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ERGODIC AUTOMORPHISMS WHOSE WEAK CLOSURE OF OFF-DIAGONAL MEASURES CONSISTS OF ERGODIC SELF-JOININGS

BY

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Abstract. Basic ergodic properties of the ELF class of automorphisms, i.e. of the class of ergodic automorphisms whose weak closure of measures supported on the graphs of iterates of T consists of ergodic self-joinings are investigated. Disjointness of the ELF class with: 2-fold simple automorphisms, interval exchange transformations given by a special type permutations and time-one maps of measurable flows is discussed. All ergodic Poisson suspension automorphisms as well as dynamical systems determined by stationary ergodic symmetric α-stable processes are shown to belong to the ELF class.

Introduction. The notion of disjointness between measure-preserving automorphisms of standard probability Borel spaces was introduced by Furstenberg [9] in 1967. Since then many results showing disjointness of some classes have been proved (see e.g. [9], [12], [14], [19], [21], [26], [28], [29], [46], [47]).

In [6] the second and the third named authors of this paper introduced the notion of ELF (¹) flow. An ELF flow is, by definition, an ergodic flow such that when we pass to the weak closure of its time-t maps considered as Markov operators of the underlying L^2-space, then all the weak limits are indecomposable Markov operators. The ELF property is interesting only in the non-mixing case, and indeed in contrast with this property, some classical weakly mixing but non-mixing special flows over irrational rotations or, more generally, over interval exchange transformations turn out to have in the weak closure of Markov operators given by their time-t maps "sufficiently" decomposable Markov operators. Such flows are often special representations of some smooth flows on surfaces and a motivation to introduce the ELF property was to prove disjointness (in the sense of Furstenberg) of such flows

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(¹) The acronym ELF comes from the French abbreviation of "ergodicité des limites faibles".

from the ELF class (see [6], [7], [8]). In particular, some classical smooth weakly mixing flows on surfaces (e.g. considered in [25]) turn out to be disjoint from the ELF class.

On the other hand, the ELF property was also introduced in the hope of expressing the fact that a given flow is of "probabilistic origin". Indeed, a first attempt to define a system to be "of probabilistic origin" might be via the Kolmogorov group property of the spectrum. However, each weakly mixing system has an ergodic extension which has the Kolmogorov group property, simply by taking the infinite direct product of the system. Therefore this spectral property is too weak to single out systems of "probabilistic origin". As noticed in [6] Gaussian flows enjoy the ELF property (this result also follows from some earlier results of [29]). The present paper and, independently, the PhD thesis of E. Roy [36] are a further confirmation of the fact that dynamical systems whose origin are well-known classes of stationary processes (see below) are inside the ELF class. We also mention that in the general case, including mixing, another joining property (satisfied for example by flows with Ratner's property [35]) has been introduced in [43] to show disjointness from Gaussian systems.

In this paper, instead of flows, we consider the ELF property for automorphisms. One of the main results of the paper states that all ergodic Poisson suspension automorphisms enjoy the ELF property. This result is a consequence of Theorem 1 below saying that Poissonian joinings of ergodic Poisson automorphisms remain ergodic; the same result is also proved in the recent, independent paper [36]. Moreover, we consider so called α-stable automorphisms, i.e. ergodic automorphisms acting on a space whose measurable structure is determined by an invariant real subspace in which all variables are symmetric α-stable ($0 < \alpha < 2$, for $\alpha = 2$ we come back to the Gaussian case). We prove (Theorem 3 below) that α-stable self-joinings of such automorphisms must necessarily be ergodic, from which the ELF property directly follows. In the aforementioned thesis [36], a further step forward is even made: it is proved that given an ergodic stationary infinitely divisible process, each infinitely divisible self-joining of the corresponding measure-preserving automorphism remains ergodic, and in particular we also obtain the ELF property in this most general case.

Furthermore, we show (Proposition 12 below) that weakly mixing but non-mixing 2-fold simple automorphisms are disjoint from the ELF class. It is also shown that the time-one maps of flows considered in [8] are disjoint from any ELF automorphism, and therefore the time-one maps of Kochergin's smooth flows from [25] are disjoint from any ELF automorphism.

Recently, some attention has been devoted to joining properties of interval exchange transformations (see e.g. [4], [5]). Here we are able to prove (see Proposition 15 below) that for almost all choices of parameters defining a three-interval exchange transformation we obtain disjointness from the ELF class. In fact, this result is a consequence of a more general statement proved in the paper. Namely given $k \geq 3$ we consider special permutations of $\{1, \ldots, k\}$ and we prove that for a.a. choices of lengths of partition intervals of $[0, 1)$ the resulting automorphisms are disjoint from all ELF automorphisms.

Some results in this paper have been obtained during the visit of the fourth-named author at Nicolaus Copernicus University in September 2003 and during the visit of the third-named author at Université de Bretagne Occidentale in the Spring 2004.

1. Preliminaries

1.1. *Factors, joinings and Markov operators.* Assume that T is an ergodic automorphism of a standard probability Borel space $(X, \mathcal{B}, \mu)$. The associated unitary action of T on $L^2(X, \mathcal{B}, \mu)$ is given by $U_T(f) = f \circ T$ (but we will often write $T(f)$ instead of $f \circ T$). We denote by $C(T)$ the *centralizer* of T, that is, the set of all automorphisms of $(X, \mathcal{B}, \mu)$ commuting with T. Endowed with the strong operator topology of $U(L^2(X, \mathcal{B}, \mu))$ the centralizer becomes a Polish group. Any T-invariant sub-σ-algebra $\mathcal{A} \subset \mathcal{B}$ is called a *factor* of T. The quotient action of T on the quotient space $(X/\mathcal{A}, \mathcal{A}, \mu|_{\mathcal{A}})$ will be denoted by $T|_{\mathcal{A}}$ or even by $\mathcal{A}$ if no confusion arises. We say that T is *rigid* if the set $\{T^n : n \in \mathbb{Z}\}$ has an accumulation point in $C(T)$. It follows that in the rigidity case the centralizer is uncountable and for some increasing sequence (q_n), $T^{q_n} \to \mathrm{Id}$. Automorphisms which have no rigidity at all are called *mildly mixing* (see [11]). More precisely, T is called *mildly mixing* if its only rigid factor is the one-point factor.

Assume now that S is another ergodic automorphism of a standard probability Borel space $(Y, \mathcal{C}, \nu)$. By a *joining* of T and S we mean any $T \times S$-invariant measure ϱ on $(X \times Y, \mathcal{B} \otimes \mathcal{C})$ whose marginals ϱ_X and ϱ_Y satisfy $\varrho_X = \varrho|_X = \mu$ and $\varrho_Y = \varrho|_Y = \nu$ respectively. The set of joinings between T and S is denoted by $J(T, S)$. Whenever the automorphism $T \times S$ acting on $(X \times Y, \mathcal{B} \otimes \mathcal{C}, \varrho)$ (for short we will also write $(T \times S, \varrho)$) is ergodic, the joining ϱ is called *ergodic* and the set of ergodic joinings is denoted by $J^{\mathrm{e}}(T, S)$. The formula

$$\int_{X \times Y} f \otimes g \, d\varrho = \int_Y \Phi_\varrho(f) \cdot g \, d\nu$$

establishes a one-to-one correspondence between the set $J(T, S)$ and the set $\mathcal{J}(T, S)$ of all Markov operators from $L^2(X, \mathcal{B}, \mu)$ to $L^2(Y, \mathcal{C}, \nu)$ intertwining U_T and U_S (see e.g. [42], [29] for more details). Recall that a positive linear operator $\Phi : L^2(X, \mathcal{B}, \mu) \to L^2(Y, \mathcal{C}, \nu)$ is called *Markov* if $\Phi(\mathbf{1}_X) = \mathbf{1}_Y$ and $\Phi^*(\mathbf{1}_Y) = \mathbf{1}_X$, and then $\Phi = \Phi_\varrho$ where $\varrho(A \times B) = \int_B \Phi(\mathbf{1}_A) \, d\nu$ for measurable sets $A \in \mathcal{B}$ and $B \in \mathcal{C}$. The set of Markov operators is closed in the weak operator topology of $B(L^2(X, \mathcal{B}, \mu), L^2(Y, \mathcal{C}, \nu))$, hence both $\mathcal{J}(T, S)$

and $J(T,S)$ are compact (on the latter set we transport the topology of $\mathcal{J}(T,S)$). Ergodic joinings correspond to so called *indecomposable* Markov operators, i.e. to the extremal points in the set $\mathcal{J}(T,S)$, which has a natural structure of a Choquet simplex. Note that the Markov operator corresponding to the product measure $\mu\otimes\nu$ equals $\Pi_{X,Y}(f)=\int_X f\,d\mu$. If one more ergodic automorphism R on $(Z,\mathcal{D},\eta)$ is given and $\Phi_\varrho\in\mathcal{J}(T,S)$, $\Phi_\kappa\in\mathcal{J}(S,R)$ then $\Phi_\kappa\circ\Phi_\varrho\in\mathcal{J}(T,R)$ and the corresponding joining of T and R will be denoted by $\kappa\circ\varrho$.

Whenever $S=T$ we will write $J_2(T)$ and $J_2^{\mathrm{e}}(T)$ instead of $J(T,T)$ and $J^{\mathrm{e}}(T,T)$ respectively. Note that if $W\in C(T)$ then the formula $\mu_W(A\times B)=\mu(A\cap W^{-1}B)$ determines a self-joining, called a *graph joining*, of T, and moreover $\mu_W\in J_2^{\mathrm{e}}(T)$ (for $W=T^n$ we speak about *off-diagonal* self-joinings). We say that T is 2-*fold simple* (see [49], [21]) if the only ergodic self-joinings of T are graph joinings or the product measure $\mu\otimes\mu$. The measure μ_{Id} will also be denoted by Δ_X or Δ_μ.

We say that T is *relatively weakly mixing* with respect to a factor $\mathcal{A}\subset\mathcal{B}$ if the self-joining λ (called the *relatively independent extension of the diagonal measure on* $\mathcal{A}$) given by

$$\lambda(A\times B)=\int_{X/\mathcal{A}} E(\mathbf{1}_A\,|\,\mathcal{A})\cdot E(\mathbf{1}_B\,|\,\mathcal{A})\,d(\mu|_{\mathcal{A}})$$

is ergodic. If $\mathcal{A}_1\subset\mathcal{A}$ is another factor and $T|_{\mathcal{A}}$ is relatively weakly mixing over $\mathcal{A}_1$ then T is still relatively weakly mixing over $\mathcal{A}_1$ (for this chain rule see e.g. [20]).

Following [9] we say that two ergodic automorphisms T and S are *disjoint* if $J(T,S)=\{\mu\otimes\nu\}$. Recall that $J^{\mathrm{e}}(T,S)=\{\mu\otimes\nu\}$ implies disjointness of T and S. Given a class $\mathcal{R}$ of ergodic automorphisms, we denote by $\mathcal{R}^\perp$ the class of all ergodic automorphisms disjoint from any member of $\mathcal{R}$. Then by a *multiplier* (see [12]) of $\mathcal{R}^\perp$ we mean an ergodic automorphism each of whose ergodic joinings with an automorphism belonging to $\mathcal{R}^\perp$ gives rise to another member of $\mathcal{R}^\perp$. The class of multipliers of $\mathcal{R}^\perp$ is then denoted by $\mathcal{M}(\mathcal{R}^\perp)$.

In what follows, we will need the following.

PROPOSITION 1 ([1]). *Let T be an ergodic automorphism of $(X,\mathcal{B},\mu)$. If $\varrho\in J_2^{\mathrm{e}}(T)$ and also $\varrho\circ\varrho\in J_2^{\mathrm{e}}(T)$ then $(T\times T,\varrho)$ is relatively weakly mixing over the two marginal factors $\mathcal{B}\otimes\{\emptyset,X\}$ and $\{\emptyset,X\}\otimes\mathcal{B}$.* ■

Assume that T is weakly mixing and $\varrho\in J_2^{\mathrm{e}}(T)$. Then directly from the chain rule for the relative weak mixing property we obtain the following.

(1) If $(T\times T,\varrho)$ is relatively weakly mixing over the marginal factors, then $(T\times T,\varrho)$ is weakly mixing.

We will also need the following simple lemma.

LEMMA 2. *Assume that T is a weakly mixing automorphism of a standard probability Borel space $(X,\mathcal{B},\mu)$. Assume that $\mathbb{N}_0\subset\mathbb{N}$ and the density of $\mathbb{N}\setminus\mathbb{N}_0$ equals zero. Assume moreover that for each $f,g\in L^2(X,\mathcal{B},\mu)$,*

$$\langle f\circ T^n,g\rangle\to\langle f,1\rangle\langle 1,g\rangle$$

as $n\to\infty$, $n\in\mathbb{N}_0$. Assume that $\varrho\in J(T)$. Then for all $f,g,h\in L^\infty(X,\mathcal{B},\mu)$,

$$\int\limits_{X\times X} f(T^nx)g(x)h(T^ny)\,d\varrho(x,y)\to\int\limits_X f(x)h(y)\,d\varrho(x,y)\int\limits_X g(x)\,d\mu(x).$$

Proof. We have

$$\begin{aligned}\int\limits_{X\times X} f(T^nx)g(x)h(T^ny)\,d\varrho(x,y)&=\int\limits_X f(T^nx)g(x)\Phi_\varrho^*(h\circ T^n)(x)\,d\mu(x)\\&=\int\limits_X (f\cdot\Phi_\varrho^*(h))\circ T^n\cdot g\,d\mu\\&\to\int\limits_X f\cdot\Phi_\varrho^*(h)\,d\mu\int\limits_X g\,d\mu=\int\limits_{X\times X} f\otimes h\,d\varrho\int\limits_X g\,d\mu.\ \blacksquare\end{aligned}$$

For more information on joinings we refer the reader to the monograph by E. Glasner [13]. For the spectral theory of dynamical systems see e.g. [3], [33].

1.2. *Sub-joinings and sub-Markov operators in infinite measure-preserving case.* Given two automorphisms T and S acting on σ-finite standard Borel spaces $(X,\mathcal{B},\mu)$ and $(Y,\mathcal{C},\nu)$ respectively, by a *sub-joining* of T and S we mean each positive σ-finite $T\times S$-invariant measure ϱ on $(X\times Y,\mathcal{B}\otimes\mathcal{C})$ whose marginals ϱ_X and ϱ_Y satisfy $\varrho_X\le\mu$ and $\varrho_Y\le\nu$. By the formula

$$\int\limits_{X\times Y} f(x)g(y)\,d\varrho(x,y)=\int\limits_Y V(f)\cdot g\,d\nu,$$

there is a one-to-one correspondence between the set of sub-joinings and the set of sub-Markov operators $V:L^2(X,\mathcal{B},\mu)\to L^2(Y,\mathcal{C},\nu)$ intertwining U_T and U_S, where by a *sub-Markov operator* we mean a positive operator $V:L^2(X,\mathcal{B},\mu)\to L^2(Y,\mathcal{C},\nu)$ such that $Vf\le 1$ for all $f\in L^2(X,\mathcal{B},\mu)$ satisfying $0\le f\le 1$, and $V^*g\le 1$ for all $g\in L^2(Y,\mathcal{C},\nu)$ satisfying $0\le g\le 1$.

REMARK 1. Note that even in the case $T=S$, although the off-diagonal measures μ_{T^n} have the property that their marginals are equal to μ (equivalently, $\int_X T^n(\mathbf{1}_A)\,d\mu=\mu(A)$ for each $A\subset X$ of finite measure), the fact that the constant function $\mathbf{1}_X$ is not integrable may cause that the marginals of a weak limit ϱ of a sequence of off-diagonal measures need not be equal to μ (nevertheless, we will have $\varrho_X\le\mu$).

