describes the increasing sequence of prime numbers in the congruence class* $1 + q\mathbb{N}$.

The proof of this theorem can be given in a few sentences, by "quotation". It is proved in [Rh] (see also [N]) that if a real polynomial $\widetilde{h}$ is such that $\widetilde{h} - \widetilde{h}(0)$ has at least one irrational coefficient, then

$$\text{the sequence } (\widetilde{h}(p))_{p\in\mathcal{P}} \text{ is u.d. mod 1.} \tag{9}$$

Now we can use the following simple trick (cf. [Mo, p. 34]):

$$\sum_{\substack{p\le n\\ p\equiv 1\,[q]}} e(h(p)) = \frac{1}{q}\sum_{j=1}^{q} e\left(-\frac{j}{q}\right)\sum_{p\le n} e(h(p)+pj/q).$$

After division by $\pi(n)$, the right side goes to zero as n goes to infinity because (9) can be applied to $\widetilde{h}(p) = h(p) + pj/q$.

Moreover, it is well known that the prime number theorem has a natural extension to the distribution of primes in arithmetic progressions: the number of primes less than n in $1+q\mathbb{N}$ is asymptotically equivalent to $\pi(n)/\varphi(q)$ as n goes to infinity.

We obtain

$$\lim_{n\to\infty} \frac{1}{\#\{p\le n : p\equiv 1\,[q]\}} \sum_{\substack{p\le n\\ p\equiv 1\,[q]}} e(h(p)) = 0.$$

This is still true when we replace h by a non-zero integer multiple of h, which, via Weyl's criterion, gives uniform distribution (mod 1) of the sequence $((h(p))_{p\in\mathcal{P},\,p\equiv 1\,[q]})$.

Proof of Proposition 1.22. This proof is parallel to that of Proposition 1.20. If f and g are proportional, we use the fact that $(f(p-1))_{p\in\mathcal{P}}$ is a vdC sequence (which is a direct consequence of the one-dimensional version of Proposition 1.19 and of Theorem 1.23). If f and g are not proportional, we deduce from Theorem 1.23 that for all large enough positive integers q, and for all $(x_1, x_2) \in \mathbb{R}^2 \setminus \mathbb{Q}^2$, the sequence $(f(p-1)x_1 + g(p-1)x_2)_{p\in\mathcal{P},\,p\equiv 1\,[q]}$ is u.d. mod 1. We conclude by Proposition 1.19. ∎

Several other examples of vdC sets are presented in Subsection 2.5.

1.3.3. *One more corollary à la Ruzsa.* Following [Ruz], we deduce from Proposition 1.18 a new combinatorial property of vdC sets.

COROLLARY 1.24 (cf. [Ruz, Corollary 3]). *Any vdC set in* $\mathbb{Z}^2$ *can be partitioned into infinitely many pairwise disjoint vdC sets.*

Proof. Let D be a vdC set in $\mathbb{Z}^2$. There exists a sequence $(I_k)_{k\ge 1}$ of pairwise disjoint finite subsets of D, and for each k, a trigonometric polynomial

P_k with spectrum in I_k and such that $P_k(0) = 1$, $P_k + 1/k > 0$. The existence of I_k and P_k can be proved by induction using the direct implication in Proposition 1.18 and the fact that, for each k, the set $D \setminus (I_1 \cup \cdots \cup I_k)$ is vdC (see Corollary 1.12). From the converse implication in Proposition 1.18, we deduce that any infinite union of the I_k's is a vdC set. We can consider an infinite family of pairwise disjoint such sets. ∎

1.4. Positive-definite multiparameter sequences and generalized vdC inequality

1.4.1. *The inequality.* We show in this subsection that the Kamae–Mendès France criterion can be formulated in terms of positive-definite sequences. This will allow us, for a given vdC set D, to obtain a quantitative van der Corput type inequality in which only correlations $\gamma(N, d)$ for $d \in D$ are involved.

PROPOSITION 1.25. *Let $(a_h)_{h\in\mathbb{Z}^2}$ be a family of complex numbers such that all but finitely many of the a_h are zero. This family is positive-definite if and only if the trigonometric polynomial $T(x) := \sum_h a_h e(h \cdot x)$, $x \in \mathbb{R}^2$, takes only nonnegative values.*

Proof. Recall that the family (a_h) of complex numbers is positive-definite if, for any family $(z_h)_{h\in\mathbb{Z}^2}$ of complex numbers, all zero but finitely many,

$$\sum_{h,h'\in\mathbb{Z}^2} a_{h-h'} z_h \overline{z}_{h'} \geq 0.$$

We will write $h = (h_1, h_2)$.

The family (a_h) is the Fourier transform of the measure having density T with respect to Lebesgue measure on the 2-torus. Thus it is clear that if the trigonometric polynomial is positive, then the family is positive-definite. In the opposite direction, suppose that (a_h) is positive-definite (and that $a_h = 0$ for all h but finitely many). For all $x \in \mathbb{R}^2$ and for all positive integers c,

$$\sum_{0\leq h,h'<(c,c)} a_{h-h'} e(h \cdot x) e(-h' \cdot x) \geq 0.$$

This can be written

$$\sum_{(-c,-c)<h<(c,c)} (c - |h_1|)(c - |h_2|) a_h e(h \cdot x) \geq 0.$$

Dividing this expression by c^2, and letting c go to infinity, we obtain

$$\sum_h a_h e(h \cdot x) \geq 0. \ ∎$$

REMARK 1.26. The Kamae–Mendès France criterion (Proposition 1.18) can now be rewritten as follows: a subset D of $\mathbb{Z}^2 \setminus \{0\}$ is a vdC set if and only if, for all $\varepsilon > 0$, there exists a positive-definite family $(a_d)_{d\in\mathbb{Z}^2}$ such that:

- all but finitely many a_d are zero;
- $a_d = 0$ whenever $d \neq 0$ and $d \notin D \cup (-D)$;
- $a_0 \leq \varepsilon$ and $\sum_d a_d = 1$.

As in the first subsection, we define

$$\gamma(N,h) := \sum_{\substack{0<n\leq N\\ 0<n+h\leq N}} u_{n+h}\overline{u}_n$$

if $h \in \mathbb{Z}^2$, $N \in \mathbb{N}^2$ and $(u_n)_{0\leq n<N}$ is a family of complex numbers. We also write

$$\|u\|_\infty := \max_n |u_n|.$$

Theorem 1.27. *Let $H \in \mathbb{N}^2$ and $(a_h)_{-H<h<H}$ be a finite positive-definite family of complex numbers, with $\sum_h a_h = 1$. Let $N \in \mathbb{N}^2$ and $(u_n)_{0<n\leq N}$ be a finite family of complex numbers. We have*

$$\Big|\sum_{0<n\leq N} u_n\Big|^2 \leq N_1N_2\Big(\sum_h a_h\gamma(N,h) + 5\|u\|_\infty^2 \sum_h (|h_1|N_2 + |h_2|N_1 + |h_1h_2|)|a_h|\Big).$$

This inequality should be compared with the "generalized van der Corput Lemma" stated in [Mo] (Chap. 2, Lemma 1).

If we consider a bounded family $(u_n)_{n\in\mathbb{N}^2}$ of complex numbers, we deduce from Theorem 1.27 the inequality

$$\left|\frac{1}{N_1N_2}\sum_{0<n\leq N} u_n\right|^2 \leq \sum_h a_h \frac{1}{N_1N_2}\gamma(N,h) + O\left(\max\left(\frac{1}{N_1},\frac{1}{N_2}\right)\right),$$

which will be utilized when describing the vdC property of Cartesian products of vdC sets.

Corollary 1.28 below, which is a direct consequence of Theorem 1.27, gives what one might call a quantitative version of the van der Corput trick. The "if" part of the Kamae–Mendès France criterion is a direct consequence of this corollary.

Corollary 1.28. *Let $(a_h)_{-H<h<H}$ be a positive-definite family of complex numbers, and $(u_n)_{n\in\mathbb{N}^2}$ be a family of complex numbers. If, for any h such that $h \neq 0$ and $a_h \neq 0$,*

$$\lim_{N_1,N_2\to\infty} \frac{1}{N_1N_2}\gamma(N,h) = 0$$

then

$$\limsup_{N_1,N_2\to\infty}\left|\frac{1}{N_1N_2}\sum_{0<n\le N}u_n\right|\le\|u\|_\infty\sqrt{a_0}.$$

Proof of Theorem 1.27. Define

$$m:=\frac{1}{N_1N_2}\sum_{0<n\le N}u_n\quad\text{and}\quad v_n:=u_n-m.$$

We have

$$\gamma(N,h)=\sum_{\substack{0<n\le N\\0<n+h\le N}}(v_{n+h}+m)(\overline{v}_n+\overline{m})=A_h+B_h+C_h+D_h,$$

where

$$A_h:=\sum_{\substack{0<n\le N\\0<n+h\le N}}v_{n+h}\overline{v}_n,\qquad B_h:=m\sum_{\substack{0<n\le N\\0<n+h\le N}}\overline{v}_n,$$
$$C_h:=\overline{m}\sum_{\substack{0<n\le N\\0<n+h\le N}}v_{n+h},\qquad D_h:=|m|^2\sum_{\substack{0<n\le N\\0<n+h\le N}}1.$$