1.3. *Cocycles and compact group extensions.* Assume that T is an ergodic automorphism of a standard probability Borel space $(X,\mathcal{B},\mu)$. Let G be a compact metric group with the σ-algebra $\mathcal{B}(G)$ of Borel sets and Haar measure m_G. Let $\varphi : X \to G$ be a measurable map. It generates a cocycle $\varphi^{(\cdot)}(\cdot) : \mathbb{Z} \times X \to G$ by the formula

$$\varphi^{(n)}(x) = \begin{cases} \varphi(T^{n-1}x)\cdot\varphi(T^{n-2}x)\cdot\ldots\cdot\varphi(x) & \text{if } n>0,\\ 1 & \text{if } n=0,\\ (\varphi(T^{-1}x)\cdot\ldots\cdot\varphi(T^{n}x))^{-1} & \text{if } n<0. \end{cases}$$

We denote by T_φ the skew product automorphism defined on $(X\times G, \mathcal{B}\otimes\mathcal{B}(G), \mu\otimes m_G)$ by the formula

$$T_\varphi(x,g) = (Tx, \varphi(x)\cdot g).$$

We call T_φ a *compact group extension* of T.

Denote by τ_g the map on $X\times G$ given by $\tau_g(x,g_1) = (x, g_1g^{-1})$. Note that $\tau_g \in C(T_\varphi)$ for each $g\in G$.

Compact group extensions have the so called *relative unique ergodicity* (RUE) property: whenever the product measure $\mu\otimes m_G$ is ergodic, it is the only T_φ-invariant measure of $(X\times G,\mathcal{B}\otimes\mathcal{B}(G))$ whose projection on X equals μ (see e.g. [10]).

We say that a cocycle $\varphi : X\to G$ is *ergodic* if T_φ considered with $\mu\otimes m_G$ is ergodic. In this case ergodic self-joinings of T_φ whose projections on $X\times X$ are Δ_X are necessarily graph joinings corresponding to τ_g, $g\in G$ (see [21]).

1.4. *Gaussian automorphisms.* An ergodic automorphism T of a standard probability Borel space $(X,\mathcal{B},\mu)$ is called *Gaussian* if there exists a U_T-invariant subspace $H\subset L^2(X,\mathcal{B},\mu)$ of real-valued functions generating $\mathcal{B}$ and such that each non-zero variable from H has a Gaussian distribution. For a joining theory of Gaussian automorphisms we refer the reader to [29] (see also [3] for a general theory of Gaussian automorphisms). In particular, it is proved in [29] that there is a special subset $J_2^{\mathrm{g}}(T)\subset J_2^{\mathrm{e}}(T)$ called the set of *Gaussian self-joinings* (for $\varrho\in J_2^{\mathrm{g}}(T)$, $(T\times T,\varrho)$ remains a Gaussian automorphism). Roughly speaking, this set corresponds to all contractions of the first chaos H intertwining the unitary action of T on H (all off-diagonal self-joinings μ_{T^n} are in $J_2^{\mathrm{g}}(T)$). It follows that $J_2^{\mathrm{g}}(T)$ is closed in the weak topology of joinings.

A Gaussian automorphism T is entirely determined by the spectral measure σ of U_T on $H^{(\mathrm{c})} = H+iH$. Moreover, T is ergodic iff σ is continuous. The maximal spectral type of T is the sum of consecutive convolutions $\sigma^{(n)}$ of σ, in particular ergodicity implies weak mixing for Gaussian automorphisms.

Each variable $f \in H$, viewed as a map $f : X \to \mathbb{R}$, is called a *Gaussian cocycle*. It is called a *Gaussian coboundary* if $f = g - g \circ T$ for some $g \in H$. The subspace H consists entirely of Gaussian coboundaries iff 1 is not in the topological support of σ ([27]). We refer the reader to [27] for more information about ergodicity of circle group extensions of the form $T_{e^{2\pi i f}}$, where f is a Gaussian cocycle.

1.5. *Integral automorphisms and special flows.* Let T be an ergodic automorphism of a standard probability Borel space $(X, \mathcal{B}, \mu)$. Assume that $f : X \to \mathbb{N}$ is a measurable function with finite integral. Let $X_f \subset X \times \mathbb{N}$ be given by $\bigcup_{n\in\mathbb{N}} X_n \times \{n\}$, where $X_n = \{x \in X : f(x) \le n\}$. Let $\mathcal{B}_f$ denote the restriction of the product σ-algebra of $\mathcal{B}$ and the σ-algebra of all subsets of $\mathbb{N}$ to the set X_f. Let μ_f denote the restriction of the product measure $\mu \otimes \sum_{n\in\mathbb{N}} \delta_{\{n\}}$ to X_f. By the *integral transformation* built over the automorphism T and under the function f we mean the transformation $T_f : (X_f, \mathcal{B}_f, \mu_f) \to (X_f, \mathcal{B}_f, \mu_f)$ defined by

$$T_f(x,k) = \begin{cases} (x, k+1) & \text{if } f(x) < k, \\ (Tx, 1) & \text{if } f(x) = k. \end{cases}$$

Suppose that $A \in \mathcal{B}$ has positive measure. It is easy to check that $(T_A)_{\tau_A}$ and T are metrically isomorphic, where $T_A : A \to A$ is the induced automorphism and $\tau : A \to \mathbb{N}$ stands for the first return time function (see [3, Chapter 1]).

Denote by $m_{\mathbb{R}}$ the Lebesgue measure on $\mathbb{R}$. Assume that $f : X \to \mathbb{R}$ is a measurable positive function such that $\int_X f\,d\mu = 1$. The *special flow* $T^f = \{(T^f)_t\}_{t\in\mathbb{R}}$ built from T and f is defined on the space $X^f = \{(x,t) \in X \times \mathbb{R} : 0 \le t < f(x)\}$ (considered with $\mathcal{B}^f$, the restriction of the product σ-algebra, and μ^f, the restriction of the product measure $\mu \otimes m_{\mathbb{R}}$ of $X \times \mathbb{R}$). Under the action of the special flow each point in X^f moves vertically at unit speed, and we identify the point $(x, f(x))$ with $(Tx, 0)$ (see e.g. [3, Chapter 11]). In the special case where $f \equiv 1$ the special flow T^f acts on $X \times [0,1)$ and is called the *suspension flow* for the automorphism T. Then we write $\widehat{T}$ instead of T^f and $(\widehat{X}, \widehat{\mathcal{B}}, \widehat{\mu})$ instead of $(X^f, \mathcal{B}^f, \mu^f)$. Let $\pi : \widehat{X} = X \times [0,1) \to X$ denote the natural projection. Then the σ-algebra $\pi^{-1}(\mathcal{B}) \subset \widehat{\mathcal{B}}$ is $(\widehat{T})_1$-invariant and $\pi : (\widehat{X}, \pi^{-1}(\mathcal{B}), \widehat{\mu}) \to (X, \mathcal{B}, \mu)$ establishes an isomorphism between automorphisms $(\widehat{T})_1$ of $(\widehat{X}, \pi^{-1}(\mathcal{B}), \widehat{\mu})$ and T of $(X, \mathcal{B}, \mu)$. Finally, notice that the flows $\widehat{T_f}$ and T^f are isomorphic whenever $f : X \to \mathbb{N}$.

LEMMA 3. *Let T be an ergodic automorphism of $(X, \mathcal{B}, \mu)$ and let $f : X \to \mathbb{N}$ be a measurable function with finite integral. Suppose that (a_n) is a sequence of integers such that $(T^f)_{a_n} \to p((T^f)_1)$ weakly, where p is a trigonometric polynomial. Then $T_f^{a_n} \to p(T_f)$ weakly.*

Proof. Since the operators $(T^f)_1$ acting on $L^2(X^f, \mathcal{B}^f, \mu^f)$ and $(\widehat{T}_f)_1$ acting on $L^2(\widehat{X}_f, \widehat{\mathcal{B}}_f, \widehat{\mu}_f)$ are unitarily isomorphic,

$$(\widehat{T}_f)_{a_n} \to p((\widehat{T}_f)_1)$$

in the weak operator topology on $L^2(\widehat{X}_f, \widehat{\mathcal{B}}_f, \widehat{\mu}_f)$. Let $\pi : \widehat{X}_f = X_f \times [0,1) \to X_f$ be the natural projection. Since $L^2(\widehat{X}_f, \pi^{-1}(\mathcal{B}_f), \widehat{\mu}_f) \subset L^2(\widehat{X}_f, \widehat{\mathcal{B}}_f, \widehat{\mu}_f)$ is an invariant subspace with respect to the operators $(\widehat{T}_f)_{a_n}$ $(n \in \mathbb{N})$, $(\widehat{T}_f)_{a_n} \to p((\widehat{T}_f)_1)$ in the weak operator topology on $L^2(\widehat{X}_f, \pi^{-1}(\mathcal{B}_f), \widehat{\mu}_f)$. Since the operators T_f on $L^2(X_f, \mathcal{B}_f, \mu_f)$ and $(\widehat{T}_f)_1$ on $L^2(\widehat{X}_f, \pi^{-1}(\mathcal{B}_f), \widehat{\mu}_f)$ are unitarily isomorphic, $T_f^{a_n} \to p(T_f)$ in the weak operator topology on $L^2(X_f, \mathcal{B}_f, \mu_f)$. ■

2. Basic properties of ELF automorphisms. An ergodic automorphism T of a standard Borel space $(X, \mathcal{B}, \mu)$ is said to have the *ELF property* if $\overline{\{\mu_{T^n} : n \in \mathbb{Z}\}} \subset J_2^{\mathrm{e}}(T)$, or equivalently, the weak closure of the set of Markov operators $\{T^n : n \in \mathbb{Z}\}$ consists of indecomposable Markov operators. For short, we will speak about *ELF automorphisms*.

It is clear that ergodic discrete spectrum automorphisms and mixing automorphisms are examples of ELF automorphisms. By what was said in Section 1.4, Gaussian automorphisms also enjoy the ELF property (see [6] for a direct proof of that fact).

The following two consequences of Proposition 1 have already been noticed in [6].

PROPOSITION 4 ([6]). *If T is an ELF automorphism and if $\varrho \in \overline{\{\mu_{T^n} : n \in \mathbb{Z}\}}$ then $(T \times T, \varrho)$ is relatively weakly mixing with respect to the two natural marginal σ-algebras.* ■

PROPOSITION 5 ([6]). *Assume that T is an ELF automorphism and let $\varrho \in \overline{\{\mu_{T^n} : n \in \mathbb{Z}\}}$. Let S be an ergodic automorphism on $(Y, \mathcal{C}, \nu)$. Assume that ϱ_1 is an ergodic joining of T and S. Then $\varrho_1 \circ \varrho$ is still ergodic.* ■

2.1. *Disjointness of ELF automorphisms from time-one maps of some measurable flows.* Proposition 5, similarly to [6], allows us to prove disjointness of the class of ELF automorphisms from automorphisms having a piece of integral Markov operator in the weak closure of its powers. Indeed, assume that S is an automorphism of $(Y, \mathcal{C}, \nu)$. Let P be a probability measure defined on the Borel σ-algebra of $C(S)$. We define a Markov operator M_P on $L^2(Y, \mathcal{C}, \nu)$ by putting

$$M_P(f) = \int_{C(S)} f \circ R \, dP(R).$$

The integral on the right hand side is meant in the weak sense, i.e. for each $f, g \in L^2(Y, \mathcal{C}, \nu)$,

$$\Big\langle \int_{C(S)} f \circ R \, dP(R), g \Big\rangle = \int_{C(S)} \langle f \circ R, g\rangle \, dP(R).$$

In order to see that this definition is correct we define

$$\langle\langle f, g\rangle\rangle = \int_{C(S)} \langle f \circ R, g\rangle \, dP(R)$$

and check that we have obtained a bilinear form on $L^2(Y, \mathcal{C}, \nu)$ which, by the Schwarz inequality, is bounded. Clearly, $M_P \in \mathcal{J}_2(S)$.

Proposition 6. *Let $S : (Y, \mathcal{C}, \nu) \to (Y, \mathcal{C}, \nu)$ be an ergodic automorphism. Assume that there exist an increasing sequence (t_n) of natural numbers and a probability Borel measure P on $C(S)$ such that*

$$S^{t_n} \to a \int_{C(S)} R \, dP(R) + (1-a)\Phi$$

in the weak operator topology on $B(L^2(Y, \mathcal{C}, \nu))$, where $a > 0$ and $\Phi \in \mathcal{J}_2(S)$. Assume that $P(\{R \in C(S) : R$ is weakly mixing$\}) > 0$. Then S is weakly mixing. If moreover P is not Dirac and either

(i) *P is concentrated on $\{S^i : i \in \mathbb{Z}\}$, or*
(ii) *P is concentrated on $\{S_t : t \in \mathbb{R}\}$, where $S_1 = S$ (i.e. we assume in particular that S is embeddable in a measurable flow),*

then S is disjoint from all ELF automorphisms.

Proof. First, let us show that S is weakly mixing. Indeed, if f is its eigenfunction then

$$\|f\|^2_{L^2} = |\langle S^{t_n} f, f\rangle| \to \Big|a \int_{C(S)} \langle f \circ R, f\rangle \, dP(R) + (1-a)\langle \Phi(f), f\rangle\Big|.$$

Since $|\langle f \circ R, f\rangle| \le \|f\|^2$ and $|\langle \Phi(f), f\rangle| \le \|f\|^2$, a convexity argument shows that we must have $\langle f \circ R, f\rangle = \|f\|^2$ for P-a.e. $R \in C(S)$ (and also $\langle \Phi(f), f\rangle = \|f\|^2$ provided $a < 1$). So for such an R, we have $f \circ R = c(R)f$ ($c(R) \in \mathbb{C}$), and since R may be taken weakly mixing, f is constant.

Let T be an ELF automorphism on $(X, \mathcal{B}, \mu)$. Let $\Psi : L^2(Y, \mathcal{C}, \nu) \to L^2(X, \mathcal{B}, \mu)$ be an indecomposable Markov operator intertwining S and T. Then $\Psi \circ S^{t_n} = T^{t_n} \circ \Psi$ and by passing to a subsequence of (t_n) if necessary, we find

$$\Psi \circ (aM_P + (1-a)\Phi) = \Phi_\varrho \circ \Psi,$$

where $\varrho = \lim_{n\to\infty} \mu_{T^{t_n}}$. In view of Proposition 5, $\Phi_\varrho \circ \Psi$ remains indecomposable. On the other hand,

$$\Psi \circ (aM_P + (1-a)\Phi) = a \int_{C(S)} \Psi \circ R\, dP(R) + (1-a)\Psi \circ \Phi,$$

and hence we must have $\Psi \circ R = \Phi_\varrho \circ \Psi$ for P-a.e. $R \in C(S)$. This means that for a set of full $P \otimes P$-measure of $(R_1, R_2) \in C(S) \times C(S)$, we have $R_2 \circ R_1^{-1} \circ \Psi^* = \Psi^*$. Notice however that both assumptions (i) and (ii) and the fact that P is not Dirac imply that for some weakly mixing element $R \in C(S)$ we have $R \circ \Psi^* = \Psi^*$ and therefore $\Psi = \Pi_{Y,X}$. ■

Suppose now that $(S_t)_{t\in\mathbb{R}}$ is a measurable, weakly mixing flow acting on $(Y, \mathcal{C}, \nu)$. Suppose that for a sequence (r_n) of real numbers with $r_n \to \infty$ we have

$$S_{r_n} \to a \int_{\mathbb{R}} S_t\, dQ(t) + (1-a)\Phi, \tag{2}$$

where Q is not Dirac. By passing to a subsequence if necessary we can assume that the sequence $(\{r_n\})$ of fractional parts of r_n converges to $0 \le b \le 1$. Since the flow is measurable, $S_{\{r_n\}} \to S_b$ in the strong operator topology. It follows that the sequence $(S_1)^{[r_n]} = S_{r_n} \circ S_{-\{r_n\}}$ converges in the weak operator topology and we have

$$(S_1)^{[r_n]} \to a \int_{\mathbb{R}} S_{t-b}\, dQ(t) + (1-a)\Phi \circ S_{-b}.$$

We have proved the following.