Since the family (a_h) is positive-definite, we have

$$\sum_h a_hA_h\ge 0.$$

The number of points n in the square $[1,N_1]\times[1,N_2]$ such that we do not have $0<n+h\le N$ is less than or equal to $|h_1|N_2+|h_2|N_1$. Since $\sum_{0<n\le N}v_n=0$ we deduce that

$$\begin{aligned}|B_h|&\le|m|(|h_1|N_2+|h_2|N_1)\|v\|_\infty\le 2|m|(|h_1|N_2+|h_2|N_1)\|u\|_\infty\\&\le 2(|h_1|N_2+|h_2|N_1)\|u\|_\infty^2.\end{aligned}$$

The same inequality holds for $|C_h|$. We also have

$$\begin{aligned}D_h&=(N_1-|h_1|)(N_2-|h_2|)|m|^2\\&\ge\frac{1}{N_1N_2}\Big|\sum_{0<n\le N}u_n\Big|^2-(|h_1|N_2+|h_2|N_1+|h_1h_2|)\|u\|_\infty^2.\end{aligned}$$

From these inequalities, we deduce that

$$\begin{aligned}\sum_h a_h\gamma(N,h)&\ge\sum_h a_hA_h+\sum_h a_hD_h-\sum_h a_h(|B_h|+|C_h|)\\&\ge\frac{1}{N_1N_2}\Big|\sum_{0<n\le N}u_n\Big|^2-5\sum_h|a_h|(|h_1|N_2+|h_2|N_1+|h_1h_2|)\|u\|_\infty^2,\end{aligned}$$

and the result follows. ■

In the next two subsections we present corollaries of Theorem 1.27.

1.4.2. *Cartesian products of vdC sets*

Corollary 1.29. *Let k, l be positive integers, and D, E be vdC sets in, respectively, $\mathbb{Z}^k$ and $\mathbb{Z}^l$. The product set $D \times E$ is a vdC set in $\mathbb{Z}^{k+l}$.*

Proof. Let us consider, as a typical example, the case $k = l = 2$. We consider two vdC sets D and E in $\mathbb{Z}^2$. Let $(u_{n,m})_{n,m\in\mathbb{Z}^2}$ be a family of complex numbers of modulus one indexed by $\mathbb{Z}^4$, and satisfying: for all $d \in D$ and all $e \in E$,

$$\lim_{\substack{N_1,N_2\to\infty\\ M_1,M_2\to\infty}} \frac{1}{N_1N_2M_1M_2} \sum_{\substack{0\le n<(N_1,N_2)\\ 0\le m<(M_1,M_2)}} u_{n+d,m+e}\overline{u}_{n,m} = 0. \tag{10}$$

It is not hard to verify that (10) is still true when $d \in (-D)$ or $e \in (-E)$.

Fix $\varepsilon > 0$. By Remark 1.26, there exist two positive-definite families (a_d) and (b_e) indexed by $\mathbb{Z}^2$ such that a_d (resp. b_e) is zero whenever d (resp. e) is outside a finite subset of $D \cup (-D) \cup \{0\}$ (resp. $E \cup (-E) \cup \{0\}$), with $a_0 < \varepsilon$, $b_0 < \varepsilon$ and $\sum_d a_d = \sum_e b_e = 1$.

It is clear from Proposition 1.25 (or from the Bochner–Herglotz theorem) that the family $(a_db_e)_{(d,e)\in\mathbb{Z}^4}$ is positive-definite. Set $P := N_1N_2M_1M_2$ and $p := \min\{N_1, N_2, M_1, M_2\}$. The generalized vdC inequality (Theorem 1.27) applied to $\mathbb{Z}^4$ gives

$$\Bigl|\sum_{\substack{0\le n<(N_1,N_2)\\ 0\le m<(M_1,M_2)}} u_{n,m}\Bigr|^2 \le P\sum_{d,e} a_db_e \sum_{\substack{0\le n,n+d<(N_1,N_2)\\ 0\le m,m+e<(M_1,M_2)}} u_{n+d,m+e}\overline{u}_{n,m} + P^2O(1/p).$$

Dividing by P^2, letting p go to infinity and using (10), we obtain

$$\limsup_{\substack{N_1,N_2\to\infty\\ M_1,M_2\to\infty}} \Bigl|\frac{1}{N_1N_2M_1M_2} \sum_{\substack{0\le n<(N_1,N_2)\\ 0\le m<(M_1,M_2)}} u_{n,m}\Bigr|^2 \le \sum_{d=0 \text{ or } e=0} a_db_e.$$

Since $\sum_{d \text{ or } e=0} a_db_e = a_0\sum_e b_e + b_0\sum_d a_d - a_0b_0 \le 2\varepsilon$, we conclude that the last limsup is zero. ∎

1.4.3. *Sequences in Hilbert space.* The goal of this subsection is to point out that generalized van der Corput inequalities can be extended from numerical sequences to sequences of vectors in a Hilbert space. One of the reasons to be interested in such extensions is that they provide useful convergence criteria for multiple ergodic averages (see for example the references mentioned at the end of the Introduction).

Let $\mathcal{H}$ be a Hilbert space and $(u_n)_{n\in\mathbb{N}^2}$ be a doubly indexed family of vectors in this space. We define, for any $h \in \mathbb{Z}^2$,

$$\gamma(N,h) := \sum_{\substack{0<n\le N \\ 0<n+h\le N}} \langle u_{n+h}, u_n\rangle \quad \text{and} \quad \|u\|_\infty := \sup_n \|u_n\|.$$

PROPOSITION 1.30. *Let $H \in \mathbb{N}^2$ and $(a_h)_{-H<h<H}$ be a finite positive-definite family of complex numbers, with $\sum_h a_h = 1$. We have*

$$\Big\| \sum_{0<n\le N} u_n \Big\|^2 \le N_1N_2\Big(\sum_h a_h\gamma(N,h) + 5\|u\|_\infty^2 \sum_h (|h_1|N_2 + |h_2|N_1 + |h_1h_2|)|a_h|\Big).$$

The proof of Proposition 1.30 is similar to the scalar case and will be omitted. Combined with Remark 1.26, this proposition leads to the following extension of the notion of vdC set to families in Hilbert space.

COROLLARY 1.31. *Let D be a vdC set in $\mathbb{Z}^2$ and $(u_n)_{n\in\mathbb{Z}^2}$ be a bounded family in $\mathcal{H}$. If*

$$\forall d \in D, \qquad \lim_{N_1,N_2\to\infty} \frac{1}{N_1N_2} \sum_{0<n\le(N_1,N_2)} \langle u_{n+d}, u_n\rangle = 0$$

then

$$\lim_{N_1,N_2\to\infty} \frac{1}{N_1N_2} \sum_{0<n\le(N_1,N_2)} u_n = 0.$$

1.5. A new spectral characterization. We work in this subsection with ordinary sequences indexed by $\mathbb{Z}$. The extension to the multidimensional case is straightforward. We have the following spectral characterization of vdC sets, which completes the classical Theorem 1.8.

THEOREM 1.32. *Let $D \subset \mathbb{Z}$. Then D is a van der Corput set if and only if any positive measure σ on the torus $\mathbb{T}$ such that $\sum_{d\in D} |\widehat{\sigma}(d)| < \infty$ is continuous.*

This result is not surprising, because we have a "parallel" fact pertaining to recurrence properties. It is not difficult to prove that if a set D is a set of recurrence, then, for any m.p.s. $(X,\mathcal{A},\mu,T)$ and any set A in $\mathcal{A}$ such that $\mu(A) > 0$, not only does there exist $d \in D$ such that $\mu(A\cap T^dA) > 0$, but also $\sum_{d\in D} \mu(A\cap T^dA) = \infty$.

Proof of Theorem 1.32. Let D be a vdC set in $\mathbb{Z}$, and fix $\varepsilon > 0$. By Remark 1.26, we know that there exists a positive-definite sequence $(a_h)_{h\in\mathbb{Z}}$

such that:

- all but finitely many a_h are zero;
- $a_h = 0$ whenever $h \neq 0$ and $h \notin D \cup (-D)$;
- $a_0 \leq \varepsilon$ and $\sum_d a_d = 1$.

Moreover, for any positive-definite sequence $(b_h)_{h\in\mathbb{Z}}$ with support in $\{-H+1, \ldots, H-1\}$ and such that $\sum_h b_h = 1$, we have the following vdC inequality (simply the one-dimensional version of Theorem 1.27): for any complex numbers $u_1, \ldots, u_N$,

$$\Big|\sum_{n=1}^{N} u_n\Big|^2 \leq N\Big(\sum_h b_h \gamma(N,h) + 5\|u\|_\infty^2 \sum_h |hb_h|\Big).$$

We apply this inequality to the sequence (a_h) after noticing that since the sequence is positive-definite, we have $|a_h| \leq a_0$. We obtain

$$\Big|\sum_{n=1}^{N} u_n\Big|^2 \leq N a_0\Big(\gamma(N,0) + \sum_{\substack{d\in D\cup(-D)\\ |d|\leq H}} |\gamma(N,d)| + 5\|u\|_\infty^2 H^2\Big).$$

Hence

$$\Big|\frac{1}{N}\sum_{n=1}^{N} u_n\Big|^2 \leq \varepsilon\Big(\frac{1}{N}\sum_{n=1}^{N} |u_n|^2 + \sum_{\substack{d\in D\cup(-D)\\ |d|\leq H}} \Big|\frac{1}{N}\gamma(N,d)\Big| + \frac{5}{N}\|u\|_\infty^2 H^2\Big). \tag{11}$$

Let σ be a probability measure on the torus such that $\sum_{d\in D} |\widehat{\sigma}(d)| < \infty$. Following Ruzsa ([Ruz]), we consider a sequence $(Y_n)_{n\in\mathbb{N}}$ of complex random variables of modulus one such that almost surely,

$$\frac{1}{N}\sum_{0<n\leq N} Y_n \to \sigma(\{0\}) \quad \text{and} \quad \frac{1}{N}\sum_{0<n\leq N} Y_{n+h}\overline{Y}_n \to \widehat{\sigma}(h).$$

(Details of the construction of such a sequence (Y_n) are given below, in Lemmas 2.3 and 2.4 and in the text that follows.)

We apply (11) to $u_n = Y_n$ and let N go to infinity. After noticing that

$$\frac{1}{N}\gamma(N,d) = \frac{1}{N}\sum_{\substack{0<n\leq N\\ 0<n+d\leq N}} Y_{n+d}\overline{Y}_n \to \widehat{\sigma}(d),$$

we obtain

$$|\sigma(\{0\})|^2 \leq \varepsilon\Big(1 + 2\sum_{d\in D} |\widehat{\sigma}(d)|\Big).$$

This proves that $\sigma(\{0\}) = 0$. ∎

2. ENHANCED VAN DER CORPUT SETS

2.1. Introduction. In this section, we introduce a new property which we call *enhanced vdC*. It is a natural concept for several reasons:

- The set of all integers is enhanced vdC, and it is often this property which is classically used in equidistribution theory and ergodic theory.
- The spectral characterization of enhanced vdC sets is given by the FC^+ property (Theorem 2.1).
- In the manner that the notion of vdC set is linked to the notion of set of recurrence, the notion of enhanced vdC set is linked to the notion of set of strong recurrence (see Subsection 3.3).

We give here the definition and spectral characterization of enhanced vdC sets in $\mathbb{Z}$, extension to $\mathbb{Z}^d$ being completely routine.

2.2. Definitions and a spectral characterization

Definition 3. An infinite set of integers D is *enhanced van der Corput* if, for any sequence $(u_n)_{n\in\mathbb{Z}}$ of complex numbers of modulus 1 such that

$$\forall d \in D, \qquad \gamma(d) := \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} u_{n+d}\overline{u}_n \quad \text{exists} \tag{12}$$

and

$$\lim_{|d|\to\infty,\, d\in D} \gamma(d) = 0,$$

we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} u_n = 0.$$

(Note that we obtain an equivalent definition if we replace lim by lim sup in (12). See Proposition 2.5.)

Definition 4. An infinite set of integers D is FC^+ if every positive measure σ on the torus $\mathbb{T}$ with $\lim_{|d|\to\infty,\, d\in D} \widehat{\sigma}(d) = 0$ is continuous.

This definition appears in [Kam-MF] and in [Bou]. We remark that in [Pe], Peres uses the notation FC^+ for sets satisfying the apparently weaker condition (S3) of Theorem 1.8. We ask in Question 1 below whether condition (S3) is actually strictly weaker than condition FC^+.

Theorem 2.1. *The notions of enhanced vdC set and* FC^+ *set coincide.*

Proof. The proof follows the lines of the spectral characterization of vdC sets. To prove that FC^+ sets are enhanced vdC, we use the following lemma, which is the one-parameter version of Lemma 1.9.

Lemma 2.2. *Let $(u_n)_{n\in\mathbb{N}}$ be a bounded sequence of complex numbers and $(N_j)_{j\in\mathbb{N}}$ be an increasing sequence of positive integers. If for all $h \in \mathbb{N}$,*

$$\gamma(h) := \lim_{j\to\infty} \frac{1}{N_j} \sum_{n=1}^{N_j} u_{n+h}\overline{u}_n \quad \textit{exists},$$

then there exists a positive measure σ on the torus such that, for all $h \in \mathbb{N}$,

$$\widehat{\sigma}(h) = \gamma(h)$$

and this measure satisfies

$$\limsup_{j\to\infty} \frac{1}{N_j}\Big|\sum_{n=1}^{N_j} u_n\Big| \le \sqrt{\sigma(\{0\})}.$$

Let D be an FC^+ set. Let (u_n) be a bounded sequence of complex numbers such that

$$\lim_{|d|\to\infty,\, d\in D}\ \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} u_{n+d}\overline{u}_n = 0.$$

There exists an increasing sequence $(N_j)_{j\in\mathbb{N}}$ of positive integers such that

- $\displaystyle\lim_{j\to\infty} \frac{1}{N_j}\Big|\sum_{n=1}^{N_j} u_n\Big| = \limsup_{N\to\infty} \frac{1}{N}\Big|\sum_{n=1}^{N} u_n\Big|,$
- $\displaystyle\forall h \in \mathbb{N},\quad \gamma(h) := \lim_{j\to\infty} \frac{1}{N_j} \sum_{n=1}^{N_j} u_{n+h}\overline{u}_n \quad \text{exists.}$

The map γ is the Fourier transform of a positive measure σ on the torus. We have $\lim_{|d|\to\infty,\, d\in D} \widehat{\sigma}(d) = 0$. By hypothesis, this forces the measure σ to be continuous. We have $\sigma(\{0\}) = 0$ and, using the above lemma, we obtain the Cesàro convergence of (u_n) to zero. The set D is enhanced vdC.

To prove that any enhanced vdC set is FC^+, the arguments of Ruzsa ([Ruz]) can be adapted and we use the following probabilistic lemmas.

Lemma 2.3. *Let $(\theta_n)_{n\in\mathbb{N}}$ be an i.i.d. sequence of random variables with values in the torus $\mathbb{T}$. Define a new sequence of complex random variables (Y_n) by*

$$Y_n := e(r\theta_m),$$

if $n = m^2 + r$, with $0 \le r \le 2m$. Then, almost surely,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} Y_n = \mathbb{P}(\theta = 0).$$

LEMMA 2.4. *Let $(X_n)_{n\in\mathbb{N}}$ be an i.i.d. sequence of bounded complex random variables. Define a new sequence of complex random variables (Z_n) by*

$$Z_n := X_m$$

if $n = m^2 + r$, with $0 \le r \le 2m$. Then, almost surely,

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} Z_n = \mathbb{E}[X].$$

Let D be an enhanced vdC set, and let σ be a positive measure on $\mathbb{T}$. We suppose that the Fourier coefficient $\widehat{\sigma}(d)$ goes to zero when d goes to infinity in D. Without loss of generality, we can suppose that σ is a probability measure, and we consider a sequence (θ_n) of independent random variables of law σ. We define, as in Lemma 2.3, the family (Y_n) of complex random variables. Let us fix $h \in \mathbb{N}$. We define $Z_n = e(h\theta_m)$ for $n = m^2 + r$ and $0 \le r \le 2m$. By Lemma 2.4 we know that, almost surely, $\lim_{N\to\infty} N^{-1}\sum_{n=1}^{N} Z_n = \mathbb{E}[e(h\theta)]$. Furthermore, the set of positive integers n such that $Y_{n+h}\overline{Y}_n = Z_n$ has full density. Thus, almost surely,

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} Y_{n+h}\overline{Y}_n = \mathbb{E}[e(h\theta)].$$

This last quantity is exactly $\widehat{\sigma}(h)$ and, by hypothesis, it goes to zero when h goes to infinity in D. Since the set D is enhanced vdC, we conclude that

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} Y_n = 0.$$

By Lemma 2.3, this means that $\mathbb{P}(\theta = 0) = 0$, that is, $\sigma(\{0\}) = 0$. The same argument can be applied to all the images of σ under translations of the torus, and we conclude that σ is a continuous measure. Hence D is FC$^+$. ∎

The spectral characterization makes it possible to give an alternative definition of enhanced vdC sets.

PROPOSITION 2.5. *An infinite set D of integers is enhanced vdC if and only if for any sequence $(u_n)_{n\in\mathbb{Z}}$ of complex numbers of modulus 1 such that*

$$\lim_{|d|\to\infty,\, d\in D} \limsup_{N\to\infty} \left| \frac{1}{N}\sum_{n=0}^{N-1} u_{n+d}\overline{u}_n \right| = 0,$$

one has

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} u_n = 0.$$

2.3. Some properties of enhanced vdC sets. From the spectral characterization we deduce various corollaries. We omit detailed proofs since they are similar to those of the corresponding statements for vdC sets (see Subsection 1.2.3).