Corollary 7. *Assume that $(S_t)_{t\in\mathbb{R}}$ is a measurable, weakly mixing flow for which* (2) *holds with Q which is not Dirac. Then the time-one map S_1 is disjoint from all ELF automorphisms.* ■

Remark 2. The assumptions of Corollary 7 are satisfied for time-one maps of some classical examples of special flows over irrational rotations and over interval exchange transformations (see [6]–[8]) and in particular it is satisfied for some smooth flows on surfaces (see [8]).

2.2. *Factors and direct products of ELF automorphisms.* The following proposition shows that the class of ELF automorphisms is closed under some basic operations.

Proposition 8. *The class of ELF automorphisms is closed under factors and inverse limits. The direct product of weakly mixing ELF automorphisms remains an ELF automorphism.*

Proof. Closedness under taking factors and inverse limits is obvious.

Assume that T_i is a weakly mixing ELF automorphism of $(X_i, \mathcal{B}_i, \mu_i)$, $i \ge 1$. Consider now $T = T_1 \times T_2 \times \cdots$ acting on $(X_1 \times X_2 \times \cdots, \mu_1 \otimes \mu_2 \otimes \cdots)$.

Suppose that $T^{n_i} \to \Phi_\varrho$ for some $\varrho \in J_2(T)$. By applying the diagonalizing procedure if necessary, we can assume that for each $j \geq 1$, $T_j^{n_i} \to \Phi_{\varrho_j}$ for some $\varrho_j \in J_2^{\mathrm{e}}(T_j)$. It easily follows that $\varrho = \varrho_1 \otimes \varrho_2 \otimes \cdots$ and because of (1), ϱ is ergodic, which completes the proof. ∎

REMARK 3. Note however that an ergodic self-joining of an ELF automorphism need not be an ELF automorphism. Indeed, even if T is mixing then an ergodic self-joining need not give rise to an ELF automorphism. For example, by Smorodinsky–Thouvenot's result from [45] it follows that given an ergodic zero entropy automorphism S and a Bernoulli automorphism T we can find an ergodic self-joining ϱ of T such that $(T \times T, \varrho)$ has S as its factor.

2.3. *Lifting the ELF property to compact group extensions.* We will now discuss the possibility of lifting the ELF property by a compact group extension. So assume that T is an ELF automorphism and let $\varphi : X \to G$ be a cocycle, where G is a compact metric group. Recall first that if T is mixing and the extension T_φ is weakly mixing then T_φ is in fact mixing (see [37]). A look at a short joining proof (due to A. del Junco) of that fact gives rise to a criterion of lifting the ELF property.

PROPOSITION 9. *Assume that T has the ELF property and $\varphi : X \to G$ is ergodic. Assume moreover that for each $\varrho \in \overline{\{\mu_{T^n} : n \in \mathbb{Z}\}}$ the cocycle $\varphi \times \varphi$ over $(T \times T, \varrho)$ is ergodic. Then T_φ has the ELF property.*

Proof. Assume that $(T_\varphi)^{m_i} \to \Phi_{\widetilde{\varrho}}$. We must show that $\widetilde{\varrho}$ is ergodic. We can assume that $m_i \to \infty$, otherwise the result is clear. We then have $T^{m_i} \to \Phi_\varrho$, where ϱ is the projection of $\widetilde{\varrho}$ on $X \times X$. Now, $\widetilde{\varrho}$ is a $T_\varphi \times T_\varphi$-invariant measure whose projection is ϱ. However, by our standing assumption the measure $\varrho \otimes m_G \otimes m_G$ has the same property and it is ergodic. The result now follows from the relative unique ergodicity property for compact group extensions. ∎

The above proof suggests that in general we have no chance to lift the ELF property and in fact we will loose this property when the base has discrete spectrum.

PROPOSITION 10. *An ergodic isometric extension $\widehat{T}$ of a discrete spectrum automorphism T has the ELF property iff the extension also has discrete spectrum.*

Proof. We can assume that T is an ergodic rotation $(Tx = x + x_0)$ of a compact metric monothetic group X. Moreover assume that $\varphi : X \to G$ is an ergodic cocycle for which T is the Kronecker factor and $\widehat{T}$ is the quotient action of T_φ on $X \times G/H$. All we need to show is that under all these assumptions $\widehat{T}$ does not have the ELF property.

To this end first choose a sequence (n_i) of density 1 such that

$$(3) \qquad U_{\widehat{T}}^{n_i} \to 0 \quad \text{weakly on } L^2(X \times G/H, m_X \otimes m_{G/H}) \ominus L^2(X, m_X),$$

which is possible because T is the Kronecker factor of $\widehat{T}$ and therefore the spectral type of $U_{\widehat{T}}$ on $L^2(X \times G/H, m_X \otimes m_{G/H}) \ominus L^2(X, m_X)$ is continuous. Since the density of (n_i) is 1, there exists a subsequence (m_i) of (n_i) such that $T^{m_i} \to \mathrm{Id}$. Indeed, given a neighbourhood $W \ni 0$ in X, by the pointwise ergodic theorem for strictly ergodic systems the average time of visiting W by the orbit of an arbitrary point of X is equal to $m_X(W)$, hence positive. Therefore we can find $n_j = n_j(W)$ so that $n_j x_0 \in W$. Letting $W \to \{0\}$ proves the claim.

It follows from (3) that $\widehat{T}^{m_i}$ converges weakly to the operator $E(\cdot \mid X)$ which corresponds to the joining $\Delta_X \otimes m_{G/H} \otimes m_{G/H}$. However, this last joining is not ergodic: the function $F(x, gH, x, g'H) = g^{-1}g'H$ is not constant but it is $\widehat{T} \times \widehat{T}$-invariant $\Delta_X \otimes m_{G/H} \otimes m_{G/H}$-a.e. Therefore, $\widehat{T}$ does not have the ELF property and the result follows. ∎

The following corollary follows directly from Proposition 10.

COROLLARY 11. *If an extension of a rotation T has the ELF property, then the extension is relatively weakly mixing over T.* ∎

REMARK 4. In [52] there are explicit constructions of ELF automorphisms which are relatively weakly mixing extensions of some irrational rotations.

Let us now show however that the criterion of Proposition 9 may work in some cases of mildly mixing ELF automorphisms which are not mixing.

We consider symmetric probability measures σ on $\mathbb{T}$ such that

(4) all weak closure points of the sequence $\{z^n : n \in \mathbb{Z}\}$ in $L^2(\mathbb{T}, \sigma)$ are in the set $\{az^n : |a| < 1, n \in \mathbb{Z}\}$.

Since σ is a symmetric measure, the numbers a in (4) have to be real. Under the above assumption, the Gaussian automorphism associated to σ has to be mildly mixing. Recall that classical Riesz products yield examples of such measures, including examples which are not Rajchman measures so that the set of weak closure points is not trivial (see [16, Ch. II, Sect. 7]).

PROPOSITION 12. *Assume that T is a mildly mixing Gaussian automorphism determined by a measure satisfying* (4) *for which a certain $a \neq 0$ is in the weak closure of characters* (*such a T is not mixing*). *Take f from the first real chaos. Assume that f is not a Gaussian coboundary. Then $\overline{T} := T_{e^{2\pi i f}}$ is still an ELF automorphism.*

Proof. Assume that

$$(T_{e^{2\pi i f}})^{n_t} \to \Phi_{\widetilde{\varrho}} \tag{5}$$

for some sequence (n_t) with $n_t \to \infty$. Then $T^{n_t} \to \Phi_\varrho$, where ϱ is the projection of $\widetilde{\varrho}$ on $X \times X$. Without loss of generality we can assume that $z^{n_t} \to a$ in the weak topology of $L^2(\mathbb{T}, \sigma)$ for some real a with $|a| < 1$. We have to prove that $\widetilde{\varrho} \in J_2(T_{e^{2\pi i f}})$ is ergodic. If now ϱ is the product measure then so is $\widetilde{\varrho}$, since $T_{e^{2\pi i f}}$ is weakly mixing ([27]) and we may apply the relative unique ergodicity property for compact group extensions to conclude. Note that ϱ cannot be a graph measure, since T is assumed to be mildly mixing. Moreover, since $z^{n_t} \to a$ in the weak topology of $L^2(\mathbb{T}, \sigma)$, T^{n_t} restricted to the first real chaos tends to multiplication by a, and hence Φ_ϱ is multiplication by a on the first real chaos. By Proposition 9 all we need to show is that $T_{e^{2\pi i f}} \times T_{e^{2\pi i f}}$ is ergodic as a $\mathbb{T} \times \mathbb{T}$-extension of $(T \times T, \varrho)$. Following Proposition 6 in [27] it is sufficient to show that the cocycle $lf(x) + mf(y)$ (with $(l, m) \neq (0, 0)$) is not a Gaussian coboundary (for the Gaussian automorphism $(T \times T, \varrho)$). If for each $r \in \mathbb{N}$ we put $f^{(r)}(x) = f(x) + f(Tx) + \cdots + f(T^{r-1}x)$ then we have

$$\begin{aligned}
\|lf^{(r)}(x) + mf^{(r)}(y)\|^2_{L^2(\varrho)} &= (l^2 + m^2)\|f^{(r)}\|^2 + 2lm\langle f^{(r)}(x), f^{(r)}(y)\rangle_{L^2(\varrho)} \\
&= (l^2 + m^2)\|f^{(r)}\|^2 + 2lm \int_X (\Phi_\varrho f^{(r)})(y) f^{(r)}(y)\, d\mu(y) \\
&= (l^2 + m^2 + 2ml \cdot a)\|f^{(r)}\|^2.
\end{aligned}$$

Now, f is not a Gaussian coboundary, so $\|f^{(r_t)}\| \to \infty$ along a subsequence (r_t) (see [27]) and since $|a| < 1$,

$$(l^2 + m^2 + 2ml \cdot a)\|f^{(r_t)}\| \to \infty$$

or equivalently

$$\|lf^{(r_t)}(x) + mf^{(r_t)}(y)\|_{L^2(\varrho)} \to \infty,$$

which means (see [27]) that indeed $lf(x) + mf(y)$ is not a Gaussian coboundary. ∎

3. Poisson automorphisms have the ELF property. In this section we will define and study a special class of self-joinings for the class of automorphisms obtained by Poisson suspension of infinite measure-preserving maps.

3.1. *Poisson suspension automorphisms.* Assume that T is an automorphism of a standard Borel space $(X, \mathcal{B}, \mu)$, where μ is σ-finite. We denote by $\widetilde{T}$ the Poisson suspension automorphism acting on $(\widetilde{X}, \widetilde{\mathcal{B}}, \widetilde{\mu})$. The points of $\widetilde{X}$ are infinite countable subsets $\widetilde{x} = \{x_n : n \geq 1\}$. Given a set $A \in \mathcal{B}$ of finite measure we define $N_A : \widetilde{X} \to \mathbb{N}$ by putting

$$N_A(\widetilde{x}) = \#\{n \in \mathbb{N} : x_n \in A\}.$$

Then we define $\widetilde{\mathcal{B}}$ as the smallest σ-algebra of subsets of $\widetilde{X}$ for which all variables N_A, $\mu(A) < \infty$, are measurable. The measure $\widetilde{\mu}$ is defined by the requirement that the variables N_A satisfy the Poisson law with parameter $\mu(A)$ and moreover that for each family of pairwise disjoint subsets of X of finite measure the corresponding variables are independent (see [24] for details). Finally, we let $\widetilde{T}$ act by the formula $\widetilde{T}(\{x_n\}) = (\{Tx_n\})$ to obtain an automorphism of $(\widetilde{X}, \widetilde{\mathcal{B}}, \widetilde{\mu})$. The space $L^2(\widetilde{X}, \widetilde{\mathcal{B}}, \widetilde{\mu})$ admits a decomposition into invariant chaos $\bigoplus_{n\geq 0} H^{(n)}$, where $H^{(0)}$ is the subspace of constants, $H = H_X = H^{(1)}$ is the subspace generated by the centred variables $N_A^0 = N_A - \mu(A)$ and $H^{(n)}$ is the orthocomplement of the sum of chaos $H^{(i)}$, $0 \leq i \leq n-1$, in the subspace generated by the products of n variables of the form N_A (see [32]). The map $\mathbf{1}_A \mapsto N_A^0$ can be extended to an isometry I of $L^2(X, \mathcal{B}, \mu)$ onto H and it conjugates U_T with $U_{\widetilde{T}}|_H$. Moreover we obtain a natural isometry between $H^{(n)}$ and the nth symmetric tensor product $H^{\odot n}$ of H under which $\odot_{i=1}^n N_{A_i}^0$ corresponds to the projection of $\prod_{i=1}^n N_{A_i}$ in $H^{(n)}$.

The operator $U_{\widetilde{T}}$ preserves the chaos and, for each $n \geq 0$, its restriction to $H^{(n)}$ corresponds to $(U_{\widetilde{T}}|_H)^{\odot n}$ by this natural isometry. In such a case, we will say that an operator *acts well on the chaos*.

If $0 \neq f \in L^2(X, \mathcal{B}, \mu)$ is an eigenfunction of U_T corresponding to c (with $|c| = 1$), then $\overline{f}$ is an eigenfunction of U_T corresponding to $\overline{c}$. Then $I(f) \odot I(\overline{f}) \in H^{(2)}$ and it is a $U_{\widetilde{T}}$-invariant function. Furthermore, if σ denotes the maximal spectral type of $U_{\widetilde{T}}$ on H (which is equal to the maximal spectral type of U_T on $L^2(X, \mathcal{B}, \mu)$) then the maximal spectral type of $U_{\widetilde{T}}$ on the nth chaos is equal to the nth convolution $\sigma^{(n)} = \sigma * \cdots * \sigma$. Recall that $\sigma^{(n)}$ is continuous iff σ is continuous.

Therefore the Poisson suspension automorphism $\widetilde{T}$ on $(\widetilde{X}, \widetilde{\mathcal{B}}, \widetilde{\mu})$ is ergodic if and only if the spectral type of T on $L^2(X, \mathcal{B}, \mu)$ is continuous; equivalently, iff there are no T-invariant subsets of X of finite positive measure or else iff $L^2(X, \mathcal{B}, \mu)$ does not contain non-zero T-invariant functions. In this case $\widetilde{T}$ is weakly mixing. Finally, note that, in particular, if a Poisson suspension automorphism is ergodic then necessarily the measure μ is infinite.

3.2. *Factors and Poisson joinings.* If X_1 is a T-invariant subset of X, then $\widetilde{T}$ is the direct product of two Poisson suspensions of T acting on X_1 and on $X \setminus X_1$, in particular, $\widetilde{T|_{X_1}}$ is a factor of $\widetilde{T}$. Assume now that S acting on another σ-finite standard Borel space $(Y, \mathcal{C}, \nu)$ is a factor of $(X_1, \mu|_{X_1}, T|_{X_1})$ in the sense that there is a measurable map $F : X_1 \to Y$ such $F_*(\mu|_{X_1}) = \nu$

and $F \circ T = S \circ F$ on X_1. Then $\widetilde{S}$ acting on $(\widetilde{Y}, \widetilde{\mathcal{C}}, \widetilde{\nu})$ is a factor of $\widetilde{T|_{X_1}}$ via the map $\widetilde{F} : \widetilde{X}_1 \to \widetilde{Y}$ given by $\widetilde{F}(\{x_n\}) = \{F(x_n)\}$.

Then the associated operator $V_{\widetilde{F}} : L^2(\widetilde{Y}, \widetilde{\nu}) \to L^2(\widetilde{X}_1, \widetilde{\mu|_{X_1}})$, $g \mapsto g \circ \widetilde{F}$, acts well on the chaos.

By *Poisson factors* of $\widetilde{T}$ we will mean factors $\widetilde{S}$ obtained as above. We will also say that the map F is a partial map of X to Y *semi-conjugating* T and S. Note that if $F : X_1 \to Y$ establishes a semi-conjugation of T and S then the associated isometry V_F is a sub-Markov operator from $L^2(Y, \nu)$ to $L^2(X_1, \mu|_{X_1})$.

Assume that T and S are automorphisms of σ-finite spaces $(X, \mathcal{B}, \mu)$ and $(Y, \mathcal{C}, \nu)$ respectively. A joining η of $\widetilde{T}$ and $\widetilde{S}$ is called a *Poisson joining* if the associated Markov operator $\widetilde{V} = \Phi_\eta$ acts well on the chaos and, via the natural isomorphisms of the first chaos H_X of $(\widetilde{X}, \widetilde{\mathcal{B}}, \widetilde{\mu})$ and H_Y of $(\widetilde{Y}, \widetilde{\mathcal{C}}, \widetilde{\nu})$ with $L^2(X, \mathcal{B}, \mu)$ and $L^2(Y, \mathcal{C}, \nu)$, the operator $\widetilde{V}|_{H_X}$ corresponds to the sub-Markov operator V associated to a sub-joining of T and S.

PROPOSITION 13. *The class of Poisson joinings between $\widetilde{T}$ and $\widetilde{S}$ is closed in the weak topology of joinings, in particular, the class of Poisson self-joinings of $\widetilde{T}$ contains the weak closure of $\{\widetilde{T}^n : n \in \mathbb{Z}\}$.*

Moreover, the relative product of $\widetilde{T}$ over a Poisson factor is a Poisson self-joining.

Proof. The first part follows directly from the fact that the set of sub-Markov operators is closed in the weak operator topology.

To prove the second part take a Poisson factor which is determined by a partial function $F : X_1 \subset X \to Y$. Then the Markov operator corresponding to the relative product over this factor is given by $V_{\widetilde{F}} \circ V^*_{\widetilde{F}} \circ P_{L^2(\widetilde{X}_1)}$. Since $V_{\widetilde{F}}$, $V^*_{\widetilde{F}}$, $P_{L^2(\widetilde{X}_1)}$ act well on the chaos and their restrictions to the first chaos can naturally be identified with V_F, V^*_F, $L^2(X) \ni f \mapsto f|_{X_1} \in L^2(X_1)$ resp., so are sub-Markov operators, the composition $V_{\widetilde{F}} \circ V^*_{\widetilde{F}} \circ P_{L^2(\widetilde{X}_1)}$ is the associated Markov operator of a Poisson self-joining. ∎

3.3. *Ergodicity of Poisson joinings.* Assume that ϱ is a sub-joining of T and S and denote by V the corresponding sub-Markov operator from $L^2(X, \mathcal{B}, \mu)$ to $L^2(Y, \mathcal{C}, \nu)$. We will now pass to a construction of a Poisson joining η of $\widetilde{T}$ and $\widetilde{S}$ corresponding to ϱ, i.e. if we put $\widetilde{V} = \Phi_\eta$ then $\widetilde{V}|_H \equiv V$. This Poisson joining turns out to be unique, so the structure of Poisson joinings will be understood.

Set $\mu' = \mu - \varrho_X$ and $\nu' = \nu - \varrho_Y$. Let us define a σ-finite standard space (Z, ϱ') as a formal disjoint union of (X, μ'), (Y, ν') and of $(X \times Y, \varrho)$. Then we define R on (Z,ϱ') by putting $R|_X = T$, $R|_Y = S$ and $R|_{X\times Y} = T\times S$. Since ϱ_X and ϱ_Y are T- and S-invariant respectively, ϱ' is R-invariant.

We now have the partial mapping $F : X\cup(X\times Y)\subset Z\to X$ which to $x\in X$ or to $(x,y)\in X\times Y$ associates x, and for each $A\subset X$ of finite measure we have $\varrho'(F^{-1}(A)) = \mu'(A)+\varrho_X(A) = \mu(A)$. Clearly, $F\circ R = T\circ F$, so F establishes a semi-conjugation of R and T. The mapping $G : Y\cup(X\times Y)\subset Z\to Y$ which to $y\in Y$ or to $(x,y)\in X\times Y$ associates y has similar properties. Hence, the two maps $\widetilde{F} : \widetilde{Z}\to\widetilde{X}$ and $\widetilde{G} : \widetilde{Z}\to\widetilde{Y}$ are factor mappings of $\widetilde{R}$ to $\widetilde{T}$ and $\widetilde{S}$ respectively. It follows that $(\widetilde{F},\widetilde{G}) : \widetilde{Z}\to\widetilde{X}\times\widetilde{Y}$ defines a joining $\eta = (\widetilde{F},\widetilde{G})_*(\widetilde{\varrho'})$ of $\widetilde{T}$ and $\widetilde{S}$, that is, for each $f\in L^2(\widetilde{X},\widetilde{\mathcal{B}},\widetilde{\mu})$ and $g\in L^2(\widetilde{Y},\widetilde{\mathcal{C}},\widetilde{\nu})$ we have

$$\int f(\widetilde{x})g(\widetilde{y})\,d\eta(\widetilde{x},\widetilde{y}) = \int f\circ\widetilde{F}\cdot g\circ\widetilde{G}\,d\widetilde{\varrho'}.$$

It follows that the Markov operator associated to η is equal to $\widetilde{V} = V^*_{\widetilde{G}}V_{\widetilde{F}}$. Hence, $\widetilde{V}$ acts well on the chaos and clearly its restriction to H_X can be naturally identified with $V^*_GV_F$. Let us now show that $V^*_GV_F = V$. Indeed, take $A\subset X$ and $B\subset Y$ of finite measure. Notice that $\mathbf{1}_A\circ F\cdot\mathbf{1}_B\circ G$ equals zero outside of $X\times Y$, and on $X\times Y$ this function is equal to $\mathbf{1}_{A\times B}$. Therefore

$$\int V^*_GV_F\mathbf{1}_A\cdot\mathbf{1}_B\,d\nu = \int\mathbf{1}_A\circ F\cdot\mathbf{1}_B\circ G\,d\varrho' = \varrho(A\times B) = \int V\mathbf{1}_A\cdot\mathbf{1}_B\,d\nu,$$

whence $V^*_GV_F = V$.

Theorem 14. *Each Poisson joining of two ergodic Poisson suspension automorphisms remains ergodic. In particular, each ergodic Poisson suspension automorphism has the ELF property.*

Proof. Notice that the second assertion follows from the first one and Proposition 13.

Assume that $\widetilde{T}$ and $\widetilde{S}$ are ergodic. It follows that X and Y have no invariant sets of finite positive measure. Let us show that in Z there are no R-invariant sets of finite positive ϱ'-measure. Indeed, suppose that $h = \mathbf{1}_C\in L^2(Z,\varrho')$ is R-invariant. Then V^*_Fh is T-invariant, so equal to zero μ-a.e. In particular for each subset $A\subset X$ of finite measure we have

$$0 = \langle V^*_Fh,\mathbf{1}_A\rangle = \langle h, V_F(\mathbf{1}_A)\rangle = \int h\cdot(\mathbf{1}_A\circ F)\,d\varrho'$$

that is, $\varrho'(C\cap F^{-1}(A)) = 0$. Since $F^{-1}(A)$ is a formal disjoint union of A and $A\times Y$, $\varrho'(C\cap X) = 0$ and by a similar argument $\varrho'(C\cap Y) = 0$ together with $\varrho'(C\cap(A\times B)) = 0$ for each $A\subset X$, $B\subset Y$ of finite measure. Therefore $\varrho'(C) = 0$. It follows that the Poisson suspension $\widetilde{R}$ of R is ergodic and therefore its factor $(\widetilde{T}\times\widetilde{S},\eta)$ remains ergodic. ∎

REMARK 5. Independently, using different arguments, the result on ergodicity of Poissonian joinings has also been proved by E. Roy in [36].

4. Self-joinings of symmetric α-stable automorphisms. In this section we will define and study α-stable self-joinings for α-stable automorphisms, i.e. automorphisms given by stationary ergodic symmetric α-stable processes (see [17], [30], [44]). We will show that each ergodic symmetric α-stable automorphism has the ELF property.

4.1. *Auxiliary lemmas.* The proofs of the following two elementary inequalities are slight adaptations of the proofs from [44, pp. 91–92].

LEMMA 15. *If $0 < \alpha < 1$ then for all $x, y \in \mathbb{R}$ we have*

$$|x|^\alpha + |y|^\alpha - |x+y|^\alpha \geq (2 - 2^\alpha)\min(|x|^\alpha, |y|^\alpha). \tag{6}$$

If $1 \leq \alpha < 2$ then for all $x, y \in \mathbb{R}$ we have

$$2(|x|^\alpha + |y|^\alpha) - (|x+y|^\alpha + |x-y|^\alpha) \geq 2(2 - 2^{\alpha/2})\min(|x|^\alpha, |y|^\alpha). \ \blacksquare \tag{7}$$

In particular, (6) implies

$$\big|\,|x+y|^\alpha - |y|^\alpha\big| \leq |x|^\alpha \quad \text{for } x, y \in \mathbb{R} \text{ and } 0 < \alpha \leq 1, \tag{8}$$

and by the Hölder inequality

$$|x+y|^\alpha \leq \max(1, 2^{\alpha-1})(|x|^\alpha + |y|^\alpha) \quad \text{for } x, y \in \mathbb{R} \text{ and } 0 < \alpha \leq 2. \tag{9}$$

The following result is a consequence of (8) and the Hölder inequality.

LEMMA 16. *Assume that $0 < \alpha \leq 2$. Let $(\Omega, \mathcal{F}, m)$ be a finite measure space. Let $(A_n)_{n\geq 1} \subset \mathcal{F}$. Assume that $(f_n), (g_n) \subset L^\alpha(\Omega, m)$ satisfy*

$$\int_{A_n} |f_n|^\alpha \, dm \to 0 \quad \text{and} \quad \int_{\Omega} |g_n|^\alpha \, dm = O(1) \quad \text{as } n \to \infty.$$

Then

$$\int_{A_n} (|f_n + g_n|^\alpha - |g_n|^\alpha) \, dm \to 0 \quad \text{as } n \to \infty. \ \blacksquare$$

4.2. *Symmetric α-stable processes.* Recall that a real random variable X has a *stable distribution* if for any $a, b > 0$ we can find $c > 0$ and a real number d such that the distributions of $aX_1 + bX_2$ and of $cX + d$ are the same, where X_1, X_2 are independent copies of X (one proves then that there exists $\alpha = \alpha(X)$, $0 < \alpha \leq 2$, such that $c = (a^\alpha + b^\alpha)^{1/\alpha}$). In what follows we will consider only the symmetric case (i.e. the distribution of X and of $-X$ are the same, cf. the Gaussian case). In this case, the characteristic function of $X \neq 0$ is of the form $E(e^{itX}) = e^{-|t|^\alpha \sigma}$ for some positive σ ($t \in \mathbb{R}$).

Let $0 < \alpha \leq 2$. Let S be an arbitrary countable set. Let $\underline{X} = (X_s)_{s\in S}$ be a process defined on a probability space $(\Omega, \mathcal{F}, P)$. We say that $\underline{X}$ is (symmetric) *α-stable* if each finite linear combination $Y = \sum_{i=1}^m a_i X_{s_i}$

($a_i \in \mathbb{R}$, $i = 1, \ldots, m$) is a symmetric α-stable variable, i.e. there exists $\sigma \geq 0$ such that $Ee^{itY} = e^{-|t|^\alpha \sigma}$ for all $t \in \mathbb{R}$ (and $\sigma > 0$ whenever $Y \neq 0$). We then write

$$(\!(Y)\!)_\alpha = \sigma^{1/\alpha}.$$

REMARK 6. For $1 \leq \alpha < 2$ ($\alpha = 2$ leads to the Gaussian case) $(\!(Y)\!)_\alpha$ turns out to be a norm in the Banach space $\bigcap_{0<r<\alpha} L^r(\Omega, P)$, and for each $0 < r < \alpha$ there exists $c = c_{\alpha,r}$ such that $(\!(Y)\!)_\alpha = c_{r,\alpha}\|Y\|_r$ for each Y which is α-stable. For $0 < \alpha < 1$ in a similar manner we obtain a Fréchet space.

The following theorem has been proved in [30, pp. 127–128].

THEOREM 17. *Assume that* $0 < \alpha \leq 2$. *Assume moreover that* $\underline{X} = (X_s)_{s\in S}$ *is an* α*-stable process. Then there exists a finite positive Borel measure* (*called a* spectral measure *of* $\underline{X}$) m *on* $\mathbb{R}^S$ *such that*

$$E \exp\Big(i \sum_{j=1}^n a_j X_{s_j}\Big) = \exp\Big(-\frac{1}{2} \int_{\mathbb{R}^S} \Big|\sum_{j=1}^n a_j x_{s_j}\Big|^\alpha dm(\underline{x})\Big)$$

for arbitrary $a_1, \ldots, a_n \in \mathbb{R}$ *and* $s_1, \ldots, s_n \in S$, *where* $\underline{x} = (x_s)_{s\in S}$.

REMARK 7. It follows that $(\!(\sum_{j=1}^n a_j X_{s_j})\!)_\alpha^\alpha = \frac{1}{2} \int_{\mathbb{R}^S} |\sum_{j=1}^n a_j x_{s_j}|^\alpha \, dm(\underline{x})$.

REMARK 8. When $0 < \alpha < 2$, the measure m is not unique.

4.3. *α-stable automorphisms.* We say that an automorphism T of a standard probability Borel space is *α-stable* if there exists a linear space B_0 of real functions on X such that

1. $\mathcal{B}(B_0) = \mathcal{B}$,
2. for each $0 \neq f \in B_0$, f is an α-stable variable,
3. B_0 is T-invariant.

The following criterion as well as the method of proof are very close to the ergodicity criteria in [17] and [15].

PROPOSITION 18. *T is ergodic iff for each* $f, g \in B$,

$$(\!(f \circ T^n - g)\!)_\alpha^\alpha \to (\!(f)\!)_\alpha^\alpha + (\!(g)\!)_\alpha^\alpha$$

along a subsequence of n's whose complement has density zero. Moreover, if T is ergodic then T is weakly mixing.

4.4. *Self-joinings of ergodic α-stable automorphisms.* From now on we assume that T is an ergodic α-stable automorphism of $(X, \mathcal{B}, \mu)$ and B is its α-stable subspace.

Assume that $\varrho \in J(T)$. We say that this self-joining is *α-stable* if the variable $F(x, y) = f(x) + g(y)$ as a variable on $(X \times X, \varrho)$ is α-stable for each $f, g \in B$.