Corollary 2.6 (Ramsey property). *If $D = D_1 \cup D_2$ is an enhanced vdC set, then at least one of the sets D_1 or D_2 is enhanced vdC.*

Corollary 2.7 (Sets of differences). *Let $D \subset \mathbb{N}$. Suppose that, for all $n > 0$ there exist $a_1 < \cdots < a_n$ such that $\{a_j - a_i : 1 \leq i < j \leq n\} \subset D$. Then D is an enhanced vdC set.*

Corollary 2.8 (Linear transformations). *Let d and e be positive integers, and L be a linear transformation from $\mathbb{Z}^d$ into $\mathbb{Z}^e$ (i.e. an $e \times d$ matrix with integer entries).*

(1) *If D is an enhanced vdC set in $\mathbb{Z}^d$ and if $0 \notin L(D)$, then $L(D)$ is an enhanced vdC set in $\mathbb{Z}^e$.*
(2) *Let $D \subset \mathbb{Z}^d$. If the linear map L is one-to-one, and if $L(D)$ is an enhanced vdC set in $\mathbb{Z}^e$, then D is an enhanced vdC set in $\mathbb{Z}^d$.*

Corollary 2.9 (Lattices are (enhanced vdC)*). *If G is any d-dimensional lattice in $\mathbb{Z}^d$, and if D is an enhanced vdC set in $\mathbb{Z}^d$, then $G \cap D$ is an enhanced vdC set in $\mathbb{Z}^d$.*

2.4. Questions

Question 1. Our intuition is that there exist vdC sets which are not enhanced vdC. Is this true? Is it possible to exhibit a particular example?

Question 2. We know (Corollary 1.31) that the notions of *vdC set for families in a Hilbert space* and of *vdC set* coincide. Is the analogous fact true for *enhanced vdC sets*?

Question 3. We know (Corollary 1.24) that any vdC set can be partitioned into infinitely many vdC sets. Is the analogous fact true for enhanced vdC sets?

Question 4. We know (Corollary 1.29) that the Cartesian product of two vdC sets is a vdC set. Is the analogous fact true for enhanced vdC sets?

2.5. Examples

2.5.1. *Ergodic sequences.* A sequence $(d_n)_{n\in\mathbb{N}}$ of integers is called *ergodic* if the following mean ergodic theorem is valid: given an *ergodic* m.p.s. $(X, \mathcal{A}, \mu, T)$ and $f \in L^2(\mu)$, the averages $N^{-1}\sum_{n=1}^{N} f \circ T^{d_n}$ converge in L^2 to $\int f\, d\mu$ when N goes to infinity.

It follows from the spectral theorem that the sequence (d_n) is ergodic if and only if, for all $x \in \mathbb{R} \setminus \mathbb{Z}$,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} e(d_n x) = 0. \tag{13}$$

PROPOSITION 2.10. *Any ergodic sequence is an enhanced vdC sequence.*

Proof. Let (d_n) be an ergodic sequence and σ a finite measure on the torus. Using the dominated convergence theorem we deduce from (13) that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \widehat{\sigma}(d_n) = \sigma(\{0\}).$$

Hence it is immediate that the sequence (d_n) is FC^+. ■

Proposition 2.10 can be used to exhibit many examples of enhanced vdC sets.

(i) In [Bos-Ko-Q-Wi] the authors consider sequences of the form $d_n = [a(n)]$ where the function a belongs to some Hardy field. They characterize those of them which are ergodic. See Theorems 3.2–3.5 and 3.8 in that paper. Here are some examples of ergodic sequences, coming from [Bos-Ko-Q-Wi]:

- $\{[bn^c] : n \in \mathbb{N}\}$, where c is irrational > 1 and $b \neq 0$;
- $\{[bn^c + dn^a] : n \in \mathbb{N}\}$, where $b, d \neq 0$, b/d is irrational, $c \geq 1$, $a > 0$ and $a \neq c$;
- $\{[bn^c(\log n)^d] : n \in \mathbb{N}\}$, where $b \neq 0$, c is irrational > 1 and d is any number;
- $\{[bn^c(\log n)^d] : n \in \mathbb{N}\}$, where $b \neq 0$, c is rational > 1 and $d \neq 0$;
- $\{[bn^c + d(\log n)^a] : n \in \mathbb{N}\}$, where $b, d \neq 0$, $c \geq 1$, and $a > 1$.

The cited paper also contains interesting examples of non-ergodic sequences. For example $[\sqrt{2}n^{3/2} + \log n]$ is not ergodic, whereas $[\sqrt{2}n^{3/2} + (\log n)^2]$ and $[\sqrt{2}n^{\pi/2} + \log n]$ are ergodic. Is $\{[\sqrt{2}n^{3/2} + \log n] : n \in \mathbb{N}\}$ an enhanced vdC set? We leave this as an open question.

(ii) In [Berg-Ha2] a mean ergodic theorem along a *tempered sequence* is proved. More precisely, it is shown (see Theorem 8.1 in [Berg-Ha2]) that, for any *tempered function* [9] g, the sequence $([g(n)])$ is ergodic. This gives a large new class of examples. For example, the function $g(x) = x^a(\cos((\log x)^b) + 2)$, where $a > 0$ and $0 < b < 1$, is a tempered function (which does not belong to any Hardy field).

[9] A real-valued function g defined on a half-line $[\alpha, \infty)$ is called a *tempered function* if there exist $k \in \mathbb{N}$ such that g is k times continuously differentiable, $g^{(k)}(x)$ tends monotonically to zero as $x \to \infty$, and $\lim_{x\to\infty} x|g^{(k)}(x)| = \infty$. This notion is classical in the theory of uniform distribution (see [Ci]).

(iii) A different type of example is provided by so-called "automatic sequences". Characterizations of ergodic automatic sequences are well known (see for example [Ma]). A typical example of such a sequence is the Morse sequence $(0, 3, 5, 6, 9, 10, \ldots)$, which consists of the integers whose sum of digits in base two is even.

(iv) As a consequence of the Wiener–Wintner ergodic theorem, we know that for any weakly mixing m.p.s. $(X, \mathcal{A}, \mu, T)$ and for any $A \in \mathcal{A}$ with $\mu(A) > 0$, for almost every $x \in X$ the sequence $\{n \in \mathbb{N} : T^n x \in A\}$ is ergodic.

In [Lem-Les-Pa-V-Wi] other types of random sequences that are almost surely ergodic are constructed, of the form $(\sum_{n=0}^{N-1} f \circ T^n)$ where f is an integer-valued function on a m.p.s. $(X, \mathcal{A}, \mu, T)$, under some conditions on the m.p.s. and the function.

2.5.2. *Polynomial sequences.* The examples given in Subsection 1.3.2 not only have the ordinary vdC property, but also the enhanced vdC property (in $\mathbb{Z}^d$). We restrict ourselves here to the one-parameter case.

The following criterion, which generalizes Proposition 2.10, is useful in obtaining additional interesting examples.

PROPOSITION 2.11. *Let $D = (d_n)_{n\in\mathbb{N}}$ be a sequence of non-zero integers. Suppose that*

(i) *for all $q \in \mathbb{N}$, $D \cap q\mathbb{Z}$ has positive upper density in D;*
(ii) *for all irrational real numbers x, the sequence $(d_n x)$ is uniformly distributed mod 1.*

Then D is an enhanced vdC sequence.

Proof. Fix $q \in \mathbb{N}$. There exists an increasing sequence $(N_k^{(q)})_{k\in\mathbb{N}}$ of positive integers such that

$$\liminf_{k\to\infty} \frac{1}{N_k^{(q)}} \#\{n \in [1, N_k^{(q)}] : q! \text{ divides } d_n\} > 0. \tag{14}$$

Define a family of uniformly bounded trigonometric polynomials with spectrum contained in D by the formula

$$P_{q,k}(x) := \frac{1}{\#\{n \le N_k^{(q)} : q! \mid d_n\}} \sum_{n \le N_k^{(q)},\, q! \mid d_n} e(d_n x). \tag{15}$$

Replacing if necessary the sequence $(N_k^{(q)})$ by a subsequence, we can suppose that, for all rational numbers y, the sequence $(P_{q,k}(y))$ converges as $k \to \infty$.

Consider now an irrational real number x. We have

$$\begin{aligned} P_{q,k}(x) &= \frac{1}{\#\{n \le N_k^{(q)} : q! \mid d_n\}} \sum_{n \le N_k^{(q)}} e(d_n x) \, \frac{1}{q!} \sum_{j=0}^{q!-1} e\Big(\frac{d_n j}{q!}\Big) \\ &= \frac{N_k^{(q)}}{\#\{n \le N_k^{(q)} : q! \mid d_n\}} \, \frac{1}{q!} \sum_{j=0}^{q!-1} \frac{1}{N_k^{(q)}} \sum_{n \le N_k^{(q)}} e\Big(d_n\Big(x + \frac{j}{q!}\Big)\Big). \end{aligned}$$

Using (14) and hypothesis (ii), we see that $\lim_{k\to\infty} P_{q,k}(x) = 0$.

We denote by g_q the pointwise limit of the sequence $(P_{k,q})_{k\in\mathbb{N}}$. For all rational numbers y, we have $P_{q,k}(y) = 1$ for all large enough q.

Letting q go to infinity, we see that the sequence (g_q) converges everywhere to the characteristic function of the rationals. Applying the dominated convergence theorem twice, we observe that, for all finite measures σ on $\mathbb{T}$,

$$\lim_{q\to\infty} \lim_{k\to\infty} \int_{\mathbb{T}} P_{q,k} \, d\sigma = \sigma(\mathbb{Q}/\mathbb{Z}).$$

Let σ be a positive measure on $\mathbb{T}$ such that $\lim_{n\to\infty} \widehat{\sigma}(d_n) = 0$. From (15), we deduce that $\lim_{k\to\infty} \int_{\mathbb{T}} P_{q,k} \, d\sigma = 0$, hence $\sigma(\mathbb{Q}/\mathbb{Z}) = 0$, and in particular $\sigma(\{0\}) = 0$.