Remark 9. According to the definition of ϱ the automorphism $T \times T$ acting on $(X \times X, \mathcal{B} \otimes \mathcal{B}, \varrho)$ is α-stable with its α-stable space being the closure of $B_0(\varrho) = \{f(x) + g(y) : f, g \in B\}$.

In this section we will prove that each ergodic α-stable automorphism has the ELF property.

Proposition 19. *Let T be an ergodic α-stable automorphism. Assume that $\Phi = \lim_{t\to\infty} U_{T^{n_t}}$, $\Phi = \Phi_\varrho$. Then ϱ is α-stable.*

Proof. Take $f, g \in B$ and $s \in \mathbb{R}$. We have

$$\begin{aligned}\int_{X\times X} e^{is(f(x)+g(y))}\, d\varrho(x,y) &= \int_{X\times X} e^{isf(x)} e^{isg(y)}\, d\varrho(x,y)\\ &= \int_X \Phi(e^{isf})(y) e^{isg(y)}\, d\mu(y) = \lim_{t\to\infty} \int_X e^{isf\circ T^{n_t}} \cdot e^{isg}\, d\mu\\ &= \lim_{t\to\infty} \int_X e^{is(f\circ T^{n_t}+g)} = \lim_{t\to\infty} e^{-|s|^\alpha ((f\circ T^{n_t}+g))_\alpha^\alpha}.\end{aligned}$$

Hence for some $\sigma \geq 0$ and any $s \in \mathbb{R}$,

$$\lim_{t\to\infty} e^{-|s|^\alpha ((f\circ T^{n_t}+g))_\alpha^\alpha} = e^{-|s|^\alpha \sigma}.$$

It follows easily that if $\sigma = 0$ then $f(x)+g(y) = 0$ for ϱ-a.e. $(x,y) \in X\times X$. ∎

From now on we fix $\mathbb{N}_0 \subset \mathbb{N}$ such that $\mathbb{N} \setminus \mathbb{N}_0$ has density zero and for each $f, g \in B$,

$$((f \circ T^n - g))_\alpha^\alpha \to ((f))_\alpha^\alpha + ((g))_\alpha^\alpha$$

as $n \to \infty$, $n \in \mathbb{N}_0$ (which uses the fact that T is weakly mixing).

Using Proposition 18 and the definition of an α-stable self-joining we obtain the following.

Lemma 20. *Assume that T is an ergodic α-stable automorphism with B its α-stable subspace and $\mathbb{N}_0$ as above. Assume that $\varrho \in J(T)$ is α-stable. Then $\varrho \in J^{\mathrm{e}}(T)$ iff for each $f, g, h, j \in B$,*

$$((f(T^n x) + g(T^n y) - h(x) - j(y)))_{\alpha,\varrho}^\alpha \to ((f(x) + g(y)))_{\alpha,\varrho}^\alpha + ((h(x) + j(y)))_{\alpha,\varrho}^\alpha$$

as $n \to \infty$, $n \in \mathbb{N}_0$. ∎

We can now apply Lemma 2 to e^{if}, e^{ig}, e^{ih} and to e^{if}, e^{ig}, e^{ij} to obtain the following.

Lemma 21. *Assume that T is an ergodic α-stable automorphism with B its α-stable subspace and $\mathbb{N}_0$ as above. Assume that $\varrho \in J(T)$ is α-stable. Then for each $f, g, h, j \in B$,*

$$((f(T^n x) + g(T^n y) - h(x)))_{\alpha,\varrho}^\alpha \to ((f(x) + g(y)))_{\alpha,\varrho}^\alpha + ((h))_\alpha^\alpha, \tag{10}$$

$$((f(T^n x) + g(T^n y) - j(y)))_{\alpha,\varrho}^\alpha \to ((f(x) + g(y)))_{\alpha,\varrho}^\alpha + ((j))_\alpha^\alpha, \tag{11}$$

as $n \to \infty$, $n \in \mathbb{N}_0$. ∎

THEOREM 22. *Assume that T is an ergodic α-stable automorphism with B its α-stable subspace and $\mathbb{N}_0$ as above. Assume that $\varrho \in J(T)$ is α-stable. Then $\varrho \in J_2^{\mathrm{e}}(T)$.*

Proof. All we need to show is that (see Lemma 20)

$$((F \circ (T\times T)^n - (H+J)))_{\alpha,\varrho}^{\alpha} \to ((F))_{\alpha,\varrho}^{\alpha} + ((H+J))_{\alpha,\varrho}^{\alpha} \tag{12}$$

as $n \to \infty$, $n \in \mathbb{N}_0$, where $F(x,y) = f(x) + g(y)$, $H(x,y) = h(x)$, $J(x,y) = j(y)$ and $f, g, h, j \in B$. In view of Lemma 21 we already have

$$((F \circ (T\times T)^n \pm H))_{\alpha,\varrho}^{\alpha} \to ((F))_{\alpha,\varrho}^{\alpha} + ((H))_{\alpha,\varrho}^{\alpha}, \tag{13}$$

$$((F \circ (T\times T)^n \pm J))_{\alpha,\varrho}^{\alpha} \to ((F))_{\alpha,\varrho}^{\alpha} + ((J))_{\alpha,\varrho}^{\alpha}, \tag{14}$$

as $n \to \infty$, $n \in \mathbb{N}_0$. In particular, for $1 \le \alpha < 2$ we have

$$((F\circ(T\times T)^n - H))_{\alpha,\varrho}^{\alpha} + ((F\circ(T\times T)^n + H))_{\alpha,\varrho}^{\alpha} \to 2(((F))_{\alpha,\varrho}^{\alpha} + ((H))_{\alpha,\varrho}^{\alpha}), \tag{15}$$

$$((F\circ(T\times T)^n - J))_{\alpha,\varrho}^{\alpha} + ((F\circ(T\times T)^n + J))_{\alpha,\varrho}^{\alpha} \to 2(((F))_{\alpha,\varrho}^{\alpha} + ((J))_{\alpha,\varrho}^{\alpha}), \tag{16}$$

as $n \to \infty$, $n \in \mathbb{N}_0$.

Let $S = \mathbb{Z} \cup \{a, b\}$ and put

$$X_t = \begin{cases} F \circ (T\times T)^n & \text{if } t = n \in \mathbb{Z}, \\ H & \text{if } t = a, \\ J & \text{if } t = b. \end{cases}$$

We hence obtain a process $(X_s)_{s\in S}$ with variables defined on $(X\times X, \varrho)$. Note that each finite linear combination of these variables has an α-stable law. It follows from Theorem 17 that there exists a finite positive Borel measure m on $\mathbb{R}^S = \mathbb{R}^{\mathbb{Z}} \times \mathbb{R}_a \times \mathbb{R}_b$ such that

$$\Big(\Big(\sum_{j=1}^{p} a_j X_{s_j}\Big)\Big)_{\alpha,\varrho}^{\alpha} = \frac{1}{2}\int_{\mathbb{R}^S} \Big|\sum_{j=1}^{p} a_j x_{s_j}\Big|^{\alpha}\, dm(\underline{x}),$$

where $\underline{x} \in \mathbb{R}^S$, $s_1, \dots, s_p \in S$ and $a_1, \dots, a_p \in \mathbb{R}$.

For $1 \le \alpha < 2$, using Lemma 15, we obtain

$$\int_{\mathbb{R}^S} (2(|x_n|^{\alpha} + |x_a|^{\alpha}) - (|x_n - x_a|^{\alpha} + |x_n + x_a|^{\alpha}))\, dm(\underline{x}) \ge \text{const} \int_{\mathbb{R}^S} \min(|x_n|^{\alpha}, |x_a|^{\alpha})\, dm(\underline{x}),$$

while for $0 < \alpha < 1$, using Lemma 15 we obtain

$$\int_{\mathbb{R}^S} (|x_n|^{\alpha} + |x_a|^{\alpha} - |x_n + x_a|^{\alpha})\, dm(\underline{x}) \ge \text{const} \int_{\mathbb{R}^S} \min(|x_n|^{\alpha}, |x_a|^{\alpha})\, dm(\underline{x}).$$

Note that both these inequalities are also true if we replace the function $\underline{x} \mapsto x_a$ by $\underline{x} \mapsto x_b$, and moreover the left hand sides tend to zero as $n \to \infty$,

$n \in \mathbb{N}_0$ (by (15) and (16)). Thus

$$\int_{\mathbb{R}^S} \min(|x_n|^\alpha, |x_a|^\alpha)\, dm(\underline{x}) \to 0 \quad \text{as } n \to \infty,\ n \in \mathbb{N}_0 \tag{17}$$

and similarly

$$\int_{\mathbb{R}^S} \min(|x_n|^\alpha, |x_b|^\alpha)\, dm(\underline{x}) \to 0 \quad \text{as } n \to \infty,\ n \in \mathbb{N}_0. \tag{18}$$

Set

$$\begin{aligned} A_{1,n} &= \{\underline{x} \in \mathbb{R}^S : |x_n| \le |x_a|\}, \\ A_{2,n} &= \{\underline{x} \in \mathbb{R}^S : |x_a| < |x_n| \text{ and } |x_n| \le |x_b|\}, \\ A_{3,n} &= \{\underline{x} \in \mathbb{R}^S : |x_a| < |x_n| \text{ and } |x_b| < |x_n|\}. \end{aligned}$$

In view of (17),

$$\int_{A_{1,n}} |x_n|^\alpha\, dm(\underline{x}) \to 0 \quad \text{as } n \to \infty,\ n \in \mathbb{N}_0.$$

Using Lemma 16 with $A_n = A_{1,n}$, $f_n(\underline{x}) = x_n$ and $g_n(\underline{x}) = -(x_a + x_b)$ we obtain

$$\int_{A_{1,n}} (|x_n - (x_a + x_b)|^\alpha - |x_a + x_b|^\alpha)\, dm(\underline{x}) \to 0,$$

whence

$$\int_{A_{1,n}} (|x_n - (x_a + x_b)|^\alpha - (|x_n|^\alpha + |x_a + x_b|^\alpha))\, dm(\underline{x}) \to 0 \tag{19}$$

as $n \to \infty$, $n \in \mathbb{N}_0$.

Using (18) we find that $\int_{A_{2,n}} |x_n|^\alpha\, dm(\underline{x}) \to 0$, and it follows by the argument as above that

$$\int_{A_{2,n}} (|x_n - (x_a + x_b)|^\alpha - (|x_n|^\alpha + |x_a + x_b|^\alpha))\, dm(\underline{x}) \to 0 \tag{20}$$

as $n \to \infty$, $n \in \mathbb{N}_0$.

Applying (17) and (18) once more we see that

$$\int_{A_{3,n}} |x_a|^\alpha\, dm(\underline{x}) \to 0 \quad \text{and} \quad \int_{A_{3,n}} |x_b|^\alpha\, dm(\underline{x}) \to 0,$$

hence, in view of (9), $\int_{A_{3,n}} |x_a + x_b|^\alpha\, dm(\underline{x}) \to 0$ as $n \to \infty$, $n \in \mathbb{N}_0$. We now use Lemma 16 with $A_n = A_{n,3}$, $f_n(\underline{x}) = -(x_a + x_b)$ and $g_n(\underline{x}) = x_n$. We hence obtain

$$\int_{A_{3,n}} (|x_n - (x_a + x_b)|^\alpha - |x_n|^\alpha)\, dm(\underline{x}) \to 0,$$

whence

$$\int_{A_{3,n}} (|x_n - (x_a + x_b)|^\alpha - (|x_n|^\alpha + |x_a + x_b|^\alpha))\, dm(\underline{x}) \to 0 \tag{21}$$

as $n \to \infty$, $n \in \mathbb{N}_0$.

Putting together (19), (20) and (21) we conclude that

$$\int_{\mathbb{R}^S} (|x_n - (x_a + x_b)|^\alpha - (|x_n|^\alpha + |x_a + x_b|^\alpha))\, dm(\underline{x}) \to 0$$

as $n \to \infty$, $n \in \mathbb{N}_0$. Thus (12) holds and our proof is complete. ∎

COROLLARY 23. *Assume that T is an ergodic α-stable automorphism. Then T has the ELF property.* ∎

REMARK 10. In the recent PhD thesis [36], E. Roy considers automorphisms given by stationary infinitely divisible (ID) processes (for simplicity of notation here and below we assume that such a process has no Gaussian part), hence in particular the class containing all symmetric α-processes ($0 < \alpha < 2$). He then studies ID-joinings of such automorphisms and proves ergodicity of such joinings whenever the joined ID-automorphisms are ergodic. It follows that ergodic ID-automorphisms have the ELF property. His method of studying ergodic properties of ID-automorphisms is completely different from the method presented in this section, and is based on a deep theorem of Maruyama (see [31]): each ID-process can be represented as a stationary process given by a certain stochastic integral in the Poisson suspension given by the Lévy measure of the original process. A study of the Poisson suspension automorphism over the Lévy measure is then the main tool of [36]. In particular, it follows from [36] that ID-automorphisms are factors of Poisson suspension automorphisms.

5. 2-fold simplicity and the ELF property. In this section we will compare the 2-fold simplicity property and the ELF property. Clearly, the interesting case is when automorphisms under consideration are weakly mixing but not mixing. In this case we will show a disjointness result.

Some auxiliary facts are needed.

LEMMA 24. *Assume that T is an ergodic automorphism of $(X, \mathcal{B}, \mu)$. Assume moreover that the closure of the set of powers of T in the weak operator topology satisfies*

$$\overline{\{T^n : n \in \mathbb{Z}\}} \subset C(T) \cup \{\Pi_X\}. \tag{22}$$

Then either

$$\overline{\{T^n : n \in \mathbb{Z}\}} \subset C(T)$$

and T has discrete spectrum, or $T^n \to \Pi_X$ and T is mixing.

Proof. Put

$$G := \overline{\{T^n : n \in \mathbb{Z}\}} \cap C(T),$$

which is a topological monothetic group (recall that on $C(T)$ the weak and strong topologies coincide). Since G is monothetic, it is either compact, or isomorphic to $\mathbb{Z}$, or not locally compact. In the first case it is well-known that T has discrete spectrum. Suppose that T is not mixing. Then there is a weak limit point of powers of T different from Π_X. In view of (22) this limit point must be a graph joining, and therefore G is not isomorphic to $\mathbb{Z}$. Now, note that by adding Π_X to G we obtain a one-point compactification of G, so G is locally compact, a contradiction. ∎

The assumptions of Lemma 24 are always satisfied for 2-fold simple ELF maps and therefore we have proved the following.

LEMMA 25. *Assume that T is weakly mixing. If T is* 2*-fold simple and has the ELF property*, *then T is mixing.* ∎

Next we turn to factors of a 2-fold simple automorphism. Recall (see [21], [49]) that a 2-fold simple map is a compact group extension of any of its non-trivial factors.

LEMMA 26. *Assume that T is a weakly mixing*, *but non-mixing*, 2*-fold simple automorphism. Then no non-trivial factor of T is an ELF automorphism.*

Proof. Suppose that $\{\emptyset, X\} \subsetneq \mathcal{A} \subset \mathcal{B}$ and $\mathcal{A}$ is an ELF factor. Note that $\mathcal{A}$ is still not mixing by Veech's theorem ([49]). Let $T|_{\mathcal{A}}$ be the quotient action of T on $(X/\mathcal{A}, \mathcal{A}, \mu|_{\mathcal{A}})$. It is an ELF automorphism, so in view of Proposition 4, each self-joining λ in the weak closure of powers of $T|_{\mathcal{A}}$ is relatively weakly mixing with respect to the two marginal σ-algebras.