We have proved that D is an FC$^+$ set. ∎

From Proposition 2.11, one can deduce the following (not too surprising) corollaries.

Corollary 2.12. *Let p be a polynomial with integer coefficients. The sequence $(p(n))_{n\in\mathbb{N}}$ is enhanced vdC if and only if for all positive integers q, there exists $n \ge 1$ such that q divides $p(n)$.*

Corollary 2.13. *Let f be a (non-zero) polynomial with integer coefficients and zero constant term. The sequences $\{(f(p-1)) : p \in \mathcal{P}\}$ and $\{(f(p+1)) : p \in \mathcal{P}\}$ are enhanced vdC.*

Let us describe one more family of examples, coming from *generalized polynomials* ([10]), dealt with in [Berg-Ha1]. Let q be an integer valued generalized polynomial. Corollary 3.5 of [Berg-Ha1] gives a sufficient condition for $(q(n))$ to be an *averaging sequence of recurrence* and this condition is the same as the hypothesis of our Proposition 2.11. In particular, averaging sequences of recurrence in [Berg-Ha1, p. 106] provide examples of enhanced vdC sets. Here are two of them:

([10]) The class of polynomial functions is obtained, starting from the constants and the identity function $x \mapsto x$, by the use of addition and multiplication. To define the class of generalized polynomials just add the greatest integer function as an allowed operation.

- For $\alpha_1, \dots, \alpha_k$ non-zero real numbers and $k \geq 3$,
$$\{[\alpha_1 n] \dots [\alpha_k n] : n \in \mathbb{N}\} \text{ is an enhanced vdC set.}$$
- For α a non-zero real number, $\{[\alpha n]n^2 : n \in \mathbb{N}\}$ is an enhanced vdC set.

3. VAN DER CORPUT SETS AND SETS OF RECURRENCE

In this section we discuss some links between the vdC property and recurrence in dynamical systems.

3.1. Sets of strong recurrence. Recall that a subset D of $\mathbb{Z}$ is a *set of recurrence* if, given any m.p.s. $(X, \mathcal{A}, \mu, T)$ and any subset A in $\mathcal{A}$ of positive μ-measure, there exists $d \in D$, $d \neq 0$, such that $\mu(A \cap T^d A) > 0$.

Definition 5. An infinite subset D of $\mathbb{Z}$ is a *set of strong recurrence* if, given any m.p.s. $(X, \mathcal{A}, \mu, T)$ and any subset A in $\mathcal{A}$ of positive μ-measure,
$$\limsup_{d \in D,\, |d| \to \infty} \mu(A \cap T^d A) > 0.$$

One of the reasons to be interested in sets of strong recurrence is that they naturally appear in combinatorial applications. See for example Theorem 4.1 in [Berg.1].

Alan Forrest ([Fo]) gave an example of a set of recurrence which is not a set of strong recurrence.

3.2. VdC sets and sets of recurrence. Recall once more the definition of a vdC set (cf. Definition 2).

A set of non-zero integers D is a van der Corput set if, for any sequence $(u_n)_{n \in \mathbb{N}}$ of complex numbers of modulus 1 such that
$$\forall d \in D, \qquad \gamma(d) := \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} u_{n+d} \overline{u}_n = 0,$$
we have
$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} u_n = 0.$$

We know that we obtain an equivalent definition if we replace in the last sentence "any sequence $(u_n)_{n \in \mathbb{N}}$ of complex numbers of modulus 1" by "any bounded sequence $(u_n)_{n \in \mathbb{N}}$ of complex numbers". (This is a consequence of the generalized vdC inequality, as Corollary 1.31 follows from Proposition 1.30.)

A set D is a set of recurrence if and only if it is *intersective*, that is, satisfies the following condition: for any set E of integers of positive upper density, one has $D \cap (E - E) \neq \emptyset$. This fact is well known (see [BM] and [Berg.1]). It is utilized in the proof of the following theorem.

THEOREM 3.1. *Let $D \subset \mathbb{Z} \setminus \{0\}$. The set D is a set of recurrence if and only if it satisfies the following van der Corput's type property*: *for any sequence $(u_n)_{n\in\mathbb{N}}$ of 0's and 1's such that*

$$\forall d \in D, \qquad \gamma(d) := \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} u_{n+d} u_n = 0$$

we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} u_n = 0.$$

It is an exercise to verify that we obtain an equivalent statement if we replace in the preceding sentence "for any sequence $(u_n)_{n\in\mathbb{N}}$ of 0's and 1's" by "for any bounded sequence $(u_n)_{n\in\mathbb{N}}$ of positive real numbers".

As a consequence of Theorem 3.1, we obtain the well known fact that any van der Corput set is a set of recurrence ([Kam-MF]). Answering a question of Ruzsa, Bourgain proved in [Bou] that there exist sets of recurrence which are not vdC.

Proof of Theorem 3.1. If D is not a set of recurrence, then there exists a set $E \subset \mathbb{N}$ such that

$$\bar{d}(E) := \limsup_{N\to\infty} \frac{1}{N} |E \cap [1, N]| > 0 \quad \text{and} \quad D \cap (E - E) = \emptyset.$$

If we consider the sequence (u_n) defined by

$$u_n = \begin{cases} 1 & \text{if } n \in E, \\ 0 & \text{if } n \notin E, \end{cases}$$

we see that

$$\forall d \in D, \qquad \frac{1}{N} \sum_{n=1}^{N} u_{n+d} u_n = 0, \quad \text{but} \quad \limsup_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} u_n > 0.$$

This proves the "if" part of the theorem.

Suppose now that D is a set of recurrence. The fact that if E is a set of positive upper density, then there exists $d \in D$ such that $\{n \in E : n + d \in E\} \neq \emptyset$, is a consequence of Furstenberg's correspondence principle. But this principle gives more ([11]): there exists $d \in D$ such that the set $\{n \in E : n + d \in E\}$ has positive upper density.

([11]) For a statement of Furstenberg's correspondence principle in the form we utilize here, see for example Theorem 1.1 in [Berg.3].

Hence if a sequence (u_n) is the indicator of a set E of positive upper density, then there exists $d \in D$ such that

$$\limsup_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} u_{n+d} u_n > 0. \blacksquare$$

The similarity and the distinction between the recurrence property and the vdC property is also illustrated by the next proposition (to be compared with the spectral characterization of vdC sets—Theorem 1.8).

If $(X, \mathcal{A}, \mu, T)$ is a m.p.s. and if $A \in \mathcal{A}$, we denote by σ_A the *spectral measure of* A, which is defined by $\mu(A \cap T^{-n}A) = \widehat{\sigma}_A(n)$ for any $n \in \mathbb{Z}$. If f is a square integrable function on X, we denote by σ_f the *spectral measure of* f, which is defined by $\int f \circ T^n \cdot f \, d\mu = \widehat{\sigma}_f(n)$ for any $n \in \mathbb{Z}$. (Of course, we have $\sigma_A = \sigma_{\mathbf{1}_A}$.)

PROPOSITION 3.2. *Let $D \subset \mathbb{Z} \setminus \{0\}$. The set D is a set of recurrence if and only if one of the following two equivalent properties is satisfied*:

- *In any ergodic m.p.s., if the Fourier transform $\widehat{\sigma}_A$ of a set A vanishes on D, then $\sigma_A = 0$.*
- *In any ergodic m.p.s., if the Fourier transform $\widehat{\sigma}_f$ of a bounded positive function f vanishes on D, then $\sigma_f = 0$.*

Proof. Suppose that D is not a set of recurrence. There exists an ergodic m.p.s. $(X, \mathcal{A}, \mu, T)$ and a set A in $\mathcal{A}$ with positive measure such that, for all $d \in D$, $\mu(A \cap T^d A) = 0$. The spectral measure σ_A of A satisfies $\widehat{\sigma}(d) = 0$ for all $d \in D$, and $\sigma_A(\{0\}) = \mu(A) \neq 0$.

Suppose that D is a set of recurrence. Let σ_f be the spectral measure of a bounded positive function f. Suppose that for all $d \in D$, $\widehat{\sigma}_f(d) = 0$. By the ergodic theorem, we have almost surely, for all $d \in D$,

$$0 = \int f \cdot f \circ T^d \, d\mu = \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} f \circ T^n \cdot f \circ T^{n-d}.$$

Using Theorem 3.1 (and more precisely the remark immediately following the theorem), we deduce that, almost surely,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} f \circ T^n = 0.$$

The ergodic theorem gives $\int f \, d\mu = 0$, hence $\sigma_f = 0$. $\blacksquare$

3.3. Enhanced vdC sets and strong recurrence. The results in this subsection indicate that the link between enhanced van der Corput sets and sets of strong recurrence is parallel to the link between van der Corput sets and sets of recurrence. However, we do not know if there exists here any example of Bourgain's type ([Bou]). Such an example would give a negative answer to the following question.

QUESTION 5 (perhaps very difficult). Is every set of strong recurrence an FC^+ set (or, equivalently, an enhanced van der Corput set)?

The following question also comes naturally.

QUESTION 6. Is there any inclusion between the collection of sets of strong recurrence and the collection of van der Corput sets?

The next theorem gives an equivalence between strong recurrence and *strong intersectivity* (which is defined by (SR2) below).

THEOREM 3.3. *Let $D \subset \mathbb{Z}$. The following assertions are equivalent*:

(SR1) *D is a set of strong recurrence.*
(SR2) *For any $E \subset \mathbb{N}$ of upper density $\bar{d}(E) > 0$, there exists $\varepsilon > 0$ and infinitely many $d \in D$ such that*

$$\bar{d}(E \cap (E + d)) > \varepsilon.$$

(SR3) *For any sequence $(u_n)_{n\in\mathbb{N}}$ of 0's and 1's such that*

$$\lim_{|d|\to\infty,\, d\in D} \limsup_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} u_{n+d}\overline{u}_n = 0$$

we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} u_n = 0.$$

Proof. It is clear that properties (SR2) and (SR3) are the same. The fact that (SR1)⇒(SR2) follows directly from Furstenberg's correspondence principle. The following proof of (SR2)⇒(SR1) has been communicated to us by Anthony Quas. Let $(X, \mathcal{A}, \mu, T)$ be a m.p.s. and $A \in \mathcal{A}$ with $\mu(A) > 0$. Let $(x_n)_{n\geq 1}$ be a sequence of random points in X chosen independently and with the law μ. We consider a new sequence in X defined by

$$(y_n) := (x_1, x_2, Tx_2, x_3, Tx_3, T^2x_3, x_4, \ldots, T^3x_4, x_5, \ldots, T^4x_5, \ldots),$$

and the random set E of numbers n such that $y_n \in A$. We claim that, almost surely,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mathbf{1}_E(n) = \mu(A). \tag{16}$$

This claim can be justified by the following law of large numbers, applied to the mutually independent random variables

$$Y_k := \left(\frac{1}{k}\sum_{j=0}^{k-1} \mathbf{1}_A(T^j x_k)\right) - \mu(A).$$

LEMMA 3.4 (Law of large numbers). *Let* (Y_k) *be a sequence of random variables such that* $\sup_k \mathbb{E}(Y_k^2) < \infty$, $\mathbb{E}(Y_k) = 0$, *and* $\mathbb{E}(Y_k Y_l) = 0$ *if* $k \neq l$. *Almost surely we have*

$$\lim_{n\to\infty} \frac{1}{n^2}\sum_{k=1}^{n} kY_k = 0. \tag{17}$$

(The convergence (17) is a direct consequence of some easy L^2 estimates. It can also be deduced from the convergence of ordinary Cesàro averages. We omit the proof.)

A similar argument using the block structure of the sequence (y_n) gives (almost surely)

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \mathbf{1}_E(n)\mathbf{1}_E(n+d) = \mu(A\cap T^{-d}A). \tag{18}$$

Assume now that condition (SR2) is satisfied. From (16) we deduce that $\bar{d}(E) > 0$, hence there exist $\varepsilon > 0$ and infinitely many $d \in D$ such that

$$\bar{d}(E\cap(E+d)) > \varepsilon,$$

which means (by (18)) that $\mu(A\cap T^{-d}A) > \varepsilon$. ∎

PROPOSITION 3.5. *Any enhanced vdC set is a set of strong recurrence.*

Proof. Let $D \subset \mathbb{Z}$ be an enhanced vdC set, let $(X, \mathcal{A}, \mu, T)$ be a m.p.s. and $A \in \mathcal{A}$ with $\mu(A) > 0$. There exists a positive measure σ on the torus such that, for all $n \in \mathbb{Z}$,

$$\widehat{\sigma}(n) = \mu(A\cap T^n A).$$

This measure has a point mass at zero: $\sigma(\{0\}) \geq \mu(A)^2$. Since the set D is FC^+, this implies that there exists $\varepsilon > 0$ such that $\widehat{\sigma}(d) > \varepsilon$ for infinitely many $d \in D$. ∎

3.4. Density notions of vdC sets and sets of recurrence. A new natural notion of vdC type set, which we will call *density vdC*, can be obtained by replacing in Definition 3 the convergence of γ to zero along the set D by the convergence of γ to zero along a subset of D which has full density in D. We will associate to it a notion of *density* FC^+ *set*. These notions are related to *averaging sets of recurrence*, as we will see below. Here are the formal definitions.

If D is an infinite set of integers, we will write $D = \{d_m : m \in \mathbb{N}\}$ with the convention that the numbers d_m are pairwise distinct and the sequence $(|d_m|)$ is non-decreasing. Let us recall that for any bounded sequence $(v(d_m))_{m\in\mathbb{N}}$ of positive numbers the following two properties are equivalent:

- $\displaystyle\lim_{M\to\infty} \frac{1}{M}\sum_{m=1}^{M} v(d_m) = 0.$
- There exists $D' \subset D$ such that
$$\lim_{M\to\infty} \frac{\#D' \cap [-M,M]}{\#D \cap [-M,M]} = 1 \quad \text{and} \quad \lim_{m\to\infty,\, d_m\in D'} v(d_m) = 0.$$

Definition 6. An infinite set of integers D is a *density vdC set* if for any sequence $(u_n)_{n\in\mathbb{Z}}$ of complex numbers of modulus 1 such that
$$\lim_{M\to\infty} \frac{1}{M}\sum_{m=1}^{M} \limsup_{N\to\infty} \left| \frac{1}{N}\sum_{n=0}^{N-1} u_{n+d_m}\overline{u}_n \right| = 0,$$
one has
$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} u_n = 0.$$
(Compare this definition with Proposition 2.5.)

Definition 7. An infinite set of integers D is a *density* FC^+ *set* if every positive measure σ on the torus $\mathbb{T}$ such that $\lim_{M\to\infty} M^{-1}\sum_{m=1}^{M}\widehat{\sigma}(d_m) = 0$ is continuous. (Compare with Definition 4. Any density FC^+ set is an FC^+ set.)

Definition 8. An infinite set of integers D is an *averaging set of recurrence* if for any m.p.s. $(X, \mathcal{A}, \mu, T)$ and $A \in \mathcal{A}$ with $\mu(A) > 0$,
$$\limsup_{M\to\infty} \frac{1}{M}\sum_{m=1}^{M} \mu(A \cap T^{-d_m}A) > 0.$$

Note that this definition differs slightly from the one given in [Berg-Ha1] where the limsup is replaced by a lim.

Any averaging set of recurrence is a set of strong recurrence.

Theorem 3.6. *The notions of a density vdC set and of a density* FC^+ *set coincide.*

The proof is similar to that of Theorem 2.1 and is omitted.

From Theorem 3.6 one can deduce for example that the class of density vdC sets has the Ramsey property.

Of course, every density vdC set is an enhanced vdC set. We do not know whether the reverse implication holds.

QUESTION 7. Do the notions of density vdC set and enhanced vdC set coincide?

Questions 2, 3 and 4 that we asked about enhanced vdC sets have obvious density vdC sets analogues.

Note also that the examples described in Subsection 2.5 can also be utilized to illustrate the notion of density vdC set. In particular we have:

- If (d_n) is an increasing ergodic sequence of integers, then the set $\{d_n\}$ is a density vdC set. This leads to the examples presented in Subsection 2.5.1.
- If an increasing sequence (d_n) of integers satisfies hypotheses (i) and (ii) of Proposition 2.11, then the set $\{d_n\}$ is a density vdC set. This leads to the "polynomial examples" presented in Subsection 2.5.2.

The following proposition establishes a link with recurrence.

PROPOSITION 3.7. *Any density vdC set is an averaging set of recurrence.*

The proof is similar to that of Proposition 3.5 and is omitted.

3.5. Nice vdC sets and nice recurrence. Another natural notion of recurrence is that of nice recurrence.

DEFINITION 9. A set D of integers is a *set of nice recurrence* if given any m.p.s. $(X, \mathcal{A}, \mu, T)$, $A \in \mathcal{A}$ with $\mu(A) > 0$, and any $\varepsilon > 0$, we have

$$\mu(A \cap T^{-d}A) \geq \mu(A)^2 - \varepsilon$$

for infinitely many $d \in D$.

The following proposition provides an equivalent definition for sets of nice recurrence.

PROPOSITION 3.8. *A set D of integers is a set of nice recurrence if and only if*

(C) *given any m.p.s. $(X, \mathcal{A}, \mu, T)$, $A \in \mathcal{A}$ with $\mu(A) > 0$, and any $\varepsilon > 0$, there exists $d \in D$, $d \neq 0$, such that $\mu(A \cap T^{-d}A) \geq \mu(A)^2 - \varepsilon$.*

Proof. We have to prove that the integer d appearing in condition (C) can be chosen arbitrarily large. Suppose that (C) is satisfied. Consider a m.p.s. $(X, \mathcal{A}, \mu, T)$ and a set $A \in \mathcal{A}$ with $\mu(A) > 0$. Denote by $(Y, \mathcal{B}, \nu, S)$ a Bernoulli scheme on two letters (Y is the set of sequences of 0's and 1's, ν is a non-trivial product measure, and S is the shift). Let k be a positive integer and B be the cylinder set in Y of all sequences beginning with a 1 followed by k 0's. We have $\nu(B) > 0$, $\nu(B \cap S^{-d}B) = 0$ if $|d| \leq k$, and $\nu(B \cap S^{-d}B) = \nu(B)^2$ if $|d| > k$. Applying the hypothesis to the product

$T \times S$ of the two dynamical systems, we find that there exists $d \in D$ such that

$$\mu \otimes \nu((A \times B) \cap (T \times S)^{-d}(A \times B)) \geq (\mu \otimes \nu(A \times B))^2 - \varepsilon \nu(B)^2,$$

hence there exists $d \in D$, $|d| > k$, such that

$$\mu(A \cap T^{-d} A) \geq \mu(A)^2 - \varepsilon. \ \blacksquare$$

The notion of sets of nice recurrence seems to be naturally related to the following definitions.