On the other hand, by Lemma 25, $T|_{\mathcal{A}}$ is not 2-fold simple, and what is more, in view of Lemma 24, in the weak closure of the powers of $T|_{\mathcal{A}}$ we must find a self-joining different from the product measure and from any graph measure. However, since this joining is ergodic, it is the image of a graph joining of T acting on $\mathcal{B}$. In other words, this joining, as an action, is isomorphic to the action of T on $\mathcal{A} \vee R\mathcal{A}$ for some $R \in C(T)$. However, since $\mathcal{B} \to \mathcal{A}$ is a compact group extension, $\mathcal{A} \vee R\mathcal{A} \to \mathcal{A}$ is a non-trivial isometric extension, so it cannot be relatively weakly mixing, a contradiction. ∎

We are now able to prove a disjointness result.

PROPOSITION 27. *Assume that T is a weakly mixing* 2*-fold simple automorphism which is not mixing. Then T is disjoint from an arbitrary ELF automorphism.*

Proof. Let S be an ELF automorphism acting on $(Y,\mathcal{C},\nu)$. Assume that $\Phi : L^2(X,\mathcal{B},\mu)\to L^2(Y,\mathcal{C},\nu)$ is a non-trivial ($\Phi\neq\Pi_{X,Y}$) indecomposable Markov operator intertwining T and S. Consider the subalgebra

$$\mathrm{span}\{\Phi^*(g_1)\cdot\ldots\cdot\Phi^*(g_n) : g_i\in L^\infty(Y,\mathcal{C},\nu),\ i=1,\ldots,n,\ n\geq 1\}$$

of $L^\infty(X,\mathcal{B},\mu)$. By a result of Zimmer ([53]) there exists $\mathcal{A}\subset\mathcal{B}$ such that

$$L^2(\mathcal{A})=\overline{\mathrm{span}}\{\Phi^*(g_1)\cdot\ldots\cdot\Phi^*(g_n) : g_i\in L^\infty(Y,\mathcal{C},\nu),\ i=1,\ldots,n,\ n\geq 1\}$$

and since the function algebra is T-invariant, $\mathcal{A}$ is a factor of T. Since Φ is non-trivial, $\mathcal{A}$ is a non-trivial factor of T. By Veech's theorem ([21], [49]), $\mathcal{A}$ is given as the fixed points of the action of a compact group $\mathcal{H}=\mathcal{H}(\mathcal{A}):=\{R\in C(T) : R|_{\mathcal{A}}=\mathrm{Id}\}$ on $\mathcal{B}$.

We will now show that the action of T on $\mathcal{A}$ has the ELF property, which is in conflict with Lemma 26. Take any sequence (n_t) and suppose that T^{n_t} (weakly) converges to a self-joining different from Π_X. We have

$$T^{n_t}\to a\int_{C(T)} R\,dP(R)+(1-a)\Pi_X,$$

where $a>0$. By passing to a further subsequence if necessary we obtain

$$\Phi\circ\Big(a\int_{C(T)} R\,dP(R)+(1-a)\Pi_X\Big)=W\circ\Phi,$$

where by Proposition 5, $W\circ\Phi$ is still indecomposable (that is, it corresponds to an ergodic joining). Since

$$\Phi\circ\Big(a\int_{C(T)} R\,dP(R)+(1-a)\Pi_X\Big)=a\int_{C(T)}\Phi\circ R\,dP(R)+(1-a)\Pi_{X,Y},$$

we have $a=1$ and

$$\int_{C(T)}\Phi\circ R\,dP(R)=W\circ\Phi.$$

It follows that for (R_1,R_2) belonging to a subset of $C(T)\times C(T)$ of full $P\otimes P$-measure we have

$$\Phi\circ R_1\circ R_2^{-1}=\Phi,$$

or equivalently

$$R_2\circ R_1^{-1}\circ\Phi^*=\Phi^*.$$

But $R_2\circ R_1^{-1}$ preserves the product of functions, and therefore $R_2\circ R_1^{-1}|_{\mathcal{A}}$ is the identity map, i.e. $R_2\circ R_1^{-1}\in\mathcal{H}$. It follows that there exists $R'\in C(T)$ such that

$$P(R'\mathcal{H})=1.$$

If now $f,g\in L^\infty(\mathcal{A})$ then

$$\langle f\circ T^{n_t},g\rangle\to\Big\langle\Big(\int\limits_{C(T)}R\,dP(R)\Big)f,g\Big\rangle=\int\limits_{C(T)}\langle Rf,g\rangle\,dP(R)=\langle R'f,g\rangle.$$

We know that the image on $\mathcal{A}\otimes\mathcal{A}$ of the measure determined by the Markov operator $\int R\,dP$ corresponds to $E(\,\cdot\,|\,\mathcal{A})\circ R'$ and since the latter is the restriction of $\mu_{R'}$ to $\mathcal{A}\otimes\mathcal{A}$, it is indecomposable. Hence $\mathcal{A}$ has the ELF property. ∎

REMARK 11. T. de la Rue [39] has shown that Gaussian automorphisms are never of rank 1. Gaussian automorphisms enjoy the ELF property. We conjecture that no weakly mixing, non-mixing rank 1 automorphism has the ELF property.

Let us recall that if T is rank 1 then by a result of Ryzhikov ([40, Thm. 3.1]) for each ergodic self-joining ϱ of T there exists a sequence (n_t) such that

$$T^{n_t}\to a\Phi_\varrho+(1-a)\Phi_\eta,$$

where $a>0$ ($\eta\in J_2(T)$). It follows that if T is rank 1 and has the ELF property, then (by Proposition 4) T is semisimple (in the sense of [20]).

We finish this section by showing that the minimal self-joinings (MSJ) automorphisms (see [37] for the definition) which are not mixing are contained in the class of multipliers of ELF$^\perp$. The proof is similar in spirit to the proof of Theorem 5.3 in [41].

PROPOSITION 28. *Let T be an MSJ automorphism which is not mixing. Then T belongs to $\mathcal{M}(ELF^\perp)$.*

Proof. Since T has the MSJ property, by the basic lemma on multipliers ([28]) all we need to show is that the Cartesian square $T\times T$ is disjoint from any ELF automorphism. Using now the criterion for disjointness from [29] (and the fact that T has the MSJ property) it is enough to show that no factor of $T^{\times\infty}$ has the ELF property. A factor of $T^{\times\infty}$ can be obtained only from permutations of finitely many coordinates ([37]), that is, it is of the form $T^{\times k}$ acting on $\underbrace{X\times\cdots\times X}_{k}/\mathcal{S}_k$ for some $k\geq 1$, where $\mathcal{S}_k$ stands for the group of all permutations of k coordinates. Suppose now that such a factor has the ELF property (recall that a factor of an ELF automorphism remains an ELF automorphism). Denoting by $\mathcal{F}$ the factor σ-algebra, we seek a contradiction.

Since T is not mixing, there exists a sequence (n_j) such that

$$T^{n_j}\to a\sum_{n=-\infty}^{\infty}a_nT^n+(1-a)\Pi_X,$$

where $0<a\leq 1$, $a_n\geq 0$, $\sum_{n=-\infty}^{\infty}a_n=1$ and either

(A) $a < 1$ and then $a_{m_0} \neq 0$ for some m_0, or
(B) $a = 1$ and then there are $m_1 \neq m_2$ such that $a_{m_1} \neq 0 \neq a_{m_2}$.

We now continue the proof assuming (B). We have

$$(T\times\cdots\times T)^{n_j} \to \Phi := a^k a_{m_1}^k T^{m_1}\otimes\cdots\otimes T^{m_1} + a^k a_{m_2}^k T^{m_2}\otimes\cdots\otimes T^{m_2} + b\Theta,$$

where Θ is a Markov operator. Since $(T\times\cdots\times T)^{n_i}$ preserves the subspace $L^2(\mathcal{F})$, so does the weak limit Φ, and since the first two summands of the limit also preserve $L^2(\mathcal{F})$, so does Θ. Therefore,

$$\Phi|_{L^2(\mathcal{F})} = a^k a_{m_1}^k T^{m_1}\otimes\cdots\otimes T^{m_1}|_{L^2(\mathcal{F})} + a^k a_{m_2}^k T^{m_2}\otimes\cdots\otimes T^{m_2}|_{L^2(\mathcal{F})} + b\Theta|_{L^2(\mathcal{F})}.$$

Since $T^{\times\infty}$ restricted to $\mathcal{F}$ has the ELF property, the Markov operator $\Phi|_{L^2(\mathcal{F})}$ is indecomposable in $\mathcal{J}_2(T^{\times\infty}|_{\mathcal{F}})$. It follows that all summands on the right hand side of the above equality are equal. In particular,

$$T^{m_1}\otimes\cdots\otimes T^{m_1}|_{\mathcal{F}} = T^{m_2}\otimes\cdots\otimes T^{m_2}|_{\mathcal{F}},$$

so $T^{m_2-m_1}\otimes\cdots\otimes T^{m_2-m_1}|_{\mathcal{F}} = \mathrm{Id}$, which is not possible.

In case (A) we proceed in the same way working in the weak limit with the operators $a^k a_{m_0}^k T^{m_0}\otimes\cdots\otimes T^{m_0}$ and $(1-a)^k \Pi_{X^k}$. ■

6. Disjointness of interval exchange transformations from ELF automorphisms. In this section we will study disjointness of interval exchange transformations from the class of ELF automorphisms.

6.1. *Interval exchange transformations. Rauzy induction.* Recall that (see e.g. [3, Chapter 5]) an *m-interval exchange transformation* is a Lebesgue measure-preserving automorphism of $[0,1)$ given by a probability vector $(\lambda_1,\ldots,\lambda_m)$ together with a permutation π of $\{1,\ldots,m\}$. The unit interval $[0,1)$ is partitioned into m subintervals of lengths $\lambda_1,\ldots,\lambda_m$ which are rearranged according to the permutation π. For some basic properties of interval exchange transformations (IET) we refer the reader to [3, Chapter 5]. Katok [22] proves that IET's have no mixing factors. In fact, an analysis of Katok's proof shows that IET's are disjoint from all mixing automorphisms (cf. [8]). An IET can be weakly mixing, and the problem of "how many" IET's are weakly mixing was one of the most important ones in this theory (see [51]). Quite recently, in a deep paper [2] Avila and Forni give a positive answer to Veech's conjecture: under some necessary restrictions on the permutation, for almost all choices of probability vectors, the corresponding IET is weakly mixing. Recall also that some IET's can be even 2-fold simple automorphisms (see [18], [4], [5]). In this section we will prove that for some special permutations almost all IET's are disjoint from all ELF automorphisms.

Fix $m > 1$, and let $\mathfrak{S}_m^0$ denote the set of all *irreducible* permutations of $\{1, \dots, m\}$, i.e. such that $\pi\{1, \dots, k\} = \{1, \dots, k\}$ implies $k = m$. Set

$$\Lambda_m = \{\lambda \in \mathbb{R}^m : \lambda_j > 0,\, 1 \le j \le m\}.$$

Given $\lambda \in \Lambda_m$ put

$$\beta_0(\lambda) = 0, \qquad \beta_j(\lambda) = \sum_{i=1}^{j} \lambda_i,$$

$$|\lambda| = \sum_{i=1}^{m} \lambda_i, \qquad I_j^\lambda = [\beta_{j-1}(\lambda), \beta_j(\lambda)) \subset I^\lambda = [0, |\lambda|),$$

for $1 \le j \le m$. We also define a vector λ^π, where $\lambda_j^\pi = \lambda_{\pi^{-1}j}$, $1 \le j \le m$.

With the notation as above, given $(\lambda, \pi) \in \Lambda_m \times \mathfrak{S}_m^0$ denote by $T = T_{(\lambda,\pi)}$ the corresponding *interval exchange transformation* of I^λ, i.e.

$$T_{(\lambda,\pi)} x = x + \beta_{\pi(i)-1}(\lambda^\pi) - \beta_{i-1}(\lambda)$$

whenever $x \in I_i^\lambda$, $1 \le i \le m$.

We will now recall the *Rauzy induction* (see the original papers [34], [48], [50], [51]). Let $Z(\lambda, \pi) = [0, \max(\beta_{m-1}(\lambda), \beta_{m-1}(\lambda^\pi)))$. Then the induced transformation $T_{Z(\lambda,\pi)} : Z(\lambda, \pi) \to Z(\lambda, \pi)$ is an m-interval exchange transformation determined by a pair $\mathfrak{I}(\lambda, \pi) \in \Lambda_m \times \mathfrak{S}_m^0$. This defines the transformation $\mathfrak{I} : \Lambda_m \times \mathfrak{S}_m^0 \to \Lambda_m \times \mathfrak{S}_m^0$ (see [34]).

For each $k = 1, \dots, m$ define a permutation τ_k by

$$\tau_k(j) = \begin{cases} j & \text{for } 1 \le j \le k, \\ j+1 & \text{for } k < j < m, \\ k+1 & \text{for } j = m. \end{cases}$$

G. Rauzy [34] has defined useful maps $a, b : \mathfrak{S}_m^0 \to \mathfrak{S}_m^0$ by

$$a(\pi) = \pi \circ \tau_{\pi^{-1}(m)}^{-1}, \qquad b(\pi) = \tau_{\pi(m)} \circ \pi.$$

These maps generate a group of maps of $\mathfrak{S}_m^0$, any orbit of which is called a *Rauzy class*. We associate to π and $c = a$ or b the $m \times m$ matrices $A(\pi, c)$ such that

$$A(\pi, a)\lambda = (\lambda_1, \dots, \lambda_{\pi^{-1}m-1}, \lambda_{\pi^{-1}m} + \lambda_{\pi^{-1}m+1}, \lambda_{\pi^{-1}m+2}, \dots, \lambda_m, \lambda_{\pi^{-1}m+1}),$$
$$A(\pi, b)\lambda = (\lambda_1, \dots, \lambda_{m-1}, \lambda_m + \lambda_{\pi^{-1}m}).$$

Define

$$c(\lambda, \pi) = \begin{cases} a & \text{if } \lambda_m < \lambda_{\pi^{-1}m}, \\ b & \text{if } \lambda_m > \lambda_{\pi^{-1}m}. \end{cases}$$

Then

$$\mathfrak{I}(\lambda, \pi) = (A^{-1}\lambda, c(\pi)),$$

where $c = c(\lambda, \pi)$ and $A = A(\pi, c)$. Let $(\lambda, \pi) \in \Lambda_m \times \mathfrak{S}_m^0$. Then $\mathfrak{I}^n(\lambda, \pi) = (\lambda^{(n)}, \pi^{(n)})$, where

$$\pi^{(n)} = c_n \circ c_{n-1} \circ \cdots \circ c_1(\pi) \quad \text{with} \quad c_k = c_k(\lambda, \pi) = c(\mathfrak{I}^{k-1}(\lambda, \pi))$$

and

$$\lambda = A^{(n)}\lambda^{(n)} \quad \text{with} \quad A^{(n)} = A(\pi, c_1)A(\pi^{(1)}, c_2) \cdots A(\pi^{(n-1)}, c_n).$$

Set $\Delta_{m-1} = \{\lambda \in \Lambda_m : |\lambda| = 1\}$, and define $\mathfrak{P} : \Delta_{m-1} \times \mathfrak{S}_m^0 \to \Delta_{m-1} \times \mathfrak{S}_m^0$ by

$$\mathfrak{P}(\lambda, \pi) = \left(\frac{A^{-1}\lambda}{|A^{-1}\lambda|}, c(\pi) \right),$$

where $c = c(\lambda, \pi)$ and $A = A(\pi, c)$. Then $\mathfrak{P}$ is essentially two-to-one and non-singular. Moreover, the inverses of $\mathfrak{P}$ are given by

$$\mathfrak{P}_c^{-1}(\lambda, \pi) = \left(\frac{A(c^{-1}(\pi), c)\lambda}{|A(c^{-1}(\pi), c)\lambda|}, c^{-1}(\pi) \right),$$

where $c \in \{a, b\}$.

Proposition 29 (Veech [49]). *Let $\mathfrak{R} \subset \mathfrak{S}_m^0$ be a fixed Rauzy class. On $\Delta_{m-1} \times \mathfrak{R}$ there exists a smooth positive σ-finite $\mathfrak{P}$-invariant measure $\mathfrak{M} = \mathfrak{M}_{\mathfrak{R}}$, with respect to which $\mathfrak{P}$ is ergodic and conservative.*

Given $(\lambda, \pi) \in \Delta_{m-1} \times \mathfrak{S}_m^0$ and $\gamma \in (0,1)$, for $\beta \in [0,1)$ define

$$\tau^-_{(\lambda,\pi,\gamma)}(\beta) = \max\{k \le 0 : T^k_{(\lambda,\pi)}(\beta) \in [0, \gamma)\}.$$

Since for a.a. $(\lambda, \pi) \in \Delta_{m-1} \times \mathfrak{S}_m^0$ the transformation $T_{(\lambda,\pi)}$ is ergodic, for a.a. $(\lambda, \pi) \in \Delta_{m-1} \times \mathfrak{S}_m^0$ the measurable function $\tau^-_{(\lambda,\pi,\gamma)} : [0,1) \to -\mathbb{N} \cup \{-\infty\}$ is almost everywhere finite. Let us consider the skew product $\mathfrak{P}_* : \Delta_{m-1} \times \mathfrak{S}_m^0 \times [0,1) \to \Delta_{m-1} \times \mathfrak{S}_m^0 \times [0,1)$ given by

$$\mathfrak{P}_*(\lambda, \pi, \beta) = \left(\mathfrak{P}(\lambda, \pi), \frac{T_{(\lambda,\pi)}^{\tau^-_{(\lambda,\pi,|Z(\lambda,\pi)|)}(\beta)}(\beta)}{|Z(\lambda, \pi)|} \right).$$

Then

$$\mathfrak{P}_*^n(\lambda, \pi, \beta) = \left(\mathfrak{P}^n(\lambda, \pi), \frac{T_{(\lambda,\pi)}^{\tau^-_{(\lambda,\pi,|(A^{(n)})^{-1}\lambda|)}(\beta)}(\beta)}{|(A^{(n)})^{-1}\lambda|} \right).$$

Let $\mathfrak{R} \subset \mathfrak{S}_m^0$ be a fixed Rauzy class. Then, as shown in [48, §3], there exist $n > 1$, $\pi_0 \in \mathfrak{R}$ and $\overline{c} = (c_1, \ldots, c_n) \in \{a, b\}^n$ such that

$$B = A(\pi_0, c_1)A(\pi_1, c_2) \cdots A(\pi_{n-1}, c_n)$$

is a positive $m \times m$ matrix, where $\pi_j = c_j \circ c_{j-1} \circ \cdots \circ c_1(\pi_0)$, $1 \le j \le n$. If we now put $\mathfrak{P}_{\bar{c}}^{-1} = \mathfrak{P}_{c_1}^{-1} \circ \ldots \circ \mathfrak{P}_{c_n}^{-1}$ then

$$\mathfrak{P}_{\bar{c}}^{-1}(\Delta_{m-1} \times \{\pi_n\}) = \left\{ \left(\frac{B\lambda}{|B\lambda|}, \pi_0 \right), \lambda \in \Delta_{m-1} \right\}.$$

Indeed, it is easy to check that $\mathfrak{P}_{\bar{c}}^{-1}(\lambda, \pi_n) = (B\lambda/|B\lambda|, \pi_0)$ for every $\lambda \in \Delta_{m-1}$.

For each $0 < \varepsilon < 1$ denote by $Y_\varepsilon \subset \Lambda_m \times \{\pi_n\}$ the set of all (λ, π_n) such that $\lambda_1 > (1-\varepsilon)|\lambda|$. Let $\Delta_{m-1}^{\mathrm{ri}}$ denote the set of all elements from Δ_{m-1} such that the only rational relations between $\lambda_1, \ldots, \lambda_m$ are multiples of $\lambda_1 + \cdots + \lambda_m = 1$. Set

$$\begin{aligned} W_{\mathfrak{R}} = {} & (\Delta_{m-1}^{\mathrm{ri}} \times \mathfrak{R} \times [0,1)) \\ & \cap \bigcap_{s \in \mathbb{N}} \bigcup_{k \ge s} \bigcup_{l \ge s} ((\mathfrak{P}^{-l}(\mathfrak{P}_{\bar{c}}^{-1}(Y_{1/k}))) \times [0,1)) \cap \mathfrak{P}_*^{-l-n}(Y_{1/k} \times [1/3, 2/3))). \end{aligned}$$

Let $m_{[0,1)}$ stand for the Lebesgue measure on $[0,1)$.

LEMMA 30. *The set W has full $\mathfrak{M} \otimes m_{[0,1)}$-measure.*

Proof. By the ergodicity and conservativity of $\mathfrak{P}$ the set

$$W' = (\Delta_{m-1}^{\mathrm{ri}} \times \mathfrak{R}) \cap \bigcap_{s \in \mathbb{N}} \bigcup_{k \ge s} \bigcup_{l \ge s} \mathfrak{P}^{-l}(\mathfrak{P}_{\bar{c}}^{-1} Y_{1/k})$$

has full $\mathfrak{M}$-measure because $\mathfrak{M}(\mathfrak{P}_{\bar{c}}^{-1} Y_{1/k}) > 0$ for every $k \in \mathbb{N}$. Since W' is the projection of W on $\Delta_{m-1} \times \mathfrak{R}$, it suffices to show that for each $(\lambda, \pi) \in W'$ the section

$$W_{(\lambda,\pi)} = \{\beta \in [0,1) : (\lambda, \pi, \beta) \in W\}$$

has full Lebesgue measure.

Fix $0 < \varepsilon < 1$ and $l \ge 1$ and suppose that $(\lambda, \pi) \in \mathfrak{P}^{-l}(\mathfrak{P}_{\bar{c}}^{-1} Y_\varepsilon)$ and $\lambda \in \Delta_{m-1}^{\mathrm{ri}}$. Then $(\lambda', \pi_n) := \mathfrak{J}^{n+l}(\lambda, \pi) \in Y_\varepsilon$ and

$$\lambda = A^{(n+l)} \lambda' = A^{(l)}(B\lambda').$$

Write $J = J_{\varepsilon,l} = I_1^{\lambda'}$ and $q = q_{\varepsilon,l} = \sum_{i=1}^m A_{i1}^{(n+l)}$. Since $\lambda \in \Delta_{m-1}^{\mathrm{ri}}$, we have $|\lambda'| = |(A^{(n+l)})^{-1}\lambda| \to 0$ as $l \to \infty$. As shown in [51], J and q satisfy the following conditions:

- $J \cap T_{(\lambda,\pi)}^j J = \emptyset$ for $1 \le j < q$,
- $T_{(\lambda,\pi)}$ is linear on $T_{(\lambda,\pi)}^j J$ for $0 \le j < q$,
- $|J \cap T_{(\lambda,\pi)}^q J| > (1 - 2\varepsilon)|J|$,
- $|\bigcup_{j=0}^{q-1} T_{(\lambda,\pi)}^j J| > 1 - \nu(B)\frac{\varepsilon}{1-\varepsilon}$,

where $\nu(B)=\max_{1\le i,j,k\le m} B_{ij}/B_{ik}$. Next consider the tower

$$\Xi_{\varepsilon,l}=\left(T^j_{(\lambda,\pi)}\left[\frac{1}{2}\,|J|,\frac{2}{3}\,|J|\right)\right)_{0\le j<q}.$$

Notice that if $\beta\in\Xi_{\varepsilon,l}$ then

$$\mathfrak{P}^{n+l}_*(\lambda,\pi,\beta)\in Y_\varepsilon\times\left[\frac{1}{2}\,\frac{|I^{\lambda'}_1|}{|I^{\lambda'}|},\frac{2}{3}\,\frac{|I^{\lambda'}_1|}{|I^{\lambda'}|}\right)\subset Y_\varepsilon\times\left[\frac{1}{2}\,(1-\varepsilon),\frac{2}{3}\right). \tag{23}$$

Now suppose that $(\lambda,\pi)\in W'$. Then there exist increasing sequences $(k_s)_{s\in\mathbb{N}}$, $(l_s)_{s\in\mathbb{N}}$ of natural numbers such that $(\lambda,\pi)\in\mathfrak{P}^{-l_s}(\mathfrak{P}^{-1}_{\bar c}Y_{1/k_s})$. By the preceding observation, $T_{(\lambda,\pi)}$ has rank 1, hence is ergodic. Moreover, as

$$\liminf_{s\to\infty} m_{[0,1)}(\Xi_{1/k_s,l_s})=1/6$$

and $T_{(\lambda,\pi)}$ is ergodic, there exists a set $\Theta_{(\lambda,\pi)}\subset I^\lambda=[0,1)$ of full $m_{[0,1)}$ measure such that for each $\beta\in\Theta_{(\lambda,\pi)}$ there exist infinitely many s such that $\beta\in\Xi_{1/k_s,l_s}$ (see King [23, Lemma 3.4 and remark after it]). Then using (23) we obtain $\Theta_{(\lambda,\pi)}\subset W_{(\lambda,\pi)}$, which completes the proof. ∎

6.2. *Disjointness theorem.* Denote by $\mathfrak{S}^r_m$ (resp. $\mathfrak{S}^l_m$) the set of all $\pi\in\mathfrak{S}^0_m$ such that $\pi(j)+1\ne\pi(j+1)$ for any $1\le j<m$ and

$$\pi(\pi^{-1}(m)+1)=\pi(m)+1\quad(\text{resp. }\pi(\pi^{-1}(1)-1)=\pi(1)-1).$$

In this section we will prove that if $\pi\in\mathfrak{S}^r_m\cup\mathfrak{S}^l_m$ then for almost every $\lambda\in\Lambda_m$ the interval exchange transformation $T_{(\lambda,\pi)}$ is disjoint from all ELF automorphisms.

Suppose that $(\lambda,\pi,\beta)\in W_{\mathfrak{R}}$ (see Section 6.1) and let $f:[0,1)\to\mathbb{R}$ be a positive function of bounded variation. For short, we will write T for $T_{(\lambda,\pi)}$. In view of the proof of Lemma 30, we can choose a sequence (J_n) of intervals whose left end-point equals 0 and an increasing sequence of natural numbers (q_n) such that

- the intervals $\{T^lJ_n, 0\le l<q_n\}$ are pairwise disjoint,
- T is linear on T^lJ_n for $0\le l<q_n$,
- $|J_n\cap T^{q_n}J_n|/|J_n|\to 1$,
- $|\bigcup_{l=0}^{q_n-1}T^lJ_n|\to 1$,
- $\beta\in\bigcup_{l=0}^{q_n-1}T^l[(1/4)|J_n|,(3/4)|J_n|)$.

Therefore T is ergodic and (q_n) is a rigidity time for T. Set $C_n:=\bigcup_{l=0}^{q_n-1}T^lJ_n$ and $b_n:=|J_n|^{-1}\int_{C_n}f(x)\,dx$. Putting $I_n:=J_n\cap T^{-q_n}J_n$ (which is also an interval) we obtain

- T^lI_n are intervals for $1\le l<2q_n$,
- $T^kI_n\cap T^{k+l}I_n=\emptyset$ for $0\le k<q_n$ and $1\le l<q_n$,
- T is linear on T^lI_n for $0\le l<2q_n$,
- $|I_n\cap T^{q_n}I_n|/|I_n|\to 1$,

- $|\bigcup_{l=0}^{q_n-1} T^l I_n| \to 1$,
- $\beta \in \bigcup_{l=0}^{q_n-1} T^{l+k}[(1/5)|I_n|, (4/5)|I_n|)$ for every $0 \le k < q_n$.

If $x \in C_n' := \bigcup_{l=0}^{q_n-1} T^l I_n$, then each element of the orbit $T^l x$, $0 \le l < q_n$, lies in exactly one interval $T^{k_l} J_n$, $0 \le k_l < q_n$. Therefore

$$\begin{aligned}|f^{(q_n)}(x) - b_n| &\le \sum_{l=0}^{q_n-1} \Big| f(T^l x) - \frac{1}{|J_n|} \int_{T^{k_l} J_n} f(t)\,dt \Big| \\ &\le \sum_{l=0}^{q_n-1} \frac{1}{|J_n|} \int_{T^{k_l} J_n} |f(T^l x) - f(t)|\,dt \\ &\le \sum_{l=0}^{q_n-1} \operatorname{Var}_{T^{k_l} J_n} f \le \operatorname{Var} f.\end{aligned}$$

Assume again that $(\lambda, \pi, \beta) \in W_{\mathfrak{R}}$ and suppose that $\beta \in I_{k+1}^{\lambda}$, where $0 \le k \le m-1$. Consider the function $f = 1 + \chi_{[\beta_k(\lambda),\beta)}$ and put $a_n := [b_n]$.

Lemma 31. *Let P be a weak limit measure of $((f^{(q_n)} - a_n)_*(m_{[0,1)}))_n$. Then P is concentrated on $\mathbb{Z} \cap [-2,2]$ and has at least two atoms.*

Proof. Since $|f^{(q_n)}(x) - a_n| < \operatorname{Var} f + 1 = 3$ for $x \in C_n'$, $f^{(q_n)} - a_n$ takes only integer values and $|C_n'| \to 1$, P is concentrated on $\mathbb{Z} \cap [-2,2]$.

Fix $n \ge 1$ and take $0 \le j < q_n$. Then $\beta_k(\lambda) \notin \bigcup_{l=0}^{q_n-1} T^l(\operatorname{Int} T^j I_n)$ and $\beta \in \bigcup_{l=0}^{q_n-1} T^{j+l}[(1/5)|I_n|, (4/5)|I_n|)$. It follows that $T^j I_n$ splits into two subintervals K_j^-, K_j^+ of size at least $|I_n|/5$ such that $f^{(q_n)} - a_n$ is constant on each of them and the values which $f^{(q_n)} - a_n$ takes on K_j^- and K_j^+ differ by 1. Since $f^{(q_n)} - a_n$ on C_n' takes values only from the set $\{-2,-1,0,1,2\}$ there exists $a \in \{-2,-1,0,1,2\}$ such that the cardinality of $A_n = \{0 \le j < q_n : (f^{(q_n)} - a_n)(K_j^-) = \{a\}\}$ is at least $q_n/5$ for infinitely many n. Moreover, there exists $\zeta = \pm 1$ such that $\#\{j \in A_n : (f^{(q_n)} - a_n)(K_j^+) = \{a+\zeta\}\} \ge \#A_n/2 \ge q_n/10$ for infinitely many n. Since $|K_j^-|, |K_j^+| \ge |I_n|/5$ and $q_n|I_n| \to 1$, we conclude that $P(\{a\}) \ge 1/25$ and $P(\{a+\zeta\}) \ge 1/50$, which completes the proof. ∎

Proposition 32. *Suppose that $(\lambda, \pi, \beta) \in W_{\mathfrak{R}}$ and $\beta \in I_{k+1}^{\lambda}$ for some $0 \le k \le m-1$. Then there exists an increasing sequence (a_n) of natural numbers and a non-trivial (i.e. with at least two non-zero frequencies) trigonometric polynomial p such that*

$$(T_f)^{a_n} \to p(T_f)$$

weakly, where $f = 1 + \chi_{[\beta_k(\lambda),\beta)}$. Moreover, T_f is weakly mixing and it is disjoint from all ELF automorphisms.