Definition 10. An infinite set D of integers is a *nice vdC set* if, for any sequence $(u_n)_{n\in\mathbb{N}}$ of complex numbers of modulus one,

$$\limsup_{N\to\infty} \left|\frac{1}{N}\sum_{n=1}^{N} u_n\right|^2 \leq \limsup_{|d|\to\infty,\, d\in D} \ \limsup_{N\to\infty} \left|\frac{1}{N}\sum_{n=1}^{N} u_{n+d}\overline{u}_n\right|.$$

Definition 11. A infinite set D of integers is a *nice* FC^+ *set* if, for any positive measure σ on the torus,

$$\sigma(\{0\}) \leq \limsup_{|d|\to\infty,\, d\in D} |\widehat{\sigma}(d)|.$$

The following proposition is similar in spirit to Proposition 3.8.

Proposition 3.9. *A set D of integers is a nice* FC^+ *set if and only if*

(C′) *for any positive measure σ on the torus and any $\varepsilon > 0$, there exists $d \in D$, $d \neq 0$, such that $|\widehat{\sigma}(d)| > \sigma(\{0\}) - \varepsilon$.*

Proof. We have to prove that the integer d appearing in condition (C′) can be chosen arbitrarily large. Suppose that (C′) is satisfied. Let k be a positive integer. There exists a positive measure ϱ on the torus such that $\widehat{\varrho}(n) = 0$ if $|n| \leq k$ and $\widehat{\varrho}(n) = \varrho(\{0\}) > 0$ if $|n| > k$. (Choose the spectral measure of the indicator of the set B in the Bernoulli scheme considered in the proof of Proposition 3.8.) We apply our hypothesis to the measure $\sigma \star \varrho$. There exists $d \in D$ such that

$$|\widehat{\sigma}(d)\widehat{\varrho}(d)| = |\widehat{\sigma \star \varrho}(d)| > \sigma \star \varrho(\{0\}) - \varepsilon\varrho(\{0\}) \geq \sigma(\{0\})\varrho(\{0\}) - \varepsilon\varrho(\{0\}),$$

hence there exists $d \in D$, $|d| > k$, such that

$$|\widehat{\sigma}(d)| > \sigma(\{0\}) - \varepsilon. \ \blacksquare$$

Question 8. What are the implications between the three properties: nice vdC, nice FC^+ and nice recurrence?

Here is what we know:

(N1) Nice $\mathrm{FC}^+ \Rightarrow$ nice recurrence.
(N2) Nice $\mathrm{FC}^+ \Rightarrow$ nice vdC.
(N3) Nice vdC $\Rightarrow$ a weak form of nice recurrence.

Let us explain what this last assertion means. If D is a nice vdC set, then for any probability measure σ on the torus,

$$\sigma(\{0\})^2 \leq \limsup_{|d|\to\infty,\, d\in D} |\widehat{\sigma}(d)|,$$

and consequently, we have the following recurrence property: given any m.p.s. $(X, \mathcal{A}, \mu, T)$, $A \in \mathcal{A}$ with $\mu(A) > 0$, and any $\varepsilon > 0$, we have

$$\mu(A \cap T^{-d}A) \geq \mu(A)^4 - \varepsilon \tag{19}$$

for infinitely many $d \in D$. (Note that the exponent 4 in (19) is not a typo. It would be "nice" to better understand the meaning of inequality (19).)

The proof of (N1) is a direct application of the spectral theorem: let $(X, \mathcal{A}, \mu, T)$ be a m.p.s. and $A \in \mathcal{A}$. There exists a positive measure σ on the torus such that

$$\forall n \in \mathbb{N}, \quad \widehat{\sigma}(n) = \mu(A \cap T^{-n}A) \quad \text{and} \quad \sigma(\{0\}) = \int_A \mu(A|\mathcal{I})\, d\mu \geq \mu(A)^2.$$

The proof of (N2) follows the lines of the spectral characterization described in Subsections 1.2.2 and 2.2. Let (u_n) be a sequence of complex numbers of modulus one and

$$M := \limsup_{|d|\to\infty,\, d\in D} \limsup_{N\to\infty} \left| \frac{1}{N} \sum_{n=1}^{N} u_{n+d}\overline{u}_n \right|.$$

There exists an increasing sequence $(N_j)_{j\geq 0}$ of positive integers such that

- $\displaystyle \lim_{j\to\infty} \frac{1}{N_j} \Big| \sum_{n=1}^{N_j} u_n \Big| = \limsup_{N\to\infty} \frac{1}{N} \Big| \sum_{n=1}^{N} u_n \Big|,$
- $\displaystyle \forall h \in \mathbb{Z}, \quad \gamma(h) := \lim_{j\to\infty} \frac{1}{N_j} \sum_{n=1}^{N_j} u_{n+h}\overline{u}_n \quad \text{exists.}$

The map γ is the Fourier transform of a positive measure σ on the torus. Suppose that D is a nice vdC set. By Lemma 2.2 we have

$$\limsup_{N\to\infty} \left| \frac{1}{N} \sum_{n=1}^{N} u_n \right|^2 \leq \sigma(\{0\}) \leq \limsup_{|d|\to\infty,\, d\in D} |\widehat{\sigma}(d)| \leq M.$$

Claim (N3) can be proved using Lemmas 2.3 and 2.4. Following the method described in Subsection 2.2, we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} Y_{n+h}\overline{Y}_n = \widehat{\sigma}(h) \quad \text{and} \quad \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} Y_n = \sigma(\{0\}).$$

Hence, if D is nice vdC, then

$$\sigma(\{0\})^2 \leq \limsup_{|d|\to\infty,\, d\in D} |\widehat{\sigma}(d)|$$

and (N3) is verified.

One more natural question concerns the Ramsey property.

Using product dynamical systems, it is easy to verify that the class of sets of recurrence and the class of sets of strong recurrence have the Ramsey property. We saw that the class of vdC sets and the class of enhanced vdC sets have this property. The other notions of vdC sets and of recurrence could be studied from this point of view.

QUESTION 9. Do the class of sets of nice recurrence and the class of nice vdC sets have the Ramsey property?

Note that the class of sets of nice recurrence has the Ramsey property if and only if the following property of simultaneous nice recurrence is valid: given any set $D \subset \mathbb{Z} \setminus \{0\}$ of nice recurrence, any m.p.s. $(X, \mathcal{A}, \mu, T)$, any sets A and B in $\mathcal{A}$, and any $\varepsilon > 0$, there exists $d \in D$ such that

$$\mu(A \cap T^{-d}A) > \mu(A)^2 - \varepsilon \quad \text{and} \quad \mu(B \cap T^{-d}B) > \mu(B)^2 - \varepsilon.$$

4. VARIATIONS ON THE AVERAGING METHOD

In this final section we provide additional remarks on some of the possible variations on the vdC theme, related to different notions of averaging which naturally appear in the theory of uniform distribution and ergodic theory. For simplicity and in order to be able to more easily stress the important points, we restrict our discussion to subsets of $\mathbb{Z}$. We want, however, to remark that many of the results in this paper can be extended to a much wider setup involving general groups and various methods of summation. (See for example [Pe], where some directions of extensions are indicated.)

4.1. Well distribution. Recall that a sequence $(x_n)_{n\in\mathbb{N}}$ of real numbers is *well distributed mod* 1 if, for any continuous function f on the torus $\mathbb{T}$, we have

$$\lim_{N-M\to\infty} \frac{1}{N-M} \sum_{n=M}^{N-1} f(x_n) = \int_{\mathbb{T}} f(t)\, dt.$$

To this notion of well distribution is naturally associated a notion of van der Corput set. Let us call it a w-vdC set: a set D of positive integers is a *w-vdC set* if, for any sequence $(u_n)_{n\in\mathbb{N}}$ of complex numbers of modulus 1 such that

$$\forall d \in D, \qquad \gamma(d) := \lim_{N-M\to\infty} \frac{1}{N-M} \sum_{n=M}^{N-1} u_{n+d}\overline{u}_n = 0$$

we have

$$\lim_{N-M\to\infty} \frac{1}{N-M} \sum_{n=M}^{N-1} u_n = 0.$$

The spectral characterization of vdC sets given in Theorem 1.8 immediately implies that any vdC set is a w-vdC set.

But the proof, coming from Ruzsa ([Ruz]), of the fact that the spectral properties (S1) and (S2) are necessary for vdC sets cannot be applied to w-vdC. This is because the law of large numbers fails dramatically when we replace averages $N^{-1}\sum_{0\le n<N}$ by moving averages $(N-M)^{-1}\sum_{M\le n<N}$.

QUESTION 10. Is every w-vdC set a vdC set?