Proof. By Theorem 6 in [8] and Lemma 31, passing to a subsequence of (a_n) if necessary we have

$$(T^f)_{a_n} \to p((T^f)_1),$$

where $p(z) = \sum_{i=-2}^{2} P(\{i\})z^{-i}$ and at least two of the numbers $P(\{i\})$, $-2 \le i \le 2$, are positive. By Lemma 3, $(T_f)^{a_n} \to p(T_f)$. Moreover, by the proof of Lemma 31 there is $i \in \{-2,-1,0,1\}$ such that $P(\{i\})$ and $P(\{i+1\})$ are positive. It now follows from an argument used in the proof of Proposition 6 that T_f is weakly mixing. Using Proposition 6 again, we conclude that T_f is disjoint from all ELF automorphisms. ∎

Since IET's associated to permutations from $\mathfrak{S}_m^l$ are isomorphic via the symmetry $x \mapsto 1-x$ to IET's associated to permutations from $\mathfrak{S}_m^r$, we focus on the latter family. It is easy to see that if $(\lambda,\pi) \in \Delta_{m-1} \times \mathfrak{S}_m^r$ then $T_{\mathfrak{P}(\lambda,\pi)}$ is an $m-1$-interval exchange transformation. Indeed, for each $j = 1,\dots,m-1$ define $i_j : \{1,\dots,m-1\} \to \{1,\dots,m\}$ and $p_j : \{1,\dots,m\} \to \{1,\dots,m-1\}$ by

$$i_j(k) = \begin{cases} k & \text{for } 1 \le k \le j, \\ k+1 & \text{for } j < k \le m-1 \end{cases}$$

and

$$p_j(k) = \begin{cases} k & \text{for } 1 \le k \le j, \\ k-1 & \text{for } j < k \le m. \end{cases}$$

Then $T_{\mathfrak{P}(\lambda,\pi)} = T_{\mathfrak{L}(\lambda,\pi)}$, where $\mathfrak{L} : \Delta_{m-1} \times \mathfrak{S}_m^r \to \Delta_{m-2} \times \mathfrak{S}_{m-1}^0$ is given by

$$\mathfrak{L}(\lambda,\pi) = \begin{cases} \Big(\frac{(\lambda_1,\dots,\lambda_{j-1},\lambda_j-\lambda_m,\lambda_{j+1}+\lambda_m,\lambda_{j+2},\dots,\lambda_{m-1})}{1-\lambda_m}, p_{\pi_m}\circ\pi\circ i_{m-1}\Big), & \lambda_m < \lambda_j, \\ \Big(\frac{(\lambda_1,\dots,\lambda_{j-1},\lambda_j+\lambda_{j+1},\lambda_{j+2},\dots,\lambda_{m-1},\lambda_m-\lambda_j)}{1-\lambda_j}, p_{m-1}\circ\pi\circ i_{j-1}\Big), & \lambda_m > \lambda_j, \end{cases}$$

with $j = \pi^{-1}(m)$. Moreover, by the definition of $\mathfrak{P}$, $T_{(\lambda,\pi)}$ is isomorphic to the integral transformation $(T_{\mathfrak{L}(\lambda,\pi)})_{f_{(\mathfrak{L}(\lambda,\pi),\beta(\lambda,\pi))}}$, where

$$f_{(\lambda,\pi,\beta)} = 1 + \chi_{[\beta_{i-1}(\lambda),\beta)} \quad \text{whenever} \quad \beta \in I_i^\lambda = [\beta_{i-1}(\lambda),\beta_i(\lambda))$$

and

$$\beta(\lambda,\pi) = \begin{cases} \dfrac{\lambda_1+\dots+\lambda_{\pi^{-1}(m)}}{1-\lambda_m} & \text{if } \lambda_m < \lambda_{\pi^{-1}(m)}, \\ \dfrac{\lambda_1+\dots+\lambda_{\pi^{-1}(m)}}{1-\lambda_{\pi^{-1}(m)}} & \text{if } \lambda_m > \lambda_{\pi^{-1}(m)}. \end{cases}$$

Consider the map $\mathfrak{L}_* : \Delta_{m-1} \times \mathfrak{S}_m^r \to \Delta_{m-2} \times \mathfrak{S}_{m-1}^0 \times [0,1)$ with

$$\mathfrak{L}_*(\lambda,\pi) = (\mathfrak{L}(\lambda,\pi),\beta(\lambda,\pi)).$$

Since $\mathfrak{L}_*$ is essentially one-to-one and its inverse is piecewise smooth,

$$\mathfrak{L}_* : \Big(\varDelta_{m-1} \times \mathfrak{S}^r_m, \mathrm{Leb}_{\varDelta_{m-1}} \otimes \sum_{\pi\in\mathfrak{S}^r_m} \delta_\pi\Big) \to \Big(\mathfrak{L}_*(\varDelta_{m-1} \times \mathfrak{S}^r_m), \mathrm{Leb}_{\varDelta_{m-2}} \otimes \sum_{\pi\in\mathfrak{S}^0_{m-1}} \delta_\pi \otimes m_{[0,1)}\Big)$$

is non-singular. Recall that $\bigcup_{\mathfrak{R}} W_{\mathfrak{R}}$ has full $\mathrm{Leb}_{\varDelta_{m-2}} \otimes \sum_{\pi\in\mathfrak{S}^0_{m-1}} \delta_\pi \otimes m_{[0,1)}$-measure. Therefore $\overline{W} = \mathfrak{L}_*^{-1}(\bigcup_{\mathfrak{R}} W_{\mathfrak{R}})$ has full $\mathrm{Leb}_{\varDelta_{m-1}} \otimes \sum_{\pi\in\mathfrak{S}^r_m} \delta_\pi$-measure.

THEOREM 33. *If* $\pi \in \mathfrak{S}^r_m \cup \mathfrak{S}^l_m$ *then for* $\mathrm{Leb}_{\varDelta_{m-1}}$*-almost every* λ *in* $\varDelta_{m-1}$ *the interval exchange transformation* $T_{(\lambda,\pi)}$ *is disjoint from all ELF transformations.*

Proof. First notice that it suffices to show that if $(\lambda, \pi) \in \overline{W}$ then $T_{(\lambda,\pi)}$ is disjoint from all ELF transformations. Assume that $(\lambda, \pi) \in \overline{W}$. Then $\mathfrak{L}_*(\lambda, \pi) = (\lambda', \pi', \beta) \in W_{\mathfrak{R}}$, where $\mathfrak{R}$ is the Rauzy class of π'. By Proposition 32, the integral transformation $(T_{(\lambda',\pi')})_{f_{(\lambda',\pi',\beta)}}$ is disjoint from all ELF transformations. On the other hand, $(T_{(\lambda',\pi')})_{f_{(\lambda',\pi',\beta)}}$ is isomorphic to $T_{(\lambda,\pi)}$, which completes the proof. ■

REFERENCES

[1] Y. Ahn and M. Lemańczyk, *An algebraic property of joinings*, Proc. Amer. Math. Soc. 131 (2003), 1711–1716.

[2] A. Avila and G. Forni, *Weak mixing for interval exchange transformations and translation flows*, Ann. of Math. 165 (2007), 637–664.

[3] I. P. Cornfeld, S. V. Fomin and Ya. G. Sinai, *Ergodic Theory*, Springer, New York, 1982.

[4] S. Ferenczi, C. Holton and L. Q. Zamboni, *Joinings of three-interval exchange transformations*, Ergodic Theory Dynam. Systems 25 (2005), 483–502.

[5] S. Ferenczi and L. Q. Zamboni, *Examples of 4-interval exchange transformations*, preprint, http://iml.univ-mrs.fr/~ferenczi/fz2.pdf.

[6] K. Frączek and M. Lemańczyk, *A class of special flows over irrational rotations which is disjoint from mixing*, Ergodic Theory Dynam. Systems 24 (2004), 1083–1095.

[7] —, —, *On symmetric logarithm and some old examples in smooth ergodic theory*, Fund. Math. 180 (2003), 241–255.

[8] —, —, *On disjointness property of some smooth flows*, ibid. 185 (2005), 117–142.

[9] H. Furstenberg, *Disjointness in ergodic theory, minimal sets and diophantine approximation*, Math. Systems Theory 1 (1967), 1–49.

[10] —, *Recurrence in Ergodic Theory and Combinatorial Number Theory,* Princeton Univ. Press, 1981.

[11] H. Furstenberg and B. Weiss, *The finite multipliers of infinite ergodic transformations*, in: Lecture Notes in Math. 668, Springer, 1978, 127–132.

[12] E. Glasner, *On the multipliers of $\mathcal{W}^{\perp}$*, Ergodic Theory Dynam. Systems 14 (1994), 129–140.

[13] —, *Ergodic Theory via Joinings*, Math. Surveys Monogr. 101, Amer. Math. Soc., 2003.

[14] E. Glasner and B. Weiss, *Processes disjoint from weak mixing*, Trans. Amer. Math. Soc. 316 (1989), 689–703.

[15] A. Gross and J. Robertson, *Ergodic properties of random measures on stationary sequences of sets*, Stochastic Process. Appl. 46 (1993), 249–265.

[16] B. Host, J.-F. Méla et F. Parreau, *Analyse harmonique des mesures*, Astérisque 135–136, 1986.

[17] A. Janicki and A. Weron, *Simulation and Chaotic Behavior of α-Stable Stochastic Processes*, Pure Appl. Math. 178, Dekker, 1994.

[18] A. del Junco, *A family of counterexamples in ergodic theory*, Israel J. Math. 44 (1983), 160–188.

[19] A. del Junco and M. Lemańczyk, *Simple systems are disjoint from Gaussian systems*, Studia Math. 133 (1999), 249–256.

[20] A. del Junco, M. Lemańczyk and M. K. Mentzen, *Semisimplicity, joinings and group extensions,* ibid. 112 (1995), 141–164.

[21] A. del Junco and D. Rudolph, *On ergodic actions whose self-joinings are graphs*, Ergodic Theory Dynam. Systems 7 (1987), 531–557.

[22] A. B. Katok, *Interval exchange transformations and some special flows are not mixing*, Israel J. Math. 35 (1980), 301–310.

[23] J. L. King, *Joining-rank and the structure of finite rank mixing transformations*, J. Anal. Math. 51 (1988), 182–227.

[24] J. F. C. Kingman, *Poisson Processes*, Oxford Stud. Probab. 3, Clarendon Press. Oxford, 1993.

[25] A. V. Kochergin, *Non-degenerate saddles and absence of mixing*, Mat. Zametki 19 (1976), 453–468 (in Russian).

[26] M. Lemańczyk and E. Lesigne, *Ergodicity of Rokhlin cocycles*, J. Anal. Math. 85 (2001), 43–86.

[27] M. Lemańczyk, E. Lesigne and D. Skrenty, *Multiplicative Gaussian cocycles*, Aequationes Math. 61 (2001), 162–178.

[28] M. Lemańczyk and F. Parreau, *Rokhlin cocycles and lifting disjointness*, Ergodic Theory Dynam. Systems 23 (2003), 1525–1550.

[29] M. Lemańczyk, F. Parreau and J.-P. Thouvenot, *Gaussian automorphisms whose ergodic self-joinings are Gaussian*, Fund. Math. 164 (2000), 253–293.

[30] M. Ledoux and M. Talagrand, *Probability in Banach Spaces* (*Isoperimetry and Processes*), Springer, Berlin, 1991.

[31] G. Maruyama, *Infinitely divisible processes*, Theory Probab. Appl. 15 (1979), 1–22.

[32] Yu. A. Neretin, *Categories of Symmetries and Infinite-Dimensional Groups*, Oxford Univ. Press, 1996.

[33] W. Parry, *Topics in Ergodic Theory*, Cambridge Univ. Press, Cambridge, 1981.

[34] G. Rauzy, *Échanges d'intervalles et transformations induites*, Acta Arith. 34 (1979), 315–328.

[35] M. Ratner, *Horocycle flows, joinings and rigidity of products*, Ann. of Math. (2) 118 (1983), 277–313.

[36] E. Roy, *Mesures de Poisson, infinie divisibilité et propriétés ergodiques*, PhD Thesis, Univ. Paris 6, 2005, http://www.proba.jussieu.fr/~roy.

[37] D. J. Rudolph, *An example of a measure-preserving map with minimal self-joinings, and application*, J. Anal. Math. 35 (1979), 97–122.

[38] D. J. Rudolph, *k-fold mixing lifts to weakly mixing isometric extensions*, Ergodic Theory Dynam. Systems 5 (1985), 445–447.
[39] T. de la Rue, *Rang des systèmes dynamiques gaussiens*, Israel J. Math. 104 (1998), 261–283.
[40] V. V. Ryzhikov, *Mixing, rank and minimal self-joining of actions with invariant measure*, Mat. Sb. 183 (1992), 133–160 (in Russian); English transl.: Russian Acad. Sci. Sb. Math. 75 (1993), 405–427.
[41] —, *Stochastic intertwinings and multiple mixing of dynamical systems*, J. Dynam. Control Systems 2 (1996), 1–19.
[42] —, *Polymorphisms, joinings and the tensor simplicity of dynamical systems*, Funktsional. Anal. i Prilozhen. 31 (1997), no. 2, 45–57 (in Russian); English transl.: Funct. Anal. Appl. 31 (1997), 109–118.
[43] V. V. Ryzhikov and J.-P. Thouvenot, *Disjointness, divisibility, and quasi-simplicity of measure-preserving actions*, Funktsional. Anal. i Prilozhen. 40 (2006), no. 3, 85–89 (in Russian); English transl,: Funct. Anal. Appl. 40 (2006), 237–240.
[44] G. Samorodnitsky and M. Taqqu, *Stable Non-Gaussian Random Processes: Stochastic Models with Infinite Variance*, Chapman and Hall, London, 1993.
[45] M. Smorodinsky and J.-P. Thouvenot, *Bernoulli factors that span a transformation*, Israel J. Math. 32 (1979), 39–43.
[46] J.-P. Thouvenot, *Some properties and applications of joinings in ergodic theory*, in: Ergodic Theory and its Connections with Harmonic Analysis, London Math. Soc. Lecture Note Ser. 205, Cambridge Univ. Press, 1995, 207–235.
[47] —, *Les systèmes simples sont disjoints de ceux qui sont infiniment divisibles et plongeables dans un flot*, Colloq. Math. 84/85 (2000), 481–483.
[48] W. A. Veech, *Interval exchange transformations*, J. Anal. Math. 33 (1978), 222–278.
[49] —, *A criterion for a process to be prime,* Monatsh. Math. 94 (1982), 335–341.
[50] —, *Gauss measures for transformations on the space of interval exchange maps*, Ann. of Math. (2) 115 (1982), 201–242.
[51] —, *The metric theory of interval exchange transformations I. Generic spectral properties*, Amer. J. Math. 106 (1984), 1331–1358.
[52] M. Wysokińska, *A class of real cocycles over an irrational rotation for which Rokhlin cocycle extensions have Lebesgue component in the spectrum*, Topol. Methods Nonlinear Anal. 24 (2004), 387–407.
[53] R. Zimmer, *Extensions of ergodic group actions*, Illinois J. Math. 20 (1976), 373–409.

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