4.2. Følner sequences. Let $F=(F_N)_{N\ge 1}$ be a Følner sequence in the space of parameters (which is $\mathbb{Z}$ in this section). Let us say that a real sequence $(x_n)_{n\in\mathbb{Z}}$ is *F-u.d. mod* 1 if, for any continuous function f on the torus $\mathbb{T}$, we have

$$\lim_{N\to\infty}\frac{1}{|F_N|}\sum_{n\in F_N} f(x_n)=\int_{\mathbb{T}} f(x)\,dx. \tag{20}$$

(We say that the sequence $(f(x_n))$ *converges to the integral of f in the F-sense* when (20) is satisfied.)

One can naturally define also the notion of F-vdC. A set D of non-zero integers is an *F-vdC set* if any sequence (x_n) such that, for all $d\in D$, the sequence $x_{n+d}-x_n$ is F-u.d. mod 1, is itself F-u.d. mod 1.

In order to compare the notion of F-vdC set with the notion of vdC set, it would be of interest to obtain a spectral characterization of F-vdC sets similar to Theorem 1.8.

Note that the sequence of correlations

$$\gamma(h):=\limsup_{N\to\infty}\frac{1}{|F_N|}\sum_{n\in F_N} u_{n+h}\overline{u}_n$$

is positive-definite, and the Følner property is exactly what is needed in order to prove a result similar to Lemma 1.9. An argument similar to the one used in the proof of the implication (S2)$\Rightarrow$(S3) allows one to establish the fact that *any vdC set is an F-vdC set.*

In the other direction we do not know any general result, but, keeping in mind the argument we used in the proof of Theorems 1.8 and 2.1, we can state the following sufficient condition: *suppose that for any probability measure on the torus $\mathbb{T}$ there exists a sequence $(Y_n)_{n\in\mathbb{N}}$ of complex numbers of modulus one such that, for all $h\in\mathbb{Z}$,*

$$\lim_{N\to\infty}\frac{1}{|F_N|}\sum_{n\in F_N} Y_n=\sigma(\{0\}) \quad \textit{and} \quad \lim_{N\to\infty}\frac{1}{|F_N|}\sum_{n\in F_N} Y_{n+h}\overline{Y}_n=\widehat{\sigma}(h);$$

then any F-vdC set is a vdC set.

We have in particular the following result (and its multiparameter extensions):

PROPOSITION 4.1. *If a Følner sequence F is such that any bounded sequence which converges in the Cesàro sense also converges in the F-sense* ([12]) *then the notions of vdC set and F-vdC set coincide.*

5. APPENDIX. A REMARK ON DIVISIBILITY OF POLYNOMIALS

Definitions

- A polynomial $p \in \mathbb{Z}[X]$ is *divisible* by an integer d if there exists $n \in \mathbb{Z}$ such that d divides $p(n)$.
- A polynomial $p \in \mathbb{Z}[X]$ is *divisible* if it is divisible by any integer.
- Polynomials $p_1, \dots, p_r \in \mathbb{Z}[X]$ are *simultaneously divisible* by an integer d if there exists $n \in \mathbb{Z}$ such that d divides $p_i(n)$, $1 \le i \le r$.
- Polynomials $p_1, \dots, p_r \in \mathbb{Z}[X]$ are *simultaneously divisible* if they are simultaneously divisible by any integer.

(Trivial examples: if $p(0) = 0$ then p is divisible; the polynomial $2X + 1$ is not divisible; the polynomials X and $X + 1$ are divisible but not simultaneously divisible.)

Known facts. Let $p_1, \dots, p_r \in \mathbb{Z}[X]$. The following assertions are equivalent:

- The sequence $(p_1(n), \dots, p_r(n))_{n\in\mathbb{N}}$ is a Poincaré recurrence sequence for finite measure preserving $\mathbb{Z}^r$ actions.
- The sequence $(p_1(n), \dots, p_r(n))_{n\in\mathbb{N}}$ is a van der Corput sequence in $\mathbb{Z}^r$.
- $p_1, \dots, p_r$ are simultaneously divisible.

In [Berg-Lei-Les], we prove that the simultaneous divisibility of polynomials $p_1, \dots, p_r$ is also a necessary and sufficient condition for multiple recurrence of the type

$$\mu(A \cap T^{p_1(n)}A \cap T^{p_2(n)}A \cap \cdots \cap T^{p_r(n)}A) > 0.$$

CLAIM. *The simultaneous divisibility of a family of polynomials is a property strictly stronger than the divisibility of any of their linear combinations. In other words, there exist two polynomials p and q in $\mathbb{Z}[X]$ such that, for any integers a and b, the polynomial $ap + bq$ is divisible but the polynomials p and q are not simultaneously divisible.*

([12]) If any bounded sequence which converges in the Cesàro sense also converges in the F-sense then the limits in the Cesàro sense and in the F-sense coincide (when they exist). This fact is left as an exercise for the reader.

Here are two facts which seem to go against this Claim. Let $p, q \in \mathbb{Z}[X]$.

- Let d be a prime number. If for all pairs (a, b) of integers, the polynomial $ap+bq$ is divisible by d, then p and q are simultaneously divisible by d.
- Let d and e be two relatively prime integers. If p and q are simultaneously divisible by d and simultaneously divisible by e, then they are simultaneously divisible by de.

These facts indicate that the key to the distinction between the simultaneous divisibility and the divisibility of linear combinations of polynomials lies with the divisibility by d^k where d is a prime number and $k > 1$.

Proof of Claim. Let us show that the polynomials

$$p(X) = (2 + X^2 + X^3)(1 + 2X) \quad \text{and} \quad q(X) = X(1 + X)(1 + 2X)$$

are not simultaneously divisible by 4 although the polynomial $ap + bq$ is divisible for all a, b in $\mathbb{Z}$.

Modulo 4, we have $p(0) = 2$ and $q(0) = 0$; $p(1) = 0$ and $q(1) = 2$; $p(2) = q(2) = 2$; $p(3) = 2$ and $q(3) = 0$. This shows that p and q are not simultaneously divisible by 4.

Let us fix a and b in $\mathbb{Z}$ and show that $ap + bq$ is divisible. It is of course enough to consider the case when a and b are relatively prime. The divisibility of $ap + bq$ by odd integers is directly given by the presence of the common factor $1 + 2X$. Let us examine divisibility by the powers of 2. We will distinguish the case when one of the two numbers a and b is even, and the case when both are odd.

First case: a or b is even (and the other is odd). Let us show by induction on k that, for all $k \geq 0$, there exists an odd number n_k such that $2^k \mid ap(n_k) + bq(n_k)$. We can choose any number n_0, and $n_1 = 1$ is OK. Suppose that the result is true for an integer $k \geq 1$. Define $l := \max\{i \geq k : 2^i \mid ap(n_k) + bq(n_k)\}$. We have $l \geq k$ and $ap(n_k) + bq(n_k) = 2^l\alpha$, with α odd. Define a new odd number by $n_{k+1} = n_k + 2^l$. Using

$$ap(X) + bq(X) = 2aX^4 + (3a + 2b)X^3 + (a + 3b)X^2 + (4a + b)X + 2a,$$

we note that, modulo 2^{l+1},

$$\begin{aligned} &ap(n_{k+1}) + bq(n_{k+1}) \\ &\qquad = ap(n_k) + bq(n_k) + 2a(4 \cdot 2^l n_k^3) + (3a + 2b)(3 \cdot 2^l n_k^2) \\ &\qquad\quad + (a + 3b)(2 \cdot 2^l n_k) + (4a + b)2^l \\ &\qquad = 2^l\alpha + a2^l n_k^2 + 2^l b = 2^l(\alpha + an_k^2 + b). \end{aligned}$$

Since $\alpha + an_k^2 + b$ is even, this shows that $2^{l+1} \mid ap(n_{k+1}) + bq(n_{k+1})$. We have $l + 1 \geq k + 1$, and n_{k+1} is odd. This concludes the induction.

SECOND CASE: a and b are odd. Let us show by induction on k that, for all $k \geq 0$, there exists an even number n_k such that $2^k \mid ap(n_k) + bq(n_k)$. We can choose any number n_0, and $n_1 = 2$ is OK. Suppose that the result is true for an integer $k \geq 1$. We define l and $n_{k+1} = n_k + 2^l$ as in the first case, but now the number n_k is even, hence we still have

$$ap(n_{k+1}) + bq(n_{k+1}) = 2^l(\alpha + an_k^2 + b) = 0 \text{ modulo } 2^{l+1},$$

and the induction process works.

In any case, we have proved that $ap+bq$ is divisible by all the powers of 2. We also know that the polynomial $ap + bq$ is divisible by any odd integer. Let us prove that it is divisible by any integer $2^k\alpha$ where α is odd. We write $ap(X) + bq(X) = (2X + 1)r(X)$. We know that $2^k \mid r(n_k)$. By the Bézout identity, there exist integers u and v such that

$$2n_k + 1 = -u2^{k+1} + v\alpha.$$

We have $\alpha \mid 2(n_k + 2^k u) + 1$ and $2^k \mid r(n_k + 2^k u)$, hence

$$2^k\alpha \mid ap(n_k + 2^k u) + bq(n_k + 2^k u).$$

This proves that the polynomial $ap + bq$ is divisible.

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Received 8 March 2007;
revised 30 March 2007 (4882